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Stability Analysis of the Characteristic Polynomials whose Coefficients are Polynomials of Interval Parameters Using Monotonicity

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In this paper, we analyze the stability of the characteristic polynomials whose coefficients are polynomials of interval parameters via monotonicity methods. Our stability conditions are based on Frazer-Duncan's theorem and all conditions can be checked using only endpoint values of interval parameters. These stability conditions are necessary and sufficient under the monotonicity assumptions.

When the monotonicity conditions do not hold on the whole parameter region, we present an interval division method and a transformation algorithm in order to apply the monotonicity conditions. Then, our stability analysis methods can be applied to all characteristic polynomials whose coefficients are polynomials of interval parameters.

Key Words: stability analysis, interval parameters, monotonicity, Frazer-Duncan's theorem

1. Introduction

In this paper, we study the stability of the following characteristic polynomial $F(s,\boldsymbol{p})$,

$$F(s,\boldsymbol{p}) = c_n(\boldsymbol{p})s^n + c_{n-1}(\boldsymbol{p})s^{n-1} + \ldots + c_0(\boldsymbol{p}) \quad (1)$$

where $\boldsymbol{p} = (p_1,\ldots,p_m)$, and $c_j(\boldsymbol{p}) = c_j(p_1,\ldots,p_m)$ is a polynomial of $p_i \in [\,\underline{p}_i,\ \overline{p}_i\,] = \boldsymbol{I}_i \subset \Re$, $i = 1,\ 2,\ \ldots,m,\ j = 0,\ 1,\ldots,n$, $\boldsymbol{p} \in \mathcal{P} = \boldsymbol{I}_1 \times \boldsymbol{I}_2 \times \cdots \times \boldsymbol{I}_m$, and $c_n(\boldsymbol{p}) > 0$

It is one of the problems of the control theory whether the characteristic polynomial (1) is stable or not for all parameter values belonging to $\mathcal{P}$. Essentially, for our present purpose of the stability analysis of (1), it is satisfactory if the Routh-Hurwitz conditions hold for any $\boldsymbol{p} \in \mathcal{P}$. But, how we can ascertain it, this is the central and very difficult problem. In this respect, it seems that the following theorem given by Frazer and Duncan is most fundamental.

Theorem 1. (Frazer and Duncan)[1),2)] The characteristic polynomial (1) is stable if and only if the following two conditions are satisfied.

1) The largest Hurwitz determinant $H_n(\boldsymbol{p})$ does not vanish for any $\boldsymbol{p} \in \mathcal{P}$.

2) There exists a $\boldsymbol{p}' \in \mathcal{P}$ where $F(s,\boldsymbol{p}')$ is stable.

Theorem 1 is equivalent to Hermite-Biehler's theorem[3),4)] and still has the deficiency that the condition 1) is difficult to verify. For the single parameter case: $m = 1$, we have the tools such as Sturm's theorem[5)] and so on if the coefficients are polynomials of the one parameter. For example, if Theorem 1 is applied to the segment of polynomials

$$F(s,\lambda) = (1-\lambda)p_0(s) + \lambda p_1(s),\ \lambda \in [0,1], \qquad (2)$$

where the highest coefficients of $p_0(s)$ and $p_1(s)$ are positive,

we can derive the following result by Hwang and Yang directly.

Theorem 2. (Hwang and Yang)[6)] The convex combination of given polynomials $p_0(s)$ and $p_1(s)$ defined in (2) is Hurwitz stable for all $\lambda \in [0,1]$ if and only if the following conditions hold:

1) One of the end-point polynomials $p_0(s)$ and $p_1(s)$ is Hurwitz stable and the other is positive at $s = 0$.

2) The entry $a_{n-1,0}(\lambda)$ of the optimal fraction free Routh array is positive for $\lambda \in [0,1]$.

The optimal fraction free Routh array is given by Jeltsch [7)], and Hurwitz determinants $H_1, H_2, H_3, \cdots, H_n$ appear in the first column of his fraction free Routh array. In this case, the entry $a_{n-1,0}(\lambda) = H_{n-1}(\lambda)$. And condition 1) of Theorem 2 means that the constant term of s in $F(s,\lambda)$ is positive and $F(s,\lambda)$ is stable at one of the endpoints

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of λ. Since $H_n(\lambda)$ is a product of $H_{n-1}(\lambda)(= a_{n-1,0}(\lambda))$ and the constant term of $F(s, \lambda)$, Theorem 2 is a segment polynomial version of Theorem 1.

Moreover, the condition 1) of Theorem 1 is the most simple one from the viewpoint of calculation of the Routh array, which is learned from the theory of optimal fraction free Routh array by Jeltsch. But, for the multi-parameter case: $m > 1$, there is not yet any result comparable with Sturm's theorem as in the one parameter case for checking condition 1) of Theorem 1 on the whole parameter region.

In order to study the multi-parameter case, we have proposed the use of monotonicity property of multivariable polynomials and presented stability conditions which we have to calculate only at the endpoints of $\mathcal{P}$ [9)~11)].

There are some cases, in which the monotonicity conditions do not hold on the whole parameter region. If the zeros of partial derivative with respect to an interval parameter do not depend on other parameters, we divide the parameter interval to subintervals so that the monotonicity conditions hold. On the other hand, if the zeros of partial derivative with respect to interval parameters depend on other parameters, we make a monotone polynomial with new independent interval parameters. The latter procedure is the modification of Sideris' method [8)] which transforms the polynomial to the multi-linear form. And the number of new independent parameters in our method is not larger than that of Sideris' method.

Finally, we will illustrate our method and stability conditions by an example. Since our stability conditions with transformation algorithm are sufficient, so we will compare the conservativeness of our stability conditions with that of Šiljak's conditions and Sideris' conditions in the same example. And the relation between our stability conditions and the edge theorem will be discussed by using two examples.

2. Monotone multi-variable polynomials

The characteristic polynomial (1) is also derived from the linear system in the state space form with structured uncertainties. In case of complex systems, interval parameters appear in polynomial form in the coefficients of the characteristic polynomials. Some of them are given in the examples in the reference 2). If we use the maximum and the minimum values in place of polynomials of interval parameters in the stability analysis, the derived stability conditions become only sufficient and have the stability margin. Since, in general, it is difficult to check the stability of the characteristic polynomial (1), several assumptions and alterations are made for the stability analysis. In this paper, we use the assumption of the monotonicity with respect to interval parameters in (1).

