

Two-Degree-of-Freedom Controller Design for Linear Parameter-Varying Systems

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Abstract:

A design strategy for Linear Parameter-Varying (LPV) systems is considered in a Two-Degree-Of-Freedom (TDOF) control framework. First, a coprime factorization for LPV systems is introduced. Second, based on the coprime factorization, a TDOF control framework of LTI systems is extended to that of LPV systems. Good tracking performance and good disturbance rejection are achieved by a feedforward controller and a feedback controller, respectively. Furthermore, each controller design problem can be formulated in terms of a linear matrix inequality related to the L2 gain performance. Finally, a simple design example is illustrated.

1. Introduction

Two-Degree-Of-Freedom (TDOF) control scheme is fundamental strategy for the design of linear time invariant (LTI) systems to deal with both command tracking and disturbance rejection independently [1,2]. Not only control performance but also controller structures can be independently dealt with in a TDOF framework [3-7].

On the other hand, over the past fifteen years, significant progress has been made in the analysis and design of linear parameter-varying (LPV) systems, and the gain scheduled control method has been established [8,9]. One of the most significant results is the design method of output-feedback gain-scheduled controllers with guaranteed L2 gain performance [10]. However, to the authors' knowledge, the general TDOF control scheme including command tracking of LPV systems has not yet been discussed. Since we cannot use transfer functions or eigenvalues of state matrices in LPV framework, the above mentioned TDOF control scheme for LTI systems cannot be applied in a straightforward manner to LPV systems.

In this paper, we extend TDOF control framework of LTI systems to that of LPV systems. As a beginning, doubly coprime factorization for LPV systems is introduced. A two-step controller design approach is then proposed. First, a feedforward LPV controller that achieves good tracking performance is designed by a model matching with L2 gain performance. Second, a feedback controller that rejects disturbances and/or model uncertainties is designed. The feedback controller

can be designed independent of the feedforward one and also meets the other L2 gain performance.

Each controller design problem can be formulated in terms of linear matrix inequality.

2. Preliminaries

In this section, doubly coprime factorization described in state space formulas for LPV systems is introduced based on a parameter-dependent Lyapunov function.

Suppose a stabilizable and detectable LPV plant $G(\theta)$ has state-space realization:

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad (1)$$

where continuous time signals $x(t) \in \mathfrak{R}^n$, $u(t) \in \mathfrak{R}^q$ and $y(t) \in \mathfrak{R}^s$ denote the state, control input and measurement output vectors, respectively. Each matrix in (1) has a compatible dimension.

Moreover we assume that the state-space matrices $A(\theta), B(\theta), C(\theta), D(\theta)$ depend affinely on bounded and continuous time-varying parameter $\theta(t) = [\theta_1(t), \theta_2(t), \dots, \theta_r(t)]^T \in \Theta$. To simplify

notation, we define $G(\theta) \triangleq \left[\begin{array}{c|c} A(\theta) & B(\theta) \\ \hline C(\theta) & D(\theta) \end{array} \right]$.

Matrices $F(\theta)$ and $L(\theta)$ can be chosen such that both $A(\theta) - B(\theta)F(\theta)$ and $A(\theta) - L(\theta)C(\theta)$ are parameter-dependent quadratically stable. That is, there exist $X(\theta) > 0$ and $Y(\theta) > 0$ satisfying the following conditions:

$$\dot{X}(\theta) + (A(\theta) - B(\theta)F(\theta))^T X(\theta) + X(\theta)(A(\theta) - B(\theta)F(\theta)) < 0, \quad (2)$$

$$\dot{Y}(\theta) + (A(\theta) - L(\theta)C(\theta))^T Y(\theta) + Y(\theta)(A(\theta) - L(\theta)C(\theta)) < 0. \quad (3)$$

