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Design of LPV Control Systems based on Youla Parameterization


## Abstract

This paper proposes one methodology to design linear parameter varying (LPV) control systems based on parameterization of all quadratically stabilizing controllers. First, conceptions of doubly coprime factorization and Youla parameterization of LTI systems are extended to LPV systems with respect to quadratic stability using state space expression. Consequently, parameterization of close-loop systems, which are affine with any quadratically stable $Q$-parameter, is described. This description enables applying $Q$-parameter approach to a variety of LPV control system designs. Above all, systematic H∞ strategy is focused on and a necessary and sufficient condition and design scheme of $Q$ to obtain $L_2$-gain performance are clarified.


## Introduction

It is well known that doubly coprime factorization is essential tool for analysis and design of linear time invariant (LTI) control systems. Youla parameterization [1, 2], that constructs the whole set of output feedback stabilizing controllers, is also based on the coprime factorization of plants. The realizable closed-loop transfer matrices are affine and readily described with free dynamic $Q$-parameter. A control system design framework based on this description is referred to as "$Q$-parameter approach". A variety of control system designs resolve itself to specify this $Q$-parameter.

In LTI systems, all stabilizing controllers can be parameterized through the use of general solution of "Bezout Identity" on related transfer functions. As for linear time varying (LTV) systems, it is no use to treat transfer functions or eigenvalues to test of stability. And then Youla parameterization or coprime factorization conceptions have not been explored systematically in LTV systems.

On the other hand, Shamma & Athans [3,4] formalized a certain type of nonlinear systems and LTV systems as linear parameter varying (LPV) systems, and succeeded in developing a control strategy for these systems based on classical gain scheduled methodology. Recently, significant progress has been made in this area. In [5], for polytopic LPV systems, a necessary and sufficient condition of quadratic stability was formulated in terms of finite linear matrix inequalities (LMI's). The underlying quadratic Lyapunov function is also used to derive bounds on robust performance measures. A unified H∞ approach is being developed that is also reducible to a LMI optimization problem [6, 7, 8]. Compared to the classical gain scheduling systems design, these approaches take into consideration the time-varying nature of plants and grow out of ad-hoc interpolation. During the last couple of years, tutorial papers and special publications concerning modern gain scheduled issues have appeared [9-13].

In this paper, first, conceptions of doubly coprime factorization and Youla parameterization of LTI systems are extended to LPV systems with respect to quadratic stability using state space expression. Consequently, parameterization of close-loop state space expression, which is affine with any stable LPV $Q$-parameters, is obtained. Accordingly, an LPV control system design resolves itself to specify the $Q$-parameter that satisfies design specifications. Namely we can apply the $Q$-parameter approach to variety of LPV control system designs. Above all, systematic H∞ strategy is focused on, then a necessary and sufficient condition and also a design scheme of $Q$ to obtain $L_2$-gain performance for LPV systems are clarified with above mentioned modern gain scheduled methodology. In particular, as to polytopic LPV systems, this condition can be transferred to be numerical convex optimization problem over finite LMI's.

The paper is organized as follows. First, the definition of LPV systems and some tools are introduced in the preliminary section. In section 2, coprime factorization of LTI systems is overviewed. Then coprime factorization of LPV systems is derived concisely, and then all quadratically stabilizing controllers of LPV systems are obtained in section 3. Design of a compensator $Q$ satisfying the $L_2$-gain performance is discussed in section 4. Finally, a numerical example is presented to illustrate the design method.

## 1. Preliminary

In this section, some notations and assumptions regarding LPV systems are introduced. Useful conceptions and several lemmas are recapped.

**Definition 1: LPV systems**

Consider an LPV plant: $G(\theta(t))$ described by state space equations as:

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \left[\begin{array}{c|cc} A(\theta(t)) & B_w(\theta(t)) & B_u \\ \hline C_z(\theta(t)) & D_{wz}(\theta(t)) & D_{uz} \\ C_y & D_{wy} & 0_{g\times q} \end{array}\right] \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix}. \tag{1}$$

Here $x \in \Re^n$: state vactor, $w \in \Re^p$: exogenous inputs (reference, disturbance, etc.), $u \in \Re^q$: control inputs, $z \in \Re^m$: controlled outputs and $y \in \Re^g$: measurable outputs. Each matrix has a compatible dimension. Moreover we have the following assumptions.

(A1) The state-space matrices $A(\theta), B_w(\theta), C_z(\theta), D_{wz}(\theta)$ depend only on bounded and continuous time-varying parameter $\theta(t) = [\theta_1(t), \theta_2(t), \cdots, \theta_r(t)]^T \in \Theta$.

(A2) The pairs $(A(\theta), B_u)$ and $(A(\theta), C_y)$ are quadratically stabilizable and quadratically detectable, respectively.

*Remark 1*

It is required that the matrices $B_u, C_y, D_{uz}, D_{wy}$ of the plant $G(\theta)$ be time invariant. However, when they are time varying, a simple means to enforces these requirements consists of filtering the control input and the output through low-pass filters having sufficiently large bandwidth. By this trick, the parameter trajectory is shifted into the state matrix $A(\theta)$ [6].

*Remark 2*

If the real parameter $\theta(t)$ is continuous real-time measurable in the polytope of vertices $\omega_1, \omega_2, \cdots, \omega_N,\ \ N = 2^r$; it can be expressed as:

$$\theta(t) \in \Theta_{co} := Co\{\omega_1, \omega_2, \cdots, \omega_N\} = \left\{ \sum_{i=1}^{N} \alpha_i(t)\omega_i : \alpha_i(t) \geq 0, \sum_{i=1}^{N} \alpha_i(t) = 1 \right\}. \tag{2}$$

Then the original LPV plants can be expressed as:

$$\begin{pmatrix} A(\theta) & B_u \\ C_y & 0 \end{pmatrix} = \sum_{i=1}^{N} \alpha_i(t) \begin{pmatrix} A_i & B_u \\ C_y & 0 \end{pmatrix} \quad \text{with} \quad \alpha_i \geq 0, \textstyle\sum_{i=1}^{N} \alpha_i = 1. \tag{3}$$

Here, $A_i = A(\omega_i)$ for $i = 1, \cdots, N$.