There are m interval parameters in (1), so we prepare the following definition for the convenience of expressions.

Definition. In this paper, we consider parameters p_1, p_2, ..., p_m; $p_i \in [\, \underline{p}_i,\ \overline{p}_i \,] = \boldsymbol{I}_i \subset \Re$, $i = 1, 2, \ldots, m$.
For the parameter region $\mathcal{P}$ defined in (1), we denote its endpoint as $\boldsymbol{p}^*$ and the set of endpoints as $\mathcal{P}^*$; $\boldsymbol{p}^* \in \mathcal{P}^*$. And we define $\hat{\boldsymbol{p}}_k = (p_1, p_2, \ldots, p_{k-1}, p_{k+1}, \ldots, p_m)$, $\hat{\boldsymbol{p}}_k \in \hat{\mathcal{P}}_k = \boldsymbol{I}_1 \times \boldsymbol{I}_2 \times \cdots \times \boldsymbol{I}_{k-1} \times \boldsymbol{I}_{k+1} \times \cdots \times \boldsymbol{I}_m$, $k = 1, 2, \ldots, m$. For the parameter region $\hat{\mathcal{P}}_k$, we also denote its endpoint as $\hat{\boldsymbol{p}}_k^*$ and the set of endpoints as $\hat{\mathcal{P}}_k^*$; $\hat{\boldsymbol{p}}_k^* \in \hat{\mathcal{P}}_k^*$. □

Definition. (Monotone polynomial of $\boldsymbol{p}$ on $\mathcal{P}$) [9)] Let $\phi(\boldsymbol{p})$ be a polynomial of $p_i \in [\, \underline{p}_i,\ \overline{p}_i \,] \subset \Re$, $i = 1, \ldots, m$. If one of the following inequalities:

$$\frac{\partial \phi(\boldsymbol{p})}{\partial p_i} \geq 0, \ p_i \in [\underline{p}_i,\ \overline{p}_i] \tag{3}$$

or

$$\frac{\partial \phi(\boldsymbol{p})}{\partial p_i} \leq 0, \ p_i \in [\underline{p}_i,\ \overline{p}_i] \tag{4}$$

holds for any fixed $\hat{\boldsymbol{p}}_i \in \hat{\mathcal{P}}_i$, $\phi(\boldsymbol{p})$ is called a monotone polynomial of p_i.
If $\phi(\boldsymbol{p})$ is monotone with respect to all p_i, $i = 1, \cdots, m$, then it is called a monotone polynomial of $\boldsymbol{p}$. □

How to check the monotonicity conditions

In the reference 9), we have provided a finite procedure with which we can check the monotonicity condition (3) or (4) by using only the endpoints $\boldsymbol{p}^* \in \mathcal{P}^*$. In some cases, our procedure requires finite but many steps of calculations with the aid of computer algebra system. In this respect, it is worth to note that there are two cases where the monotonicity can be checked immediately.

Case 1: If the partial derivative is in the form of

$$\frac{\partial \phi(\boldsymbol{p})}{\partial p_i} = \psi(\hat{\boldsymbol{p}}_i), \tag{5}$$

where $\psi(\hat{\boldsymbol{p}}_i)$ is a polynomial of $\hat{\boldsymbol{p}}_i$,

$\phi(\boldsymbol{p})$ is monotone in p_i. The proof is obvious.

Case 2: If the partial derivative is calculated as

$$\frac{\partial \phi(\boldsymbol{p})}{\partial p_i} = \psi_1(p_i)\psi_2(\hat{\boldsymbol{p}}_i), \tag{6}$$

where $\psi_1(p_i)$ is a polynomial of p_i

and $\psi_2(\hat{\boldsymbol{p}}_i)$ is a polynomial of $\hat{\boldsymbol{p}}_i$,

the monotonicity of $\phi(\boldsymbol{p})$ with respect to p_i can be checked by $\psi_1(p_i)$. If one of the following conditions:

$$\psi_1(p_i) \geq 0 \text{ for } p_i \in [\underline{p}_i,\ \overline{p}_i] \tag{7}$$

or

$$\psi_1(p_i) \leq 0 \text{ for } p_i \in [\underline{p}_i,\ \overline{p}_i] \tag{8}$$

holds, $\phi(\boldsymbol{p})$ is a monotone polynomial of p_i.

The case 1 is a special case of the case 2, where $\psi_1(p_i)$ is constant. And if the polynomial $\phi(\boldsymbol{p})$ is linear or affine or multi-linear with respect to $\boldsymbol{p}$, $\phi(\boldsymbol{p})$ is in the case 1. So we separate the case 1 from the case 2.

In the case 2, if $\psi_1(p_i)$ changes its sign on the interval $[\underline{p}_i,\ \overline{p}_i]$, we can divide the interval into subintervals and $\psi_1(p_i)$ is monotone on each subinterval. In addition, the conditions (7) or (8) can be checked by using Sturm's theorem at the endpoints of p_i.

Of course, there are other cases where we can check easily the monotonicity of $\phi(\boldsymbol{p})$. The typical case is given in the section 5.1.

3. Stability analysis with monotonicity assumptions

In this section, we derive the stability conditions under the monotonicity assumptions. In our previous papers [9)~11)], we have used the notion of monotonicity for multiparameter polynomials to derive stability conditions. If a polynomial is monotone with respect to interval parameters, we can derive its maximum and minimum values on the whole parameter region using the value of polynomial at the endpoints of the parameter region $\mathcal{P}$. Thus, if a polynomial satisfies the monotonicity conditions, we can show its positivity on the whole parameter region using only the endpoint values of parameters. Applying this idea to the condition 1) of Theorem 1, we obtain the stability conditions as follows.