The LPV plant $G(\theta)$ is said to have doubly coprime factorization if there exist right coprime factorization $G(\theta) = N(\theta)M(\theta)^{-1}$ and left coprime factorization $G(\theta) = \tilde{M}(\theta)^{-1}\tilde{N}(\theta)$, where a set of matrices: $N(\theta)$, $M(\theta)$, $\tilde{N}(\theta)$, $\tilde{M}(\theta)$, $U(\theta)$, $V(\theta)$, $\tilde{U}(\theta)$ and $\tilde{V}(\theta)$ can be chosen such that

$$\begin{bmatrix} \tilde{V}(\theta) & \tilde{U}(\theta) \\ -\tilde{N}(\theta) & \tilde{M}(\theta) \end{bmatrix} \begin{bmatrix} M(\theta) & -U(\theta) \\ N(\theta) & V(\theta) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \quad (4)$$

A particular set of stable state-space realizations of these matrices can be chosen such that

$$\begin{bmatrix} M(\theta) & -U(\theta) \\ N(\theta) & V(\theta) \end{bmatrix} \triangleq \left[\begin{array}{c|cc} A(\theta) - B(\theta)F(\theta) & B(\theta) & L(\theta) \\ \hline & -F(\theta) & I & 0 \\ C(\theta) - D(\theta)F(\theta) & D(\theta) & I \end{array} \right], \quad (5)$$

$$\begin{bmatrix} \tilde{V}(\theta) & \tilde{U}(\theta) \\ -\tilde{N}(\theta) & \tilde{M}(\theta) \end{bmatrix} \triangleq \left[\begin{array}{c|cc} A(\theta) - L(\theta)C(\theta) & B(\theta) - L(\theta)D(\theta) & L(\theta) \\ \hline & F(\theta) & I & 0 \\ & -C(\theta) & -D(\theta) & I \end{array} \right], \quad (6)$$

where, $A(\theta) - B(\theta)F(\theta)$ and $A(\theta) - L(\theta)C(\theta)$ satisfies (2) and (3) respectively.

3. TDOF controller design for an LPV plant

In this section, the conventional TDOF control structure of LTI systems will be extended to that of LPV systems using doubly coprime factorization of an LPV plant. It is well known that LTI feedback control systems can be constructed as in Fig. 1, in which signals r , u , d and y denote reference inputs, control inputs, output disturbances, and controlled outputs, respectively. In addition, K_{ff1} and K_{ff2} are feedforward controllers, K_{fb} is a feedback controller, and G is a plant.

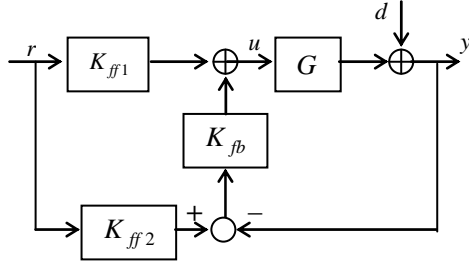


Fig.1. Configuration of general LTI control systems

In this configuration, if K_{ff1} is set to be zero and K_{ff2} is set to be a unit matrix, then Fig. 1 represents a One-Degree-Of-Freedom (ODOF) control system. Otherwise, Fig. 1 represents a TDOF control system. In particular, if K_{ff1} is set to be MK_{ff} and K_{ff2} is set to be NK_{ff} , where M and N are the right and left factors of right coprime factorization of the plant, then the feedback and feedforward controllers can be designed independently. This TDOF configuration of LTI systems can be easily extended to that of LPV systems with doubly coprime factorization of the LPV plant introduced in Section 2. In Fig. 2, we have $u = MK_{ff}r + K_{fb}e$ and $e = NK_{ff}r - Gu - d$. The substitution the former equation into the latter equation, considering $G = NM^{-1}$ and $y = Gu + d$, we finally obtain the following equation:

$$y = N(\theta)K_{ff}(\theta)r + (I + G(\theta)K_{fb}(\theta))^{-1}d. \quad (7)$$

We see that controller: K_{fb} works only for disturbance d and as a consequence, we can say that a feedback controller and a feedforward controller can be designed independently. Namely, tracking performance and disturbance rejection (as well as robustness against model uncertainties) are independently achieved by a feedforward controller: $K_{ff}(\theta)$, and a feedback controller: $K_{fb}(\theta)$,

respectively.