In this case, such LPV systems are called polytopic LPV systems.

**Definition 2**

The $L_2$-gain of an operator having input $w$ and output $z$ is defined as

$$\sup_{w \neq 0, \|w\|_2 < \infty} \frac{\|z\|_2}{\|w\|_2} \tag{4}$$

where $\|z\|_2 = (\int_0^\infty z^T(t) z(t) dt)^{1/2}$ is the $L_2$-norm of the signal $z$.

**Lemma 1: Quadratic stability**

Considering an LPV system, described as $\dot{x}(t) = A(\theta)x(t)$, a necessary and sufficient condition for quadratic stability of this system is there exists $P > 0$ such that

$$A^T(\theta)P + PA(\theta) < 0 \quad . \tag{5}$$

*Remark 3*

For polytopic LPV systems, we have the equivalent conditions for (5) as

$$A_i^T P + PA_i < 0, \quad i = 1, \cdots, N. \tag{6}$$

Lemma 2

Suppose matrices $A_{11}(\theta)$ and $A_{22}(\theta)$ are quadratically stable, every continuous and bounded block triangle matrix whose diagonal matrices consist of $A_{11}(\theta)$ and $A_{22}(\theta)$ is quadratically stable.

Proof:

Consider a lower triangle matrix: $A(\theta)=\begin{bmatrix} A_{11}(\theta) & 0 \\ A_{21}(\theta) & A_{22}(\theta)\end{bmatrix}$. According to the assumption, there exist bounded positive definite matrices $P_1, P_2$ and positive real numbers $r_1, r_2$, respectively, which satisfy

$$A_{11}^T(\theta)P_1 + P_1A_{11}(\theta) \le -r_1 I, \quad -r_2 I \le A_{22}^T(\theta)P_2 + P_2A_{22}(\theta) < 0. \tag{7}$$

Since $A_{21}(\theta)$ is also bounded, there exists a positive real number $r_3$, which satisfies

$$P_2A_{21}(\theta) \le r_3 I\,. \tag{8}$$

According to the lemma 1, if the following inequality given by $P=\begin{bmatrix} P_1 & 0 \\ 0 & \lambda P_2\end{bmatrix} > 0$ and $A(\theta)$ is satisfied, then $A(\theta)$ is said to be quadratically stable.

$$A^T(\theta)P + PA(\theta) = \begin{bmatrix} P_1A_{11}(\theta)+A_{11}^T(\theta)P_1 & \lambda A_{21}^T(\theta)P_2 \\ \lambda P_2A_{21}(\theta) & \lambda P_2A_{22}(\theta)+\lambda A_{22}^T(\theta)P_2\end{bmatrix} < 0\,. \tag{9}$$

The rest of the proof, we find positive real number $\lambda > 0$, which satisfies (9). Using Schur Complement, inequality (9) is equivalent to the following simultaneous inequalities as

$$\lambda(P_2A_{22}(\theta)+A_{22}^T(\theta)P_2) < 0$$

$$\pi = P_1A_{11}(\theta)+A_{11}^T(\theta)P_1 - \lambda(A_{21}^T(\theta)P_2)[P_2A_{22}(\theta)+A_{22}^T(\theta)P_2]^{-1}(P_2A_{21}(\theta)) < 0 \tag{10}$$

The first inequality clearly holds with any $\lambda > 0$. On second inequality, from (7) and (8), we obtain

$$\pi \le -r_1 I + \lambda r_2 (A_{21}^T(\theta)P_2)(P_2A_{21}(\theta) \le -r_1 I + \lambda r_2 r_3^2 I\,. \tag{11}$$

The right hand side inequality holds for any $0 < \lambda < r_1 r_2^{-1} r_3^{-2}$.

The upper triangle matrices can be deduced in the same way. ■

## 2 Q-parameter approach to LTI systems

In this section, coprime factorization and Youla parameterization of LTI systems are remineded and then Q-parameter approach to general LTI feedback control systems shown in Fig.1 is summarized.

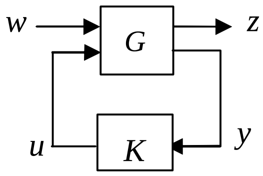


Fig.1 General feedback control systems

For LTI system ( $\theta$ is frozen), original plant $G_{22}(s)$ that is transfer function of $G(s)$ between $u$ and $y$ has realization derived from (1) as

$$G_{22}(s) \triangleq \left[\begin{array}{c|c} A & B_u \\ \hline C_y & 0 \end{array}\right] \tag{12}$$

where the pairs $(A, B_u)$ and $(A, C_y)$ are also stabilizable and detectable respectively. The doubly coprime factorization over RH∞ is given by

$$G_{22} = N_r D_r^{-1} = D_l^{-1} N_l \tag{13}$$

where

$$\begin{bmatrix} D_r & \hat{X}_l \\ N_r & \hat{Y}_l \end{bmatrix} \cdot \begin{bmatrix} \hat{Y}_r & -\hat{X}_r \\ -N_l & D_l \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{14}$$

It is well known that the $G_{22}(s)$-$K(s)$ feedback system is stable if and only if $G_{22} = N_r D_r^{-1} = D_l^{-1} N$ and $K(s) = \hat{Y}_r^{-1} \hat{X}_r = \hat{X}_l \hat{Y}_l^{-1}$ satisfies Bezout Identity shown as (14). The general solution of $(\hat{Y}_r, \hat{X}_r)$ or $(\hat{X}_l, \hat{Y}_l)$ is described as a particular solution and kernels. Namely, all stabilizing controllers $K(s)$ are parameterized from the particular solution $K_0(s) = Y_r^{-1} X_r = X_l Y_l^{-1}$ as follows:

$$K(s) = (X_l + D_r Q)(Y_l + N_r Q)^{-1} = (Y_r + QN_l)^{-1}(X_r + QD_l) \text{ for any } Q \in RH_\infty . \tag{15}$$

The central controller is in $RH_\infty$ and has realizations as

$$\begin{bmatrix} D_r & X_l \\ N_r & Y_l \end{bmatrix} \triangleq \left[\begin{array}{c|cc} A + B_u F & B_u & -L \\ \hline F & I & 0 \\ C_y & 0 & I \end{array}\right],$$

$$\begin{bmatrix} Y_r & -X_r \\ -N_l & D_l \end{bmatrix} \triangleq \left[\begin{array}{c|cc} A+LC_y & -B_u & L \\ \hline F & I & 0 \\ C_y & 0 & I \end{array}\right] \tag{16}$$

where both $A+B_uF$ and $A+LC_y$ are stable.