Since the condition 1) of Theorem 1 is expressed by means of Hurwitz determinant, we denote Hurwitz determinants as $H_1(\boldsymbol{p})$, $H_2(\boldsymbol{p})$, $\cdots$, $H_n(\boldsymbol{p})$.

$$H_n(\boldsymbol{p}) = \begin{vmatrix} c_{n-1}(\boldsymbol{p}) & c_{n-3}(\boldsymbol{p}) & c_{n-5}(\boldsymbol{p}) & \cdots \\ c_n(\boldsymbol{p}) & c_{n-2}(\boldsymbol{p}) & c_{n-4}(\boldsymbol{p}) & \cdots \\ 0 & c_{n-1}(\boldsymbol{p}) & c_{n-3}(\boldsymbol{p}) & \cdots \\ 0 & c_n(\boldsymbol{p}) & c_{n-2}(\boldsymbol{p}) & \cdots \\ 0 & 0 & c_{n-1}(\boldsymbol{p}) & \cdots \\ 0 & 0 & c_n(\boldsymbol{p}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} \tag{9}$$

Then, we have the stability conditions:

Theorem 3. We assume $H_{n-1}(\boldsymbol{p})$ and $c_0(\boldsymbol{p})$ are monotone in $\boldsymbol{p}$, then the characteristic polynomial (1) is stable if and only if the following two conditions hold.

1) $H_{n-1}(\boldsymbol{p})$ and $c_0(\boldsymbol{p})$ are positive at all endpoints $\boldsymbol{p}^* \in \mathcal{P}^*$.

2) $F(s,\boldsymbol{p})$ is stable at one of the endpoints $\boldsymbol{p}^* \in \mathcal{P}^*$.

Proof. We assume $c_n(\boldsymbol{p})$ is positive in (1). If $F(s,\boldsymbol{p})$ is stable, all coefficients of $F(s,\ \boldsymbol{p})$ are positive and all $H_i(\boldsymbol{p}) > 0$, $i = 1, 2, \cdots, n$ on the whole parameter region $\mathcal{P}$. Then all conditions of Theorem 3 hold.

Under the monotonicity condition of $H_{n-1}(\boldsymbol{p})$ and $c_0(\boldsymbol{p})$, the minimum value of $H_{n-1}(\boldsymbol{p})$ and $c_0(\boldsymbol{p})$ with respect to $\boldsymbol{p}$ are found at some endpoints in $\mathcal{P}^*$, respectively. Then, if the condition 1) of Theorem 3 holds, $H_{n-1}(\boldsymbol{p})$ and $c_0(\boldsymbol{p})$ are positive on the whole parameter region $\mathcal{P}$. Since $H_n(\boldsymbol{p}) = H_{n-1}(\boldsymbol{p})c_0(\boldsymbol{p})$, if the condition 1) of Theorem 3 holds, then the condition 1) of Theorem 1 is satisfied on the whole parameter region $\mathcal{P}$. The condition 2) of Theorem 3 is equivalent to the condition 2) of Theorem 1. Thus, under the assumption of monotonicity, condition 1) and 2) of Theorem 3 are necessary and sufficient for the stability of $F(s,\ \boldsymbol{p})$ on the whole parameter region $\mathcal{P}$ □

4. Interval division method and transformation algorithm

Unfortunately, there are some characteristic polynomials which do not satisfy the monotonicity conditions of Theorem 3 or are difficult to verify the monotonicity. Next, we will consider how to apply our monotonicity conditions to the general cases.

4.1 Interval division method

If $f(\boldsymbol{p})$ is not monotone in p_i and zeros of $\partial f(\boldsymbol{p})/\partial p_i = 0$ do not depend on other interval parameters p_k $(i \neq k)$, we divide the interval of p_i at zeros of $\partial f(\boldsymbol{p})/\partial p_i = 0$.

Interval division method If a polynomial $f(\boldsymbol{p})$ is not monotone in p_i, and ℓ zeros $(pz_{i1} < pz_{i2} < \ldots < pz_{i\ell})$ of $\partial f(\boldsymbol{p})/\partial p_i = 0$ do not depend on other interval parameter p_k, $i \neq k$, we divide the parameter interval of p_i into subintervals: $[\underline{p}_i, pz_{i1}]$, $[pz_{i1}, pz_{i2}]$, $\ldots$, $[pz_{i\ell}, \overline{p}_i]$.

Then, $f(\boldsymbol{p})$ is monotone in p_i on each subintervals. □

When we apply this method to $H_{n-1}(\boldsymbol{p})$ or $c_0(\boldsymbol{p})$ which is not monotone in $\boldsymbol{p}$, the conditions of Theorem 3 remain necessary and sufficient on the subintervals.

4.2 Transformation algorithm

In this section, we consider a case where zeros of $\partial f(\boldsymbol{p})/\partial p_i = 0$ depend on the other interval parameters p_k $(i \neq k)$. In this case, we cannot apply the above mentioned interval division method.

Let us consider a simple example of Sideris' method,

$$g = p_1^5 - p_1 p_2,\ p_1 \in [0,\ 1],\ p_2 \in [0,\ 2]. \tag{10}$$

In this case, we have the following derivatives

$$\frac{\partial g}{\partial p_1} = 5p_1^4 - p_2 \tag{11}$$

$$\frac{\partial g}{\partial p_2} = -p_1. \tag{12}$$

g is monotone in p_2(monotone case 1), but not monotone in p_1. In this case, when p_2 is fixed($p_2 \neq 0$), the partial derivative (11) changes its sign on the interval $p_1 \in [0, 1]$. And zeros of $\partial g/\partial p_1 = 0$ depend on p_2, then we cannot divide the interval of p_1 to subintervals.

Sideris and Peña[8] showed how to calculate the maximum value and the minimum value of g with respect to $\boldsymbol{p}$. They introduced new independent parameters and modified g to g':

$$g' = p_1'p_2'p_3'p_4'p_5' - p_1'p_6' \tag{13}$$
$$\text{where } p_1' \in [0,\ 1],\ p_2' \in [0,\ 1],\ p_3' \in [0,\ 1]$$
$$p_4' \in [0,\ 1],\ p_5' \in [0,\ 1],\ p_6' \in [0,\ 2].$$

In this modification, g is embedded in g', and g' is multilinear with respect to the parameter p_1', p_2', p_3', p_4', p_5', and p_6'. In this case, $[\min(g), \max(g)] \subseteq [\min(g'), \max(g')]$ on the parameter region.