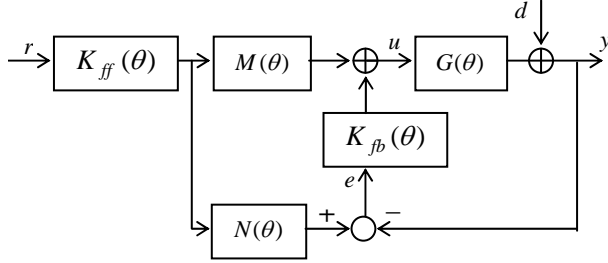


Fig.2. Two-degree-of-freedom structure of LPV systems

With the Linear Fractional Transformation (LFT), the design of the gain-scheduled output-feedback controller in [10] provides internal stability and guarantees L2 gain bound for the augmented LPV plant. Here, the design of each LPV controller, $K_{ff}(\theta)$ and $K_{fb}(\theta)$, can be reduced to the design of gain-scheduled controller by considering each augmented plant in LFT configuration.

In the remainder of this section, a two-step controller design approach is introduced. First, a feedforward controller: $K_{ff}(\theta)$ that achieves good tracking performance is designed by the model matching with L2 gain performance. Second, a feedback controller: $K_{fb}(\theta)$ that rejects disturbances and model uncertainties, is also designed with the other L2 gain performance. Each controller design problem can be formulated in terms of a linear matrix inequality.

3.1. Augmented plant for feedforward controllers

We consider the output error between the target signal and the controlled output. Accordingly, the augmented plant: $P_{ff}(\theta)$ for the tracking problem can be constructed as shown in Fig. 3 with

reference model: $T(\theta)$ and a weighting function: $W_r(\theta)$.

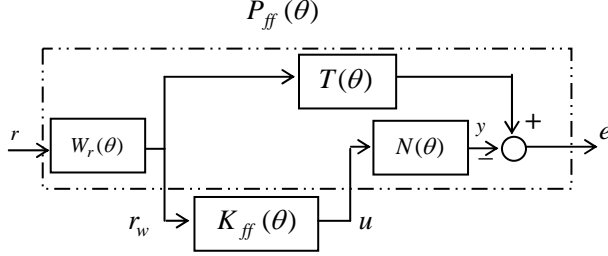


Fig.3. LFT configuration for the command tracking problem

According to Fig. 3, the augmented plant has the following relationship:

$$\begin{bmatrix} e \\ r_w \end{bmatrix} = P_{ff}(\theta) \begin{bmatrix} r \\ u \end{bmatrix} = \begin{bmatrix} T(\theta)W_r(\theta) & -N(\theta) \\ W_r(\theta) & 0 \end{bmatrix} \begin{bmatrix} r \\ u \end{bmatrix}. \quad (8)$$

Here, the $P_{ff}(\theta)$ is derived with the state space realization of $W_r(\theta) \triangleq \begin{bmatrix} A_{wr}(\theta) & B_{wr}(\theta) \\ C_{wr}(\theta) & D_{wr}(\theta) \end{bmatrix}$,

$$T(\theta) \triangleq \begin{bmatrix} A_t(\theta) & B_t(\theta) \\ C_t(\theta) & D_t(\theta) \end{bmatrix} \text{ and } N(\theta) \triangleq \begin{bmatrix} A(\theta) - B(\theta)F(\theta) & B(\theta) \\ C(\theta) - D(\theta)F(\theta) & D(\theta) \end{bmatrix} \text{ as}$$

$$P_{ff}(\theta) \triangleq \begin{bmatrix} A_{wr} & B_{wr}C_t & 0 & B_{wr}D_t & 0 \\ 0 & A_t & 0 & B_t & 0 \\ 0 & 0 & A - BF & 0 & -B \\ \hline C_{wr} & D_{wr}C_t & C - DF & D_{wr}D_t & -D \\ C_{wr} & 0 & 0 & D_{wr} & 0 \end{bmatrix}. \quad (9)$$

Hereinafter, the dependent parameter θ in matrix factors will be omitted.