The closed-loop transfer matrix between $w$ and $z$ can be expressed in the form of lower Linear Fractional Transformation (LFT) by

$$F_l(G,K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}\,. \tag{17}$$

We substitute $K(I-G_{22}K)^{-1} = -(X_l + D_rQ)D_l$ into the above equation; the closed-loop transfer matrix is written in affine form with $Q$-parameter as

$$\begin{aligned} F_l(G,K) &= G_{11} - G_{12}X_lD_lG_{21} - G_{12}D_rQD_lG_{21} \\ &= T_{11} + T_{12}QT_{21} \end{aligned} \tag{18}$$

where

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \triangleq \left[\begin{array}{cc|cc} A+B_uF & B_uF & B_w & B_u \\ 0 & A+LC_y & -B_w-LD_{wy} & 0 \\ \hline C_z+D_{uz}F & D_{uz}F & D_{wz} & D_{uz} \\ 0 & -C_y & D_{wy} & 0 \end{array}\right]. \tag{19}$$

Consequently, design of a control system resolves itself to fix the $Q$ in order to satisfy design specifications. For instance, $Q$-parameter approach on $H_\infty$ control is minimizing $\infty$-norm of $F_l(G,K)$ selecting a suitable $Q$ such that

$$\inf_K \|F_l(G,K)\| = \inf_Q \|T_{11} + T_{12}QT_{21}\|_\infty \le \gamma\,. \tag{20}$$

## 3. Youla parameterization of LPV systems

In this section, conception of doubly coprime factorization is extended to LPV systems. Based on this factorization, we introduce a way of obtaining all quadratically stabilizing controllers construction for LPV systems.

### 3.1 Quadratically stabilizing observer-based controller for LPV system

The original LPV plant $G_{22}(\theta)$ can be expressed as

$$\begin{aligned} \dot{x} &= A(\theta)x + B_uu \\ y &= C_yx \end{aligned} \tag{21}$$

**Lemma 3**

A quadratically stabilizing observed-based controller for LPV plant (21) can be formulated as

$$\dot{\hat{x}} = A(\theta)\hat{x} + B_u u + L(\theta)(C_y \hat{x} - y)$$
$$u = F(\theta)\hat{x} \tag{22}$$

where $F(\theta), L(\theta)$ is continuous function of dependent parameter $\theta(t)$.

Moreover, $F(\theta) = V(\theta)P_f^{-1}$ and $L(\theta) = P_l^{-1}W(\theta)$ satisfy the following LMI's as

$$P_f > 0, \quad A(\theta)P_f + P_f A^T(\theta) + B_u V(\theta) + V^T(\theta)B_u^T < 0,$$

$$P_l > 0, \quad A^T(\theta)P_l + P_l A(\theta) + W(\theta)C_y + C_y^T W^T(\theta) < 0. \tag{23}$$

Proof: if the controller (22) is substituted into plant (21), the closed-loop state matrix can be expressed as

$$A_{cl}(\theta) = \begin{bmatrix} A(\theta) + B_u F(\theta) & B_u F(\theta) \\ 0 & A(\theta) + L(\theta)C_y \end{bmatrix} \tag{24}$$

Based on (23) and lemma 1, we see that $A(\theta) + B_u F(\theta)$ and $A(\theta) + L(\theta)C_y$ are quadratically stable. Thus, using lemma 2, the system (24) is said to be quadratically stable.■

*Remark 4*

For polytopic LPV systems, the equivalent conditions of (23) are written as the following finite LMI's

$$P_f > 0, \quad A_i P_f + P_f A_i^T + B_u V_i + V_i^T B_u^T < 0, \quad i = 1, \cdots, N,$$

$$P_l > 0, \quad A_i^T P_l + P_l A_i + W_i C_y + C_y^T W_i^T < 0, \quad i = 1, \cdots, N \tag{25}$$

where $F_i = V_i P_f^{-1}$ and $L_i = P_l^{-1} W_i$. We can construct $F(\theta)$ and $L(\theta)$ as

$$F(\theta) = \sum_{i=1}^{N} \alpha_i(t) F_i \quad \text{and} \quad L(\theta) = \sum_{i=1}^{N} \alpha_i(t) L_i . \tag{26}$$

The quadratically stabilizing observer-based controller for a LPV plant shown in Fig.1 can be expressed as Fig.2.