In this paper, we modify Sideris' method to make the monotone polynomials with respect to interval parameters. In this example, if term $-p_2$ does not exist in (11), then g is monotone with respect to parameter p_1. So we introduce new independent parameters $\tilde{p}_1$, $\tilde{p}_2$, and $\tilde{p}_3$ with $p_1 = \tilde{p}_1$ and $\tilde{p}_2 = p_2$. Then we modify the second term "$-\tilde{p}_1\tilde{p}_2$" to "$-\tilde{p}_3\tilde{p}_2$", where $\tilde{p}_3 \in [0,\ 1]$ and denote the new polynomial $\tilde{g}$ by

$$\tilde{g} = \tilde{p}_1^5 - \tilde{p}_2\tilde{p}_3 \tag{14}$$
$$\text{where } \tilde{p}_1 \in [0,\ 1],\ \tilde{p}_2 \in [0,\ 2],\ \tilde{p}_3 \in [0,\ 1].$$

Then, we have derivatives with respect to $\tilde{p}_i$, $i = 1, 2, 3$,

$$\frac{\partial \tilde{g}}{\partial \tilde{p}_1} = 5\tilde{p}_1^4,\ \frac{\partial \tilde{g}}{\partial \tilde{p}_2} = -\tilde{p}_3,\ \frac{\partial \tilde{g}}{\partial \tilde{p}_3} = -\tilde{p}_2. \tag{15}$$

We can see that $\tilde{g}$ is monotone in all parameters $\tilde{p}_1$, $\tilde{p}_2$, and $\tilde{p}_3$. And g is embedded in $\tilde{g}$. Now, we prepare one definition in order to describe the above mentioned procedure.

Definition. (Monotonization index) Let us consider the polynomial

$$\Phi(\boldsymbol{p}) = p_i^\alpha \Psi_\alpha(\hat{\boldsymbol{p}}_i) + p_i^\beta \Psi_\beta(\hat{\boldsymbol{p}}_i) + \cdots + p_i^\zeta \Psi_\zeta(\hat{\boldsymbol{p}}_i)$$
$$+ p_i^\eta \Psi_\eta(\hat{\boldsymbol{p}}_i) + X(\hat{\boldsymbol{p}}_i) \tag{16}$$

where $\alpha > \beta > \cdots > \zeta > \eta \geq 1$,

$\alpha,\ \beta,\ \cdots,\ \zeta,\ \eta \in \mathcal{Z}$, and $\Psi_\alpha,\ \Psi_\beta,\ \cdots,$ $\Psi_\zeta,\ \Psi_\eta,\ X$ are polynomials of $\hat{\boldsymbol{p}}_i$,

$\boldsymbol{p} = (p_1, \ldots, p_m)$.

For this polynomial, we set a monotonization index mz_i for parameter p_i as

$$mz_i = \alpha - \eta. \tag{17}$$

□

Using this definition, we will propose a transformation algorithm. If the polynomial is not monotone with respect to an interval parameter and the interval division method cannot be applied to the parameter, we will use the following procedure.

Algorithm

Step 1. Firstly, the order of terms in the polynomial should be set by the descending order of powers of p_i. Then, the polynomial is in the form of (16), which will be assumed in the following procedure. We calculate the monotonization index mz_i.

Step 2. If the monotonization index mz_i is smaller than η; the lowest degree term of p_i ($mz_i < \eta$), then **go to Step 4**. Or else (if $mz_i \geq \eta$), **go to Step 3**.

Step 3. In this step, $mz_i \geq \eta$. And we transpose the lowest degree term of p_i: $p_i^\eta \to q_i^{\eta-1}q_{m+1}$ and in other terms all $\boldsymbol{p}$ is transposed to $\boldsymbol{q}$; $\boldsymbol{q} = (q_1, \ldots, q_m)$, $q_j \in [\underline{p}_j, \overline{p}_j]$, $j = 1, 2, \ldots, m$, and $q_{m+1} \in [\underline{p}_i, \overline{p}_i]$. When we apply this step to (16), we have

$$\widetilde{\Phi}(\boldsymbol{q}) = q_i^\alpha \Psi_\alpha(\hat{\boldsymbol{q}}_i) + q_i^\beta \Psi_\beta(\hat{\boldsymbol{q}}_i) + \cdots + q_i^\zeta \Psi_\zeta(\hat{\boldsymbol{q}}_i)$$
$$+ q_i^{\eta-1} q_{m+1} \Psi_\eta(\hat{\boldsymbol{q}}_i) + X(\hat{\boldsymbol{q}}_i) \tag{18}$$

Then we check the monotonicity of the transposed polynomial $\widetilde{\Phi}(\boldsymbol{q})$ with respect to q_i. If the transposed polynomial is monotone in q_i, **the algorithm ends**, or else we transpose again parameters $\boldsymbol{q} \to \boldsymbol{p}$ with added new interval parameter $p_{m+1} \in [\underline{p}_i, \overline{p}_i]$ and set $\widetilde{\Phi} \to \Phi$, then **go to Step 1**.

If the lowest degree of p_i is 1 ($\eta = 1$) in the polynomial Φ, the lowest degree of q_i in the transposed polynomial $\widetilde{\Phi}(\boldsymbol{q})$ becomes ζ. And if the transposed parameter p_{m+1} does not appear in the lowest degree term of p_i, we transpose $p_i \to q_{m+1}$ using the above mentioned rule. If there are transposed parameter $p_{m+1}, \ldots, p_{m+k}$, we transpose $p_i \to q_{m+k+1}$.