The design problem of the feedforward controller: $K_{ff}(\theta)$ obtaining L2 gain performance related to r and e is formulated in terms of the linear matrix inequality mentioned later in Subsection 3.3. Note that the $K_{ff}(\theta)$ itself should be a stable controller. In particular, when the inverse of $N(\theta)$ is proper and stable, a simple feedforward controller: $K_{ff}(\theta)$ designed by exact model matching is obtained as $K_{ff}(\theta) = N^{-1}(\theta) \cdot T(\theta)$ and is realized as

$$K_{ff}(\theta) \triangleq \left[\begin{array}{cc|c} A - BD^{-1}C & -BD^{-1}C_t & -BD^{-1}D_t \\ \hline 0 & A_t & B_t \\ \hline D^{-1}C - F & D^{-1}C_t & D^{-1}D_t \end{array} \right]. \quad (10)$$

3.2. Augmented plant for feedback controllers

The standard gain-scheduled control methodology can be applied to the design of a feedback controller $K_{fb}(\theta)$. In this case, the augmented plant is constructed as shown in Fig. 4 with a weighting function: $W_d(\theta)$.

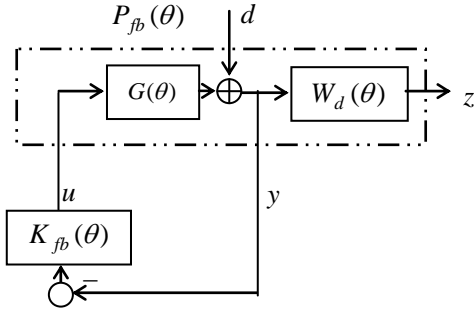


Fig.4. LFT configuration for the disturbance rejection problem

According to Fig. 4, the augmented plant has the following relationship

$$\begin{bmatrix} z \\ y \end{bmatrix} = P_{fb}(\theta) \begin{bmatrix} d \\ u \end{bmatrix} = \begin{bmatrix} W_d & W_d G \\ I & G \end{bmatrix} \begin{bmatrix} d \\ u \end{bmatrix}, \quad (11)$$

and is derived with the state space realization of $W_d(\theta) \triangleq \begin{bmatrix} A_{wd} & B_{wd} \\ C_{wd} & D_{wd} \end{bmatrix}$ as

$$P_{fb}(\theta) \triangleq \left[\begin{array}{cc|cc} A_{wd} & B_{wd}C & B_{wd} & B_{wd}D \\ \hline 0 & A & 0 & B \\ \hline C_{wd} & D_{wd}C & D_{wd} & D_{wd}D \\ \hline 0 & C & I & D \end{array} \right]. \quad (12)$$

Likewise the design of $K_{ff}(\theta)$, the design problem of $K_{fb}(\theta)$ obtaining L2 gain performance related to d and z , can also be formulated in terms of the linear matrix inequality.

3.3. Controller design

The necessary and sufficient conditions for the existence of the LPV controller that assure both internal stability and L2 gain performance have been clarified in [10]. Here, both the augmented plants $P_{ff}(\theta)$ and $P_{fb}(\theta)$ are assured to have the following formulation with adequate

substitution, such as $A_p = \begin{bmatrix} A_{wd} & B_{wd}C \\ 0 & A \end{bmatrix}$,

$$P(\theta) \triangleq \left[\begin{array}{c|cc} A_p & B_{p1} & B_{p2} \\ \hline C_{p1} & D_{p11} & D_{p12} \\ \hline C_{p2} & D_{p21} & D_{p22} \end{array} \right]. \quad (13)$$

After the augmented plant (13) is given, controllers can be obtained by means of the well-known procedure given by Theorem 2.1 in Reference [10]. In particular, if it is assumed that the rate of variation $\dot{\theta}$ is unbounded, the result is rephrased as the following.