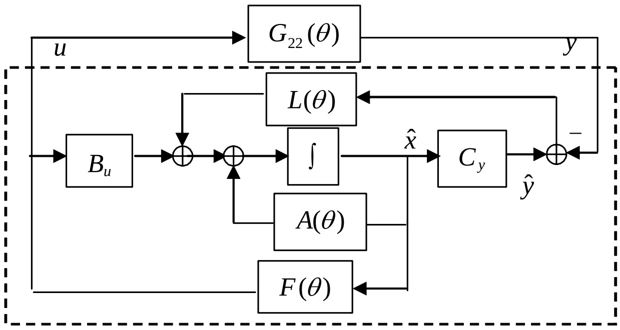


Fig.2. Quadratically stabilizing observed-based controller

**Definition 3**: Coprime factorization of LPV plants:

Now we can define the coprime factorization of LPV plants that resemble the case of LTI systems. The doubly coprime factorization of LPV plants $G_{22}(\theta)$ is given by

$$G_{22}(\theta) = N_r(\theta)D_r^{-1}(\theta) = D_l^{-1}(\theta)N_l(\theta) \tag{27}$$

where

$$\begin{bmatrix} D_r(\theta) & X_l(\theta) \\ N_r(\theta) & Y_l(\theta) \end{bmatrix} \triangleq \left[\begin{array}{c|cc} A(\theta)+B_uF(\theta) & B_u & -L(\theta) \\ \hline F(\theta) & I & 0 \\ C_y & 0 & I \end{array}\right],$$

$$\begin{bmatrix} Y_r(\theta) & -X_r(\theta) \\ -N_l(\theta) & D_l(\theta) \end{bmatrix} \triangleq \left[\begin{array}{c|cc} A(\theta)+L(\theta)C_y & -B_u & L(\theta) \\ \hline F(\theta) & I & 0 \\ C_y & 0 & I \end{array}\right] \tag{28}$$

where $F(\theta)$ and $L(\theta)$ satisfy the LMI's (23). (especially (25) for polytopic LPV plants.)

We can derive the following expressions

$$\begin{bmatrix} D_r(\theta) & X_l(\theta) \\ N_r(\theta) & Y_l(\theta) \end{bmatrix} \cdot \begin{bmatrix} Y_r(\theta) & -X_r(\theta) \\ -N_l(\theta) & D_l(\theta) \end{bmatrix} \triangleq \left[\begin{array}{cc|c} A+B_uF(\theta) & \begin{bmatrix} B_u & -L(\theta) \end{bmatrix}\begin{bmatrix} F(\theta) \\ C_y \end{bmatrix} & \begin{bmatrix} B_u & -L(\theta) \end{bmatrix} \\ 0 & A+LC_y & \begin{bmatrix} -B_u & L(\theta) \end{bmatrix} \\ \hline \begin{bmatrix} F(\theta) \\ C_y \end{bmatrix} & \begin{bmatrix} F(\theta) \\ C_y \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \end{array}\right]$$

$$\triangleq \left[\begin{array}{cc|c} A+B_uF(\theta) & B_uF(\theta)-L(\theta)C_y & \begin{bmatrix} B_u & -L(\theta) \end{bmatrix} \\ 0 & A+LC_y & \begin{bmatrix} -B_u & L(\theta) \end{bmatrix} \\ \hline \begin{bmatrix} F(\theta) \\ C_y \end{bmatrix} & \begin{bmatrix} F(\theta) \\ C_y \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \end{array}\right]. \tag{29}$$

With similar transformation matrix $T = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$ and eliminating both stable uncontrollable and unobservable modes, we finally obtain

$$\begin{bmatrix} D_r(\theta) & X_l(\theta) \\ N_r(\theta) & Y_l(\theta) \end{bmatrix} \cdot \begin{bmatrix} Y_r(\theta) & -X_r(\theta) \\ -N_l(\theta) & D_l(\theta) \end{bmatrix} \triangleq \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{30}$$

This relationship supports that (27) is said to be coprime factorization.

3.2 All quadratically stabilizing controllers for LPV systems

In this subsection, parameterization of all quadratically stabilizing controllers for LPV systems is investigated.

**Theorem 1**

Let LPV plants $G(\theta)$ be described in (1) and controllers $K(\theta)$ be connected as Fig.1. Then, all quadratically stabilizing controllers can be parameterized as $F_l(M(\theta), Q(\theta))$ for any quadratically stable $Q(\theta)$ with $M(\theta)$,

where

$$M(\theta) \triangleq \left[\begin{array}{c|cc} A(\theta)+B_uF(\theta)+L(\theta)C_y & -L(\theta) & B_u \\ \hline F(\theta) & 0 & I \\ -C_y & I & 0 \end{array}\right] \tag{31}$$

and the adjustable dynamics $Q(\theta)$ with any dimensions should be quadratically stable one of dependent parameter $\theta(t)$ such as

$$Q(\theta) \triangleq \left[\begin{array}{c|c} A_Q(\theta) & B_Q(\theta) \\ \hline C_Q(\theta) & D_Q(\theta) \end{array}\right] \tag{32}$$

where there exists $P_Q > 0$ such that

$$A_Q^T(\theta)P_Q + P_QA_Q(\theta) < 0. \tag{33}$$

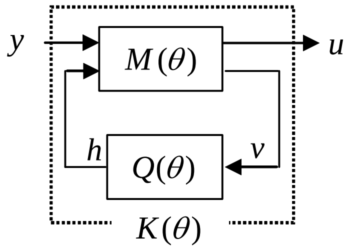


Fig.3 Structure of all quadratically stabilizing controllers

Proof:

*Sufficiency*:

Using (31) and (32), the controller $K(\theta)$ is derived as

$$K(\theta)=\left[\begin{array}{cc|c} A(\theta)+B_uF(\theta)+L(\theta)C_y-B_uD_Q(\theta)C_y & B_uC_Q(\theta) & B_uD_Q(\theta)-L(\theta) \\ -B_Q(\theta)C_y & A_Q(\theta) & B_Q(\theta) \\ \hline F(\theta)-D_Q(\theta)C_y & C_Q(\theta) & D_Q(\theta) \end{array}\right]. \tag{34}$$

Connecting (34) to (21), the closed loop system can be expressed as

$$\begin{bmatrix}\dot{x}\\ \dot{e}\\ \dot{x}_Q\end{bmatrix}=\begin{bmatrix} A(\theta)+B_uF(\theta) & B_uF(\theta)-B_uD_Q(\theta)C_y & B_uC_Q(\theta)\\ 0 & A(\theta)+L(\theta)C_y & 0\\ 0 & -B_Q(\theta)C_y & A_Q(\theta)\end{bmatrix}\begin{bmatrix}x\\ e\\ x_Q\end{bmatrix} \tag{35}$$

where $e=\hat{x}-x$, $x_Q$ is state vector of the $Q(\theta)$.