Step 4. In this step, $mz_i < \eta$. We transpose the highest degree term $p_i^\alpha \to q_i^{\alpha-1}q_{m+1}$ and in other terms all $\boldsymbol{p}$ is transposed to $\boldsymbol{q}$; $\boldsymbol{q} = (q_1, \ldots, q_m)$, $q_i \in [\underline{p}_i, \overline{p}_i]$, $i = 1, 2, \ldots, m$, and $q_{m+1} \in [\underline{p}_i, \overline{p}_i]$. If we apply this step to (16), we have

$$\overline{\Phi}(\boldsymbol{q}) = q_i^{\alpha-1} q_{m+1} \Psi_\alpha(\hat{\boldsymbol{q}}_i) + q_i^\beta \Psi_\beta(\hat{\boldsymbol{q}}_i) + \cdots$$
$$+ q_i^\zeta \Psi_\zeta(\hat{\boldsymbol{q}}_i) + q_i^\eta \Psi_\eta(\hat{\boldsymbol{q}}_i) + X(\hat{\boldsymbol{q}}_i) \tag{19}$$

Then we check the monotonicity of the transposed polynomial $\overline{\Phi}(\boldsymbol{q})$ with respect to q_i. If the transposed polynomial is monotone in q_i, **the algorithm ends**, or else we transpose again parameters $\boldsymbol{q} \to \boldsymbol{p}$ with added new interval parameter $p_{m+1} \in [\underline{p}_i, \overline{p}_i]$ and set $\overline{\Phi} \to \Phi$, then

apply this Step 4 repeatedly.

If we have applied this Step 4 to (16) $(\alpha - \beta)$ times, we have

$$\overline{\Phi}(q) = q_i^\beta \overline{\Psi}_\beta(\hat{q}_i) + \cdots + q_i^\eta \Psi_\eta(\hat{q}_i) + X(\hat{q}_i) \quad (20)$$

$$\text{where } \overline{\Psi}_\beta(\hat{q}_i) = q_{m+1}q_{m+2}\cdots q_{m+\alpha-\beta}\Psi_\alpha(\hat{q}_i) + \Psi_\beta(\hat{q}_i)$$

If we cannot verify the polynomial is monotone in p_i in this Step 4, we should apply this step $(\alpha - \eta)$ times, we have

$$\overline{\Phi}(q) = q_i^\eta \overline{\Psi}_\eta(\hat{q}_i) + X(\hat{p}_i) \quad (21)$$

$$\begin{aligned}\text{where } \overline{\Psi}_\eta(\hat{q}_i) &= q_{m+1}q_{m+2}\cdots q_{m+\alpha-\eta}\Psi_\alpha(\hat{q}_i) \\ &+ q_{m+\alpha-\beta+1}\cdots q_{m+\alpha-\eta}\Psi_\beta(\hat{q}_i) + \cdots \\ &+ q_{m+\alpha-\zeta+1}\cdots q_{m+\alpha-\eta}\Psi_\zeta(\hat{q}_i) + \Psi_\eta(\hat{q}_i).\end{aligned}$$

In this case,

$$\frac{\partial \overline{\Phi}(q)}{\partial q_i} = \eta q_i^{\eta-1}\overline{\Psi}_\eta(\hat{q}_i). \quad (22)$$

If $\eta - 1$ is even (η is odd), $\overline{\Phi}(q)$ is obviously monotone in q_i. If $\eta - 1$ is odd and the interval $[\underline{p}_i, \overline{p}_i]$ do not contain 0, $\overline{\Phi}(q)$ is also monotone in q_i.

If $\eta - 1$ is odd and the interval $[\underline{p}_i, \overline{p}_i]$ contains 0, we divide the interval at 0. Then, $\overline{\Phi}(q)$ is monotone on $[\underline{p}_i, 0]$ and $[0, \overline{p}_i]$. In this paper, we consider the positivity of polynomial in Theorem 3 and, in this case, η is even for $\overline{\Phi}(q) = q_i^\eta \overline{\Psi}_\eta(\hat{q}_i) + X(\hat{q}_i)$. Therefore, if $|\underline{p}_i| > \overline{p}_i$, we should check its positivity only at 0 and $\underline{p}_i$, so we can reduce the interval to $[\underline{p}_i, 0]$. Conversely, if $|\underline{p}_i| \le \overline{p}_i$, we should check its positivity at only 0 and $\overline{p}_i$, and reduce the interval to $[0, \overline{p}_i]$. If the transposed polynomial is monotone in q_i or finally, the form of (21), then **the algorithm terminates.**

If the transposed parameter p_{m+1} do not exist in the highest degree term, we transpose $p_i \to q_{m+1}$ using the above mentioned rule. If there are transposed parameter $p_{m+1}, \ldots, p_{m+k}$, we transpose $p_i \to q_{m+k+1}$. □

Remark 1. (Step 3): If we have applied only Step 3 to (16) $(\alpha - \beta)$ times, we have

$$\widetilde{\Phi}(q) = q_i^\alpha \Psi_\alpha(\hat{q}_i) + q_i^\beta \Psi_\beta(\hat{q}_i) + \cdots + q_i^\zeta \Psi_\zeta(\hat{q}_i) + \widetilde{X}(\hat{q}_i) \quad (23)$$

$$\text{where } \widetilde{X}(\hat{q}_i) = X(\hat{q}_i) + q_{m+1}q_{m+2}\cdots q_{m+\eta}\Psi_\eta(\hat{q}_i)$$

If (23) is not monotone in q_i and $mz_i \ge \zeta$, we apply Step 3 again. If (23) is not monotone and the inequality $mz_i < \zeta$ is satisfied, we go to Step 4.

In the following, we will explain Step 4 is finished even if the intermediate polynomials are not monotone. If we apply Step 4 to (16), mz_i always decreases and the same type of inequality $mz_i < \eta$ remains valid until the algorithm of Step 4 comes to the end in finite steps.

On the contrary, in Step 3, both the monotonization index mz_i and the lowest degree term of p_i may change simultaneously. Hence, according to the value of mz_i − (the lowest degree of p_i), we go to Step 3 or Step 4. So long as the value is positive or equal to 0, we apply the procedures of Step 3, repeatedly. In this case, if we apply only Step 3 for (16), we have

$$\widetilde{\Phi}(q) = q_i^\alpha \Psi_\alpha(\hat{q}_i) + \widetilde{X}(\hat{q}_i) \quad (24)$$

$$\begin{aligned}\text{where } \widetilde{X}(\hat{q}_i) &= X(\hat{q}_i) + q_{m+1}q_{m+2}\cdots q_{m+\eta}\Psi_\eta(\hat{q}_i) \\ &+ q_{m+1}q_{m+2}\cdots q_{m+\zeta}\Psi_\zeta(\hat{q}_i) + \cdots \\ &+ q_{m+1}q_{m+2}\cdots q_{m+\beta}\Psi_\beta(\hat{q}_i).\end{aligned}$$

This polynomial (24) is similar to (21). As the space is limited, we only note that transposing η to α, the same argument of the monotonicity in (21) can be applied to (24) and the algorithm ends in finite steps.