Theorem

Consider the LPV plant (13) depend affinely on bounded and continuous time-varying parameter $\theta(t) \in \Theta$. There exists a gain-scheduled output-feedback controller (14) enforcing internal stability and a bound γ on the L2 gain of the closed loop system composed of (13) and (14), whenever there exist constant symmetric matrices X and Y and a parameter-dependent quadruple of state-space data $(\hat{A}_k(\theta), \hat{B}_k(\theta), \hat{C}_k(\theta), D_k(\theta))$ holds (15) and (16).

$$K(\theta) \triangleq \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array} \right]. \quad (14)$$

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} \geq 0, \quad (15)$$

$$\begin{bmatrix} XA_p + \hat{B}_k C_{p2} + (*) & * & * & * \\ \hat{A}_k^T + A_p + B_{p2} D_k C_{p2} & A_p Y + B_{p2} \hat{C}_k + (*) & * & * \\ (XB_{p1} + \hat{B}_k D_{p21})^T & (B_{p1} + B_{p2} D_k D_{p21})^T & -\gamma I & * \\ C_{p1} + D_{p12} D_k C_{p2} & C_{p1} Y + D_{p21} \hat{C}_k & D_{p11} + D_{p12} D_k D_{p21} & -\gamma I \end{bmatrix} < 0. \quad (16)$$

Terms denoted by * are induced by symmetry.

In such case, a gain-scheduled controller of the form (14) is readily obtained with the following two-step scheme.

1. Solve for H, J the factorization problem $I - XY = HJ^T$.
2. Compute A_k, B_k, C_k with

$$A_k = H^{-1}(X + H + \hat{A}_k - X(A_p - B_{p2} D_k C_{p2})Y - \hat{B}_k C_{p2} Y - XB_{p2} \hat{C}_k)J^{-T},$$

$$B_k = H^{-1}(\hat{B}_k - XB_{p2} D_k),$$

$$C_k = (\hat{C}_k - D_k C_{p2} Y)J^{-T}.$$

4. Example

4.1. Plant modeling

A simple example of a parameter-varying unstable plant that can be viewed as a mass-spring-damper system with time-varying spring stiffness is considered. The state space equation of this unstable un-weighted LPV plant is as follows [11]:

$$A(\theta) = \begin{bmatrix} 0 & 1 \\ -0.5-0.5\theta & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [-1 \quad 0], \quad D = 0.$$

Here the scope of time-varying parameter $\theta(t)$ is varying in polytopic spaces

$\Theta_1 := \text{Co}\{-1,1\}$ and the trajectory of this parameter is assumed as $\theta = e^{-0.1t} \cdot \cos(100t)$.

4.2. Controller design

Blow we obtain a feedforward and a feedback controller and also a One-degree-of-freedom controller for comparison.

1) Feedforward controller

The reference model is chosen as follows: $T = \frac{1}{s^2 / \omega^2 + 2s\zeta / \omega + 1}$, where $\omega = 22.36 \text{ rad/s}$

and $\zeta = 0.8$. A simple feedforward controller $K_{ff}(\theta)$ with exact model matching is obtained

as $K_{ff}(\theta) \triangleq \sum_{i=1}^2 \alpha_i(t) \left[\begin{array}{c|c} A_{ki} & B_{ki} \\ \hline C_{ki} & D_{ki} \end{array} \right]$, where $\alpha_1(t) = (1 - \theta(t)) / 1.998$ and $\alpha_2(t) = (\theta(t) + 0.998) / 1.998$

with $A_{k1} = A_{k2} = \begin{bmatrix} -35.78 & -7.81 \\ 64 & 0 \end{bmatrix}$, $B_{k1} = B_{k2} = \begin{bmatrix} 128 \\ 0 \end{bmatrix}$, $C_{k1} = [131.46 \quad 30.32]$ $C_{k2} = [131.5 \quad 30.30]$

and $D_{k1} = D_{k2} = -500$.