Using Lemma 2 twice in succession, we can find $P_{cl}$>0 with adequate $\lambda_f>0$ and $\lambda_q>0$ formed as

$$P_{cl}=\begin{bmatrix}\lambda_f P_f & 0 & 0\\ 0 & P_l & 0\\ 0 & 0 & \lambda_q P_q\end{bmatrix} \tag{36}$$

where $P_f>0, P_l>0$ and $P_q>0$ with compatible dimensions satisfy (5) for system (35).

*Necessity*:

We will show that arbitrary stabilizing controller $K(\theta)$ can be expressed with an appropriate quadratically stable $Q_0(\theta)$ as $K(\theta)=F_l(M(\theta),Q_0(\theta))$.

First, consider $Q_0(\theta)=F_l(\hat{M}(\theta),K(\theta))$, where $\hat{M}(\theta)$ is realized as

$$\hat{M}(\theta) = \left[\begin{array}{c|cc} A(\theta) & -L(\theta) & B_u \\ \hline -F(\theta) & 0 & I \\ C_y & I & 0 \end{array}\right]. \tag{37}$$

Since (2, 2) input-output relation of $\hat{M}(\theta)$ is same as it of original plant (21), then $K(\theta)$ stabilizes not only the plant but also the $\hat{M}(\theta)$ . Accordingly $Q_0 = F_l(\hat{M}(\theta), K(\theta))$ is quadratically stable.

The substitution of the $Q_0$ into $F_l(M(\theta), Q_0(\theta))$ yields $F_l(M(\theta), Q_0(\theta)) = F_l(M(\theta), F_l(\hat{M}(\theta), K(\theta))) = F_l(J_{tmp}(\theta), K(\theta))$,

where $J_{tmp}$ is obtained using star product formula as

$$J_{tmp} \triangleq \left[\begin{array}{cc|cc} A(\theta)+L(\theta)C_y+B_uF(\theta) & -B_uF(\theta) & -L(\theta) & B_u \\ L(\theta)C_y & A(\theta) & -L(\theta) & B_u \\ \hline F(\theta) & -F(\theta) & 0 & I \\ -C_y & C_y & I & 0 \end{array}\right]. \tag{38}$$

Similar transform with $T = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$ and eliminating stable uncontrollable and unobservable mode, we have

$$J_{tmp} \triangleq \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{39}$$

Consequently, the relation that we want to show is deduced.

$$F_l(M(\theta), Q_0(\theta)) = F_l(J_{tmp}(\theta), K(\theta)) = K(\theta). \blacksquare$$

### 4. Design of function $Q$ to achieve L2 gain performance

So far, parameterization of close-loop LPV systems with any stable $Q(\theta)$ is obtained. Based on this parameterization, LPV control system designs resolve themselves to settle this $Q$ that satisfies design specifications. In this section, a systematic H∞ strategy is focused on. Then a necessary and sufficient condition and also a design scheme of $Q$ to obtain $L_2$-gain performance is clarified with LMI methodology. In particular, as to polytopic LPV systems, this condition can be transferred to be numerical convex optimization problem over finite LMI's.

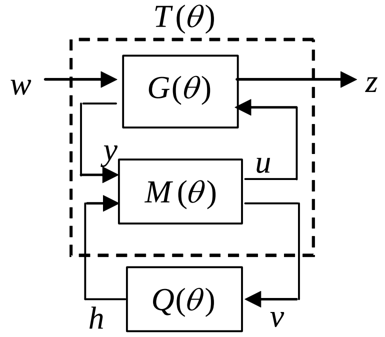


Fig.4 $Q$-parameter scheme of LPV control systems

Fig.4 shows construction of LPV control systems. Here, $G(\theta)$ satisfies assumptions (A1)-(A2) and $M(\theta)$ is an observer-based controller mentioned in Lemma 3. Then, generalized plant $T(\theta)$ is derived as

$$\left[\begin{array}{c|cc} A_t(\theta) & B_{t1}(\theta) & B_{t2} \\ \hline C_{t1}(\theta) & D_{t11}(\theta) & D_{t12} \\ C_{t2} & D_{t21} & D_{t22} \end{array}\right] = \left[\begin{array}{c|cc} \begin{bmatrix} A(\theta)+B_uF(\theta) & B_uF(\theta) \\ 0 & A(\theta)+L(\theta)C_y \end{bmatrix} & \begin{bmatrix} B_w(\theta) \\ -B_w(\theta)-L(\theta)D_{wy} \end{bmatrix} & \begin{bmatrix} B_u \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} C_z(\theta)+D_{uz}F(\theta) & D_{uz}F(\theta) \end{bmatrix} & D_{wz}(\theta) & D_{uz} \\ \begin{bmatrix} 0 & -C_y \end{bmatrix} & D_{wy} & 0 \end{array}\right]$$

(40)

We have the following necessary and sufficient condition for existence of $\gamma$-suboptimal compensators $Q(\theta)$ for the generalized plant $T(\theta)$.

**Theorem 2**

Consider LPV plant (40). Let $N_R$ and $N_s$ denote bases of the null spaces of $(B_{t2}^T, D_{t12}^T)$ and $(C_{t2}, D_{t21})$ respectively. There exists a "quadratically stable" LPV compensator $Q(\theta)$ guaranteeing $\|z\|_2 \le \gamma\|w\|_2$ along all parameter trajectories, if and only if there exist two symmetric matrices $(R,S)$ in $\Re^{2n\times 2n}$ satisfying the following LMI's,

$$\begin{pmatrix} N_R & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} A_t(\theta)R+RA_t^T(\theta) & RC_{t1}^T(\theta) & B_{t1}(\theta) \\ C_{t1}(\theta)R & -\gamma I & D_{t11}(\theta) \\ B_{t1}^T(\theta) & D_{t11}^T(\theta) & -\gamma I \end{pmatrix} \begin{pmatrix} N_R & 0 \\ 0 & I \end{pmatrix} < 0,$$

$$\begin{pmatrix} N_s & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} SA_t(\theta)+A_t^T(\theta)S & SB_{t1}(\theta) & C_{t1}^T(\theta) \\ B_1^T(\theta)S & -\gamma I & D_{t11}^T(\theta) \\ C_{t1}(\theta) & D_{t11}(\theta) & -\gamma I \end{pmatrix} \begin{pmatrix} N_S & 0 \\ 0 & I \end{pmatrix} < 0, \qquad (41)$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \ge 0.$$