Therefore, the algorithm terminates for all polynomials in finite steps. □

When we apply this algorithm to $H_{n-1}(\boldsymbol{p})$ or $c_0(\boldsymbol{p})$ which is not monotone in $\boldsymbol{p}$ or cannot verify the monotonicity, we can use Theorem 3 for the stability analysis, but conditions become sufficient. If the number of additional independent parameters increases, the stability margin also increases in general. On this point of view, the stability margin of our method is not larger than that of Sideris' method, because the number of new independent parameters of our algorithm does not exceed that of Sideris' method.

5. Examples

Let us consider the stability of the following characteristic polynomial:

$$\begin{aligned}F(s,\boldsymbol{p}) &= s^3 + (3p_1^3 + p_1^2 p_2 + p_1 p_2 + 3p_1 + 10)s^2 \\ &+ (4p_1^2 + p_2^2 + 15)s + 6p_1 p_2 + 17 \quad (25)\end{aligned}$$

Source of this characteristic polynomial is the reference 12), in which Barmish showed Sideris' method. In this example, we will check the stability of (25) for two sets of interval parameters p_1 and p_2. And we will compare our method with Šiljak's method and Sideris' method for the polynomial (25) in this example.

Now, we apply our method to this characteristic polynomial. We have

$$\begin{aligned}H_2(\boldsymbol{p}) &= 133 + 45p_1 + 40p_1^2 + 57p_1^3 + 12p_1^5 + 9p_1p_2 \\ &+ 15p_1^2p_2 + 4p_1^3p_2 + 4p_1^4p_2 + 10p_2^2 + 3p_1p_2^2\end{aligned}$$

Table 1 $H_2(\boldsymbol{p})$ and $c_0(\boldsymbol{p})$ at $\boldsymbol{p}^*$

p_1	p_2	$H_2(\boldsymbol{p})$	$c_0(\boldsymbol{p})$
0	0	133	17
0	1	143	17
1	0	287	17
1	1	337	23

$$+\ 3p_1^3p_2^2 + p_1p_2^3 + p_1^2p_2^3 \tag{26}$$

$$c_0(\boldsymbol{p}) = 17 + 6p_1p_2. \tag{27}$$

5.1 Monotone case: $p_1 \in [0,1]$, $p_2 \in [0,1]$.

Firstly, we check the monotonicity of $H_2(\boldsymbol{p})$ and $c_0(\boldsymbol{p})$ with respect to $\boldsymbol{p}$.

$$\begin{aligned}\frac{\partial H_2(\boldsymbol{p})}{\partial p_1} &= 45 + 80p_1 + 171p_1^2 + 60p_1^4 + 9p_2 + 30p_1p_2 \\ &+\ 12p_1^2p_2 + 16p_1^3p_2 + 3p_2^2 + 9p_1^2p_2^2 + p_2^3 \\ &+\ 2p_1p_2^3 > 0\end{aligned} \tag{28}$$

$$\begin{aligned}\frac{\partial H_2(\boldsymbol{p})}{\partial p_2} &= 9p_1 + 15p_1^2 + 4p_1^3 + 4p_1^4 + 20p_2 + 6p_1p_2 \\ &+\ 6p_1^3p_2 + 3p_1p_2^2 + 3p_1^2p_2^2 \geq 0\end{aligned} \tag{29}$$

And for $c_0(\boldsymbol{p})$, we also have

$$\frac{\partial c_0(\boldsymbol{p})}{\partial p_1} = 6p_2,\ \frac{\partial c_0(\boldsymbol{p})}{\partial p_2} = 6p_1. \tag{30}$$

Then, we can see that $H_2(\boldsymbol{p})$ and $c_0(\boldsymbol{p})$ satisfy the monotonicity condition for $p_1 \in [0,1]$ and $p_2 \in [0,1]$, because all coefficients in (28), (29), and (30) are positive. Then we have **Table 1**. From Table 1, we can learn that $H_2(\boldsymbol{p})$ and $c_0(\boldsymbol{p})$ are positive on the whole parameter region $\mathcal{P}$. At any endpoints of $\boldsymbol{p}$, the characteristic polynomial is stable. Thus, the characteristic polynomial (25) is stable with $p_1 \in [0,1]$ and $p_2 \in [0,1]$. In this case, our conditions are necessary and sufficient.

5.2 Non-monotone case: $p_1 \in [-0.8, 0.4]$ **and** $p_2 \in [-6,3]$.

For intervals $p_1 \in [-0.8, 0.4]$ and $p_2 \in [-6,3]$, we check the monotonicity of $H_2(\boldsymbol{p})$. In this case, if p_2 is fixed at -6, (28) is positive at $p_1 = -0.8$ and negative at $p_1 = 0.4$. And also if p_1 is fixed at 0.4, (29) is positive at $p_2 = 3$ and negative at $p_2 = -6$. Then, we can see that $H_2(\boldsymbol{p})$ is not monotone in $\boldsymbol{p}$ on given intervals. Firstly we transform $H_2(\boldsymbol{p})$ with respect to p_1 using our algorithm.

$$\begin{aligned}H_2(\boldsymbol{p}) &= 12p_1^5 + 4p_1^4p_2 + p_1^3(57 + 4p_2 + 3p_2^2) \\ &+p_1^2(40 + 15p_2 + p_2^3) + p_1(45 + 9p_2 \\ &+3p_2^2 + p_2^3) + 10p_2^2 + 133.\end{aligned} \tag{31}$$

In this case, the monotonization index $mz_1 = 4$ and the lowest degree of p_1 is 1, then we apply Step 3 in the algorithm. Applying Step 3 three times, we have

$$\begin{aligned}\widetilde{H}_2(\boldsymbol{q}) &= 12q_1^5 + 4q_1^4q_2 + q_1^3(57 + 4q_2 + 3q_2^2) \\ &+q_3q_4(40 + 15q_2 + q_2^3) + q_3(45 + 9q_2 \\ &+3q_2^2 + q_2^3) + 10q_2^2 + 133\end{aligned} \tag{32}$$

where $q_1 \in [-0.8, 0.4]$, $q_2 \in [-6,3]$,

$q_3 \in [-0.8, 0.4]$, $q_4 \in [-0.8, 0.4]$.