2) Feedback controller

Here, the LTI weighting function is chosen as $W_d = \frac{s^2 + 101s + 100}{s^2 + 0.105s + 0.0005}$. Using Matlab

LMI toolbox [13], we obtained a suboptimal feedback controller with $\gamma = 2.21$ as

$$K_{fb}(\theta) \triangleq \sum_{i=1}^2 \alpha_i(t) \left[\begin{array}{c|c} A_{ki} & B_{ki} \\ \hline C_{ki} & D_{ki} \end{array} \right] \quad \text{with} \quad A_{k1} = A_{k2} = \begin{bmatrix} 54083 & 3.88 & -1116 & 2.96e5 \\ -21042 & -2.06 & 434.1 & -1.15e5 \\ -2.60e7 & -1867 & 5.37e5 & -1.42e8 \\ -1.40e5 & -35.15 & 2902 & -7.69e5 \end{bmatrix},$$

$$B_{k1} = B_{k2} = \begin{bmatrix} -0.117 \\ -0.704 \\ -4.73e-4 \\ 41.06 \end{bmatrix}, \quad C_{k1} = [-29425164.8 \quad -2111.4 \quad 607108.4 \quad -160955776.9],$$

$$C_{k2} = [-29425164.8 \quad -2111.4 \quad 607108.2 \quad -160955721.5], \quad D_{k1} = D_{k2} = 0.$$

3) One-degree-of-freedom controller

The reference model is chosen to be the same as that used in the feedforward controller. In this case,

we also obtain a simple controller with exact model matching is realized as

$$K_o(\theta) \triangleq \sum_{i=1}^2 \alpha_i(t) \begin{bmatrix} A_{ki} & B_{ki} \\ C_{ki} & D_{ki} \end{bmatrix}, \quad \text{with } A_{k1} = A_{k2} = \begin{bmatrix} -35.78 & 2 \\ 0 & 0 \end{bmatrix}, \quad B_{k1} = \begin{bmatrix} 138.98 \\ 0 \end{bmatrix}, \quad B_{k2} = \begin{bmatrix} 138.98 \\ -1.953 \end{bmatrix},$$

$$C_{k1} = C_{k2} = [128 \quad 0] \quad \text{and} \quad D_{k1} = D_{k2} = -500.$$

4.3. Simulation results

Step reference response and the magnitude of the control signal with the following output disturbance are compared for each control system:

$$d(t) = \begin{cases} 0 & 0 < t < 2.5 \\ -0.2 & t \geq 2.5 \end{cases}.$$

Because the ODOF and TDOF control systems achieve exact model matching, they both maintain nominal response for the reference input, but the TDOF control system realizes better disturbance rejection than the ODOF control system. On the other hand, the magnitude of the control signal for the TDOF control system is larger than that of the ODOF control system. The trade-off between these inconsistent requirements can be considered in the TDOF control system.