Proof:

From (40), it is clear that the pairs $(A_t(\theta), B_{t2})$ and $(A_t(\theta), C_{t2})$ are quadratically stabilizable and quadratically detectable, respectively. Then the conditions (41) can be obtained straightforward from application of [15]. It must be noted that the plant is not $G(\theta)$ but $T(\theta)$. The resulting $\gamma$-suboptimal compensator $Q(\theta)$ quadratically stabilizes LPV generalized plant $T(\theta)$. This means $Q(\theta)$ also quadratically stabilize LPV original plant $G(\theta)$ with $M(\theta)$ in Fig.4. According to theorem 1, $Q(\theta)$ itself is also quadratically stable. ■

The compensator $Q(\theta)$ can be constructed by means of procedures mentioned in Theorem 2.3 of [16] applying to generalized plant $T(\theta)$. A possible solution of quadratically stable compensator $Q$ satisfying above LMI's can be constructed as the following two-step scheme.

1. Solve for $N, M$, the factorization problem $I - RS = NM^T$.
2. Compute $A_Q(\theta), B_Q(\theta), C_Q(\theta)$ with

$$A_Q(\theta) = N^{-1}(\hat{A}_Q(\theta) - R(A_t(\theta) - B_{t2}D_Q(\theta)C_{t2})S - \hat{B}_Q(\theta)C_{t2}S - RB_{t2}\hat{C}_Q(\theta))M^{-T} \quad (42)$$

$$\text{and } B_Q(\theta) = N^{-1}(\hat{B}_Q(\theta) - RB_{t2}D_Q(\theta)),\ C_Q(\theta) = (\hat{C}_Q(\theta) - D_Q(\theta)C_{t2}S)M^{-T} \quad (43)$$

where $D_Q(\theta)$ is solution to $\delta_{\max}(D_{t11} + D_{t12}D_Q(\theta)D_{t21}) < \gamma$ and set $D_{cl}(\theta) = D_{t11}(\theta) + D_{t12}D_Q(\theta)D_{t21}$,

$\hat{B}_Q(\theta)$ and $\hat{C}_Q(\theta)$ are solutions of the linear matrix equations

$$\begin{bmatrix} 0 & D_{t21} & 0 \\ D_{t21}^T & -\gamma I & D_{cl}^T \\ 0 & D_{cl} & -\gamma I \end{bmatrix}\begin{bmatrix} \hat{B}_Q^T \\ * \end{bmatrix} = -\begin{bmatrix} C_{t2} \\ B_{t1}^T R \\ C_{t1} + D_{t12}D_Q C_{t2} \end{bmatrix}$$

$$\begin{bmatrix} 0 & D_{t12}^T & 0 \\ D_{t21} & -\gamma I & D_{cl} \\ 0 & D_{cl}^T & -\gamma I \end{bmatrix}\begin{bmatrix} \hat{C}_Q \\ * \end{bmatrix} = -\begin{bmatrix} B_{t2}^T \\ C_{t1}S \\ (B_{t1} + B_{t12}D_Q D_{t21})^T \end{bmatrix} \quad (44)$$

and

$$\hat{A}_Q(\theta) = -(A_t(\theta) + B_{t2}D_Q(\theta)C_{t2})^T +$$

$$\begin{bmatrix} RB_{t1}(\theta) + \hat{B}_Q(\theta)D_{t21} & (C_{t1}(\theta) + D_{t12}D_Q(\theta)C_{t2})^T \end{bmatrix}\begin{bmatrix} -\gamma I & D_{cl}^T \\ D_{cl} & -\gamma I \end{bmatrix}^T \begin{bmatrix} (B_{t1} + B_{t2}D_Q D_{t21})^T \\ C_{t1}S + D_{t12}\hat{C}_Q \end{bmatrix} \quad (45)$$

Generally, the solvability for conditions (41) is difficult to be obtained since it is infinite LMI's problem, however, as to polytopic LPV systems, these conditions can be transferred to finite vertex LMI's problems.

*Remark 5*

For polytopic LPV systems with trajectories in the polytope

$$\theta(t) \in \Theta_{co} := Co\{\omega_1, \omega_2, \cdots, \omega_N\} = \left\{\sum_{i=1}^{N} \alpha_i(t)\omega_i : \alpha_i(t) \geq 0, \sum_{i=1}^{N} \alpha_i(t) = 1\right\}, \quad (46)$$

the generalized plant has the expression as

$$\left[\begin{array}{c|cc} A_t(\theta) & B_{t1}(\theta) & B_{t2} \\ \hline C_{t1}(\theta) & D_{t11}(\theta) & D_{t12} \\ C_{t2} & D_{t21} & D_{t22} \end{array}\right] = \sum_{i=1}^{N} \alpha_i(t) \left[\begin{array}{c|cc} A_{ti} & B_{t1i} & B_{t2} \\ \hline C_{t1i} & D_{t11i} & D_{t12} \\ C_{t2} & D_{t21} & 0 \end{array}\right] \quad \text{with} \quad \alpha_i \geq 0, \sum_{i=1}^{N} \alpha_i = 1. \quad (47)$$

The corresponding conditions (41) can be written equivalently as the following finite $2N+1$ LMI's,

$$\begin{pmatrix} N_R & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} A_{ti}R + RA_{ti}^T & RC_{t1i}^T & B_{t1i} \\ C_{1i}R & -\gamma I & D_{t11i} \\ B_{t1i}^T & D_{t11i}^T & -\gamma I \end{pmatrix} \begin{pmatrix} N_R & 0 \\ 0 & I \end{pmatrix} < 0,$$

$$\begin{pmatrix} N_s & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} SA_{ti} + A_{ti}^T S & SB_{t1i} & C_{t1i}^T \\ B_{1i}^T S & -\gamma I & D_{t11i}^T \\ C_{t1i} & D_{t11i} & -\gamma I \end{pmatrix} \begin{pmatrix} N_S & 0 \\ 0 & I \end{pmatrix} < 0, \quad i = 1, \cdots, N, \quad (48)$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0.$$

The controller $Q(\theta)$ in this case also can be constructed with interpolation of vertex controllers $Q_i$ with time varying coefficients $\alpha_i$'s.