Then, monotonization index mz_1 becomes 2 and the lowest degree of q_1 is 3, so we apply Step 4. Applying Step 4 twice, we have

$$\begin{aligned}\overline{H}_2(\boldsymbol{q}) &= q_1^3(57 + 4q_2 + 3q_2^2 + 12q_3q_4 + 4q_2q_3) \\ &+q_3q_4(40 + 15q_2 + q_2^3) + q_3(45 + 9q_2 \\ &+3q_2^2 + q_2^3) + 10q_2^2 + 133\end{aligned} \tag{33}$$

where $q_1 \in [-0.8, 0.4]$, $q_2 \in [-6,3]$,

$q_3 \in [-0.8, 0.4]$, $q_4 \in [-0.8, 0.4]$.

In this case, $\overline{H}_2$ is monotone in q_1:

$$\frac{\partial \overline{H}_2(\boldsymbol{q})}{\partial q_1} = 3q_1^2(57 + 4q_2 + 3q_2^2 + 12q_3q_4 + 4q_2q_3). \tag{34}$$

Next, we consider the condition with respect to p_2. Now we consider the following (transposed) polynomial

$$\begin{aligned}H_2(\boldsymbol{p}) &= p_2^3p_3(1 + p_4) + p_2^2(10 + 3p_1^3 + 3p_3) \\ &+p_2(4p_1^3 + 9p_3 + 4p_1^3p_3 + 15p_3p_4) \\ &+12p_1^3p_3p_4 + 57p_1^3 + 45p_3 + 40p_3p_4\end{aligned} \tag{35}$$

where $p_1 \in [-0.8, 0.4]$, $p_2 \in [-6,3]$,

$p_3 \in [-0.8, 0.4]$, $p_4 \in [-0.8, 0.4]$.

In this case, $mz_2 = 2$ and the lowest degree of p_2 is 1. Then, we apply Step 3. We have

$$\begin{aligned}\widetilde{H}_2(\boldsymbol{q}) &= q_2^3q_3(1 + q_4) + q_2^2(10 + 3q_1^3 + 3q_3) \\ &+q_5(4q_1^3 + 9q_3 + 4q_1^3q_3 + 15q_3q_4) \\ &+12q_1^3q_3q_4 + 57q_1^3 + 45q_3 + 40q_3q_4\end{aligned} \tag{36}$$

where $q_5 \in [-6,3]$

The monotonization index mz_2 changes to 1 and the lowest degree of q_2 becomes 2. Then, we apply Step 4. We have

$$\begin{aligned}\overline{H}_2(\boldsymbol{q}) &= q_2^2(q_3q_5(1 + q_4) + 10 + 3q_1^3 + 3q_3) \\ &+q_5(4q_1^3 + 9q_3 + 4q_1^3q_3 + 15q_3q_4) \\ &+12q_1^3q_3q_4 + 57q_1^3 + 45q_3 + 40q_3q_4\end{aligned} \tag{37}$$

$$\frac{\partial \overline{H}_2}{\partial q_2} = 2q_2(q_3q_5(1 + q_4) + 10 + 3q_1^3 + 3q_3) \tag{38}$$

where $q_1 \in [-0.8, 0.4]$, $q_2 \in [-6,3]$,

$q_3 \in [-0.8, 0.4]$, $q_4 \in [-0.8, 0.4]$,

$q_5 \in [-6,3]$.

Then (37) is monotone on the divided interval $q_2 \in [-6, 0]$, $[0, 3]$. According to Step 4, we can reduce the interval of q_2 to $[-6,\ 0]$. And in (37), other parameters q_3, q_4, and q_5 appear multi-linearly, thus (37) is monotone in $\boldsymbol{q}$. Then, we have **Table 2**. In this case, (37) is positive at

Table 2 $\overline{H}_2(\boldsymbol{q})$ at $\boldsymbol{q}^*$

q_1	q_2	q_3	q_4	q_5	$\overline{H}_2(q)$
-0.8	-6	-0.8	-0.8	-6	304.846
-0.8	-6	-0.8	-0.8	3	315.157
-0.8	-6	-0.8	0.4	-6	604.443
-0.8	-6	-0.8	0.4	3	107.758
-0.8	-6	0.4	-0.8	-6	478.789
-0.8	-6	0.4	-0.8	3	445.986
-0.8	-6	0.4	0.4	-6	328.99
-0.8	-6	0.4	0.4	3	549.685
-0.8	0	-0.8	-0.8	-6	51.9824
-0.8	0	-0.8	-0.8	3	114.133
-0.8	0	-0.8	0.4	-6	144.219
-0.8	0	-0.8	0.4	3	10.4144
-0.8	0	0.4	-0.8	-6	148.165
-0.8	0	0.4	-0.8	3	89.4416
-0.8	0	0.4	0.4	-6	102.046
-0.8	0	0.4	0.4	3	141.301
0.4	-6	-0.8	-0.8	-6	419.24
0.4	-6	-0.8	-0.8	3	400.52
0.4	-6	-0.8	0.4	-6	685.659
0.4	-6	-0.8	0.4	3	209.71
0.4	-6	0.4	-0.8	-6	543.416
0.4	-6	0.4	-0.8	3	556.232
0.4	-6	0.4	0.4	-6	410.206
0.4	-6	0.4	0.4	3	651.637
0.4	0	-0.8	-0.8	-6	104.168
0.4	0	-0.8	-0.8	3	137.288
0.4	0	-0.8	0.4	-6	163.227
0.4	0	-0.8	0.4	3	50.1584
0.4	0	0.4	-0.8	-6	150.584
0.4	0	0.4	-0.8	3	137.48
0.4	0	0.4	0.4	-6	121.054
0.4	0	0.4	0.4	3	181.045

Table 3 $c_0(\boldsymbol{p})$ at $\boldsymbol{p}^*$

p_1	p_2	$c_0(\boldsymbol{p})$
-0.8	-6	45.8
-0.8	3	2.6
0.4	-6	2.6
0.4	3	24.2

all endpoints $\boldsymbol{q}^*$, then we learn that (37) is positive on the given region. And $H_2(\boldsymbol{p})$ is positive on the given region $\mathcal{P}$, because $H_2(\boldsymbol{p})$ is embedded in (37).