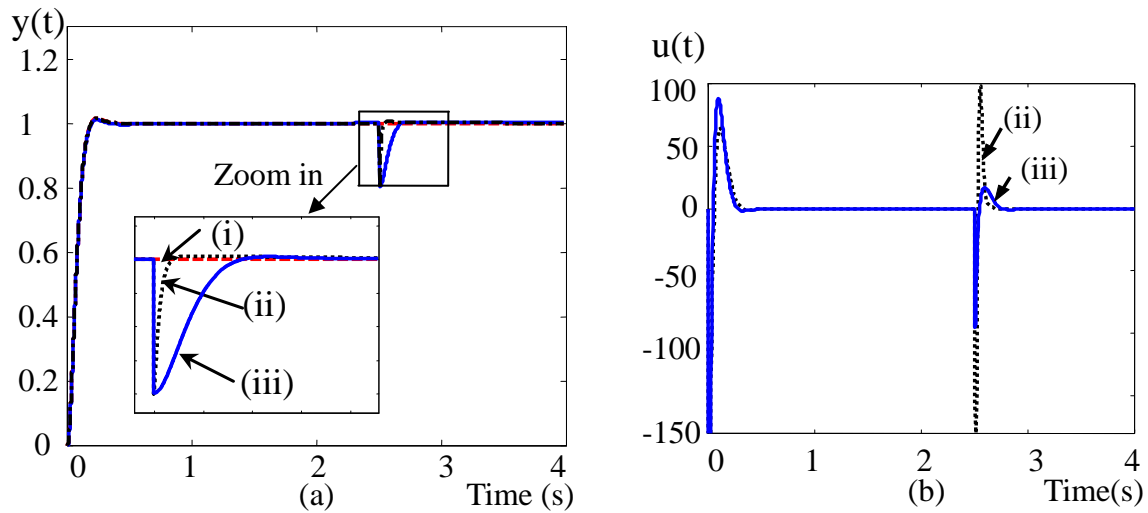


Fig.5 (a): Step reference responses with output disturbance: (b): Magnitude of control input

(i) Nominal response without disturbance, (ii) TDOF control system, (iii) ODOF control system.

5. Conclusions

We have developed a two-step design strategy for LPV systems in a TDOF framework. The first step is to design a feedforward controller that achieves good tracking performance by a model matching with L2 gain performance. The second step is to design a feedback controller with the other L2 gain performance that rejects disturbances. Each controller design problem is formulated in terms of a linear matrix inequality and both problems can be solved by the adequate gain scheduled control system design method. We herein focus on only L2 gain performance in designing the controllers. The Q -parameter approach will provide more practical validity for deducing the solution and covers more general control system designs, including multi-objective and/or switching systems

for LPV plants. Based on these results, the Q -parameter approach is also applicable to LPV control systems, and this approach will be considered in another paper.

References

- [1] Horowitz, I. M., *Synthesis of Feedback Systems*, Academic Press (1963).
- [2] Wolovich, W. A., *Linear Multivariable Systems*, Springer-Verlag (1974).
- [3] Lang, G. and G. Ham, "Conditional Feedback System - A New Approach to Feedback Control-," *Trans. American Institute of Electrical Engineers*, Vol. 74, pp. 152-161 (1953).
- [4] Hara, S. and T. Sugie, "Independent parameterization of two-degree-of-freedom compensators in general robust tracking systems," *IEEE Trans. Automatic Control*, Vol.33, No. 1, pp.59-67 (1988).
- [5] Morari, M. and E. Zafiriou, *Robust Process Control*, Prentice Hall (1989).
- [6] Limebeer, D. J. N., E. M. Kasenally and J. D. Perkins, "On the design of robust two-degree-of-freedom-controllers," *Automatica*, Vol.29, No.1, pp.157-168 (1993).
- [7] Yali, A. R. and T. Eisaka, "Robust Compensator Design for Exploiting Existing Control System," *IEE Proc.-Control Theory and Applications*, Vol. 147, No.1, pp. 71-79 (2000).
- [8] Rugh, W. J. and J. S. Shamma, "Research on gain scheduling," *Automatica*, Vol.36, No.10, pp.1401-1425 (2000).
- [9] Leith, D. J. and W. E. Leithead, "Survey of gain-scheduling analysis and design," *Int. J. Control*,

Vol.73, No.11, pp.1001-1025 (2000).

[10] Apkarian, P. and R. J. Adams, "Advanced Gain-Scheduling Techniques for Uncertain Systems,"

IEEE Trans. on Control Systems Technology, Vol.6, No.1, pp.21-31 (1998).

[11] Xie. W. and T. Eisaka, "Design of LPV control system based on Youla Parameterization," *IEE*

Proc.-Control Theory and Applications, Vol.151, No.4, pp.465-472 (2004).

[12] Apkarian, P, P. Gahinet and G. Becker, "Self-Scheduled H infinity Control of Linear Parameter

Varying System: a Design Example", *Automatica*, Vol. 31, No.9, pp. 1251-1261 (1995).

[13] Gahinet, P., A. A. Nemirovski, J. Laub and M. Chilali, *LMI control toolbox for use with Matlab*,

User's guide, The Math Works Inc Natick, MA, USA (1995)