## 5. Example

A classical example of parameter-varying unstable plant that can be viewed as a mass-spring-damper system with time-varying spring stiffness is considered. The state space equation of this unstable un-weighted LPV plant is as follows

$$A(\theta) = \begin{bmatrix} 0 & 1 \\ -0.5 - 0.5\theta & -0.2 \end{bmatrix}, B_w = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_z = \begin{bmatrix} -1 & 0 \end{bmatrix}, \; D_{wz} = 1, D_{uz} = 0$$

$$C_y = \begin{bmatrix} -1 & 0 \end{bmatrix}, \; D_{wy} = 1, \; D_{uy} = 0. \quad (49)$$

Here the scope of nominal time-varying parameter $\theta(t)$ is in polytopic spaces $\Theta_1 := Co\{-1,1\}$.

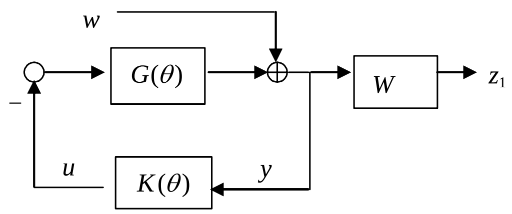


Fig.5 Block diagram for design

### 5.1 A quadratic stabilizing observed-based controller

According to the following LMI's

$$P_f > 0, \quad A_i P_f + P_f A_i^T + B_u V_i + V_i^T B_u^T < 0, \quad i = 1,2 ,$$

$$P_l > 0, \quad A_i^T P_l + P_l A_i + W_i C_y + C_y^T W_i^T < 0, \quad i = 1,2 \tag{50}$$

Using Matlab toolbox [14], We get

$$P_f = \begin{bmatrix} 81.9 & -32.7 \\ -32.7 & 81.9 \end{bmatrix} , F_1 = \begin{bmatrix} -0.43 & -0.87 \end{bmatrix} \text{ and } F_2 = \begin{bmatrix} -1.43 & -0.87 \end{bmatrix}$$

and $$P_l = \begin{bmatrix} 149.6 & -33.7 \\ -33.7 & 163.2 \end{bmatrix}, \; L_1 = \begin{bmatrix} 0.75 & 0.11 \end{bmatrix}^T \quad \text{and} \quad L_2 = \begin{bmatrix} 0.75 & 1.11 \end{bmatrix}^T \tag{51}$$

which satisfy the above six linear inequalities. Consequently, we can obtain a quadratically stabilizing controller

$$\begin{aligned} \dot{\hat{x}} &= A(\theta)\hat{x} + B_u u + L(\theta)(C_y \hat{x} - y) \\ u &= F(\theta)\hat{x} + h \end{aligned} \tag{52}$$

by setting $F(\theta)$ and $L(\theta)$ as $F(\theta) = \sum_{i=1}^{2} \alpha_i(t) F_i$ and $L(\theta) = \sum_{i=1}^{2} \alpha_i(t) L_i$ , where $\alpha_1(t) = (1-\theta(t))/2$ and $\alpha_2(t) = (1+\theta(t))/2$ .

### 5.2 The controller satisfying the constrained conditions

In Fig.5 we choose the weighting function as

$$W = \frac{50}{s+1} . \tag{53}$$

This weight has bandwidth $1\,rad/s$, so it might be used to get good tracking. The constrained condition is chosen, such that $\|z_1\|_2 \le \gamma \|w\|_2$, where $\gamma$ is set as 0.55.

Generalized plant $T(\theta)$ also can be derived as

$$\left[\begin{array}{c|cc} A_t(\theta) & B_{t1}(\theta) & B_{t2} \\ \hline C_{t1}(\theta) & D_{t11}(\theta) & D_{t12} \\ C_{t2} & D_{t21} & D_{t22} \end{array}\right] =$$

$$\left[\begin{array}{ccccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1.42-\theta & -1.07 & -0.92-0.5\theta & -0.87 & 0 & 0 & 1 \\ 0 & 0 & -0.75 & 1 & 0 & 0.75 & 0 \\ 0 & 0 & -1.11-\theta & -0.2 & 0 & -0.61+0.5\theta & 0 \\ -50 & 0 & 0 & 0 & -1 & 50 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \tag{54}$$

Since, $T(\theta)$ is polytopic in the spaces $\Theta_1 := Co\{-1,1\}$, the process of obtaining $Q$ is transferred to finite vertex LMI's and we have the following fifth order quadratically stable $Q$.

$$A_{Q1} = \begin{bmatrix} -6.99e3 & -10.9 & -11.9 & -1.61e3 & -2.12e6 \\ -4.23e3 & -0.91 & 0.98 & -977 & 1.28e6 \\ 8.97e3 & -1.16 & -2.38 & 2.07e3 & 2.73e6 \\ 2.79e4 & -0.95 & -7.43 & 6.46e3 & 8.52e6 \\ -14.3 & 55.9 & 67.0 & -8.83 & -2.12e4 \end{bmatrix}, \quad B_{Q1} = \begin{bmatrix} 16.8 \\ -0.18 \\ 1.06 \\ 3.75 \\ -88.2 \end{bmatrix}$$

$$A_{Q2} = \begin{bmatrix} -6.99e3 & -10.9 & -12.0 & -1.61e3 & -2.12e6 \\ -4.23e3 & -0.20 & 1.85 & -977 & 1.28e6 \\ 8.97e3 & -1.70 & -3.05 & 2.07e3 & 2.73e6 \\ 2.79e4 & -0.64 & -7.13 & 6.46e3 & 8.52e6 \\ -14.3 & 55.9 & 67.0 & -8.83 & -2.12e4 \end{bmatrix}, \quad B_{Q2} = \begin{bmatrix} 16.7 \\ 0.20 \\ 0.77 \\ 3.89 \\ -88.2 \end{bmatrix}$$

$$C_{Q1} = \begin{bmatrix} 3.05e4 & 1.44 & -5.68 & 7.04e3 & 9.28e6 \end{bmatrix},$$

$$C_{Q2} = \begin{bmatrix} 3.05e4 & 1.16 & -6.08 & 7.04e3 & 9.28e6 \end{bmatrix}$$

$$A_Q(\theta) = \sum_{i=1}^{2} \alpha_i(t) A_{Qi}, \quad B_Q(\theta) = \sum_{i=1}^{2} \alpha_i(t) B_{Qi} \quad \text{and} \quad C_Q(\theta) = \sum_{i=1}^{2} \alpha_i(t) C_{Qi}. \tag{55}$$

The controller $K(\theta)$ can be constructed as (34), which guarantees stability of closed-loop system and $L_2$ gain performance $\gamma$=0.55.

## 6. Conclusions

One methodology to design LPV control systems through the use of Youla-parameterization has been proposed. First, conceptions of doubly coprime factorization and Youla parameterization of LTI systems have been extended to LPV systems with respect to quadratic stability. In LTI systems, internal stability can be assured from Bezout Identity. Instead, as for LTV systems, stability is assured through Lyapunov inequality.

The next step, parameterization of close-loop systems with any quadratically stable $Q$-parameter in affine fashion has been described. Consequently, $Q$-parameter approach can be applicable to variety of LPV control system designs. Among them, a systematic H∞ strategy has been focused on and a necessary and sufficient condition and a design scheme of $Q$ to obtain $L_2$-gain performance have been clarified with LMI methodology.

When we just focus on $L_2$-gain performance, we can obtain controllers straightforward using gain scheduled methodology [6], [16]. The $Q$-parameter approach, however, have more practical validity to deduce the solution. Namely, time-varying nevanlinna-pick methodology [17] is also applied to solve corresponding Nehari problem. Moreover, $Q$-parameter approach covers more general control system designs including multi-objective and/or switching systems. This merit is cased by two-step or separate structure shown in Fig.6 and affine fashion of the free parameter. The Object-oriented design framework, that is, design of a module controller depending on each control specification is excellent with $Q$-parameter approach.

References


1. YOULA, D.C., JABR, H., and BONGIORNO, J.J.: 'Modern Wiener-Hopf design of optimal control controllers: part II : The multivariable case', *IEEE Trans. Automatic. Control,* 1976, **21**, pp. 319-338
2. KUČERA, V.: 'Discrete linear Control: the Polynomial Equation Approach' (New York: Wiley, 1979).
3. SHAMMA, J.S., and ATHANS, M.: 'Analysis of nonlinear gain-scheduled control systems', *IEEE Trans., on Automatic Control,* 1990, **35**, pp.898-907
4. SHAMMA, J.S., and ATHANS, M.: 'Guaranteed properties of gain-scheduled control for Linear parameter-varying plants', *Automatica*, 1991, **27**, pp.559-564
5. BOYD, S., GHAOUI, L.E., FERON, E., and BALAKRISHNAN, V.: 'Linear Matrix Inequality in Systems and Control Theory' (SIAM Studies in Applied Mathematics, Vol.15, SIAM, Philadelphia, 1994)
6. APKARIAN, P., GAHINET, P., and BECK, G.: 'Self-scheduled H∞ control of linear parameter varying systems: a design example', *Automatica*, 1995, **31**, (9), pp.1251-1261

7. APKARIAN, P., GAHINET, P.: 'A convex characterization of gain scheduled H infinite controllers', *IEEE Trans. on Automatic. Control*, 1995, **40**, pp.853-864
8. BECKER, G., PACKARD, A., PHILBRICK, D., and BLAS, G.: 'Control of Parametrically-dependent linear systems: a single quadratic Lyaponov approach', *Proc. American Control Conf., San Francisco*, 1993, pp.2795-2799
9. WU, F.: 'A generalized LPV system analysis and control synthesis framework,' *INT.J.Control.*, 2001, **74**, (7), pp. 745-759
10. SCHERER, C.W.: 'LPV control and full block multipliers', *Automatica, 2001*, **37**, pp. 361-375
11. WANG, F., and BALAKRISHNAN. V.: 'Improved stability analysis and gain-scheduled Controller synthesis for parameter-dependent systems', *IEEE Trans on Automatic Control*, 2002,**47**, (5), pp.720-734
12. RUGH, W.J., and SHAMMA. J.S.: 'Research on gain scheduling', *Automatica*, 2000, **36**, pp.1401-1425
13. International Journal of Robust and Nonlinear Control, (Vol.12, Issue 9, Special Issue on Gain Scheduling, 2002)
14. GAHINET, P., NEMIROVSKII, A., LAUB, A.J., and CHILALI, M.: 'LMI control toolbox' (Natick, MA: Mathworks, 1995)
15. GAHINET, P., and APKARIAN, P.: 'A Linear Matrix Inequality Approach to H∞ control', *Int., J.Robust Nonlinear Control,* 1994, **4**, pp.421-448
16. APKARIAN, P., and ADAMS, R.J.: 'Advanced Gain-Scheduling Techniques for Uncertain Systems', *IEEE Trans on Control Systems Technology*, 1998, **6**, (1), pp.21-31
17. Gohberg, I. (Ed.): 'Time-varying systems and Interpolation', (Birkhausen, Operator Theory Vol.56, 1992)