Finally, we check $c_0(\boldsymbol{p})$. From (30), we can see that $c_0(\boldsymbol{p})$ is monotone in $\boldsymbol{p}$. When we check the value of $c_0(\boldsymbol{p})$ at $\boldsymbol{p}^*$, we have **Table 3** and learn that $c_0(\boldsymbol{p})$ is positive at all endpoints $\boldsymbol{p}^* \in \mathcal{P}^*$. In this example, at any endpoints of $\boldsymbol{p}$, the characteristic polynomial is stable. So the characteristic polynomial (25) is stable on the given parameter region $\mathcal{P}$.

5.3 Comparison of our method with Šiljak's method and Sideris' method

In the reference 13), Šiljak and Stipanović made the sum of the square of the real part and the square of the imaginary part of $F(j\omega, \boldsymbol{p})$ and derived each minimum value of coefficients with respect to $\boldsymbol{p}$. For these coefficients, Šiljak applied Sideris' method and the iteration algorithm, named Bernstein subdivision algorithm [14], in order to calculate the minimum value of each coefficients of squared polynomial with respect to $\boldsymbol{p}$. Making the polynomial with the minimum value of each coefficients with respect to $\boldsymbol{p}$, then Šiljak calculated his modified Routh array [13),15)]. Applying Šiljak's positivity method to (25), we have the following squared polynomial:

$$\begin{aligned} g(\omega,\boldsymbol{p}) &= 289 + 204p_1p_2 + 36p_1^2p_2^2 + (-115 - 102p_1 \\ &+ 120p_1^2 - 102p_1^3 + 16p_1^4 - 154p_1p_2 - 70p_1^2p_2 \\ &- 36p_1^4p_2 + 30p_2^2 - 4p_1^2p_2^2 - 12p_1^3p_2^2 + p_2^4)\omega \\ &+ (70 + 60p_1 + p_1^2 + 60p_1^3 + 18p_1^4 + 9p_1^6 \\ &+ 20p_1p_2 + 26p_1^2p_2 + 6p_1^3p_2 + 6p_1^4p_2 + 6p_1^5p_2 \\ &- 2p_2^2 + p_1^2p_2^2 + 2p_1^3p_2^2 + p_1^4p_2^2)\omega^2 + \omega^3 \end{aligned} \quad (39)$$

For the intervals $p_1 \in [0,1]$ and $p_2 \in [0,1]$, the minimum values of coefficients with respect to $\boldsymbol{p}$ are calculated by numerical method. Then, we also have

$$\underline{g}(\omega) = \omega^3 + 68\omega^2 - 428\omega + 289 \quad (40)$$

and the modified Routh array(**Table 4**).

And for the intervals $p_1 \in [-0.8, 0.4]$ and $p_2 \in [-6, 3]$, we have

$$\underline{g}(\omega) = \omega^3 - 57.783424\omega^2 - 187.960336\omega + 6.76 \quad (41)$$

and the modified Routh array(**Table 5**).

Table 4 The modified Routh array with $p_1 \in [0,1]$ and $p_2 \in [0,1]$

-1	68	428	289
-3	136	428	
22.67	285.33	289	
173.76	466.24		
224.50	289		
242.56			
289			

Table 5 The modified Routh array with $p_1 \in [-0.8, 0.4]$ and $p_2 \in [-6, 3]$

-1	-57.783424	187.992436	6.76
-3	-115.566848	187.992436	
-19.261141	125.3282906	6.76	
-135.087232	186.9395388		
98.67389283	6.76		
196.1941619			
6.76			

The first columns of both modified Routh array(Table 4, Table 5) have one sign variation, then the characteristic polynomial is not $\mathbf{R}_+$-positive. Thus, we cannot ascertain its stability using Šiljak's method.

In this example, since there are only two interval parameters, we can see its stability by other methods, for example, some graphical methods. And we can learn that the characteristic polynomial (25) with all interval parameter sets of p_1 and p_2 in this example is stable.

When we apply Šiljak's method to the characteristic polynomials (25), the degree of each interval parameters in the coefficients of (39) become to doubled degree of (25). Therefore, the complexity of calculation in the stability analysis also increased. Šiljak compared the stability margin of two methods by the stability test of the example and showed that using Bernstein subdivision algorithm has smaller stability margin than using Sideris' method[13]. From the same point of view, our method is less conservative than Šiljak's method.

In case of $p_1 \in [-0.8, 0.4]$ and $p_2 \in [-6, 3]$, if we apply Sideris' method to the coefficients of (25), we have

$$\begin{aligned} F'(s,p') &= s^3 + (3p_1'p_2'p_3' + p_1'p_2'p_4' + p_1'p_4' + 3p_1' \\ &\quad +10)s^2 + (4p_1'p_2' + p_4'p_5' + 15)s \\ &\quad +6p_1'p_4' + 17 \qquad (42) \end{aligned}$$

$$\text{where } p_1' \in [-0.8, 0.4],\ p_2' \in [-0.8, 0.4],\ p_3' \in [-0.8, 0.4],\ p_4' \in [-6, 3],\ p_5' \in [-6, 3].$$

In this case, the characteristic polynomial(42) is unstable at $p_1' = 0.4$, $p_2' = 0.4$, $p_3' = 0.4$, $p_4' = 3$, $p_5' = -6$. Thus, our method is also less conservative than Sideris' method.

If we apply Sideris' method to $H_2(\boldsymbol{p})$ for $p_1 \in [-0.8, 0.4]$ and $p_2 \in [-6, 3]$ instead of our methods, then we have

$$\begin{aligned} H_2'(\boldsymbol{p}') &= 133 + 45p_1' + 40p_1'p_2' + 57p_1'p_2'p_3' + 10p_6'p_7' \\ &\quad +12p_1'p_2'p_3'p_4'p_5' + 9p_1'p_6' + p_1'p_2'p_6'p_7'p_8' \\ &\quad +4p_1'p_2'p_3'p_6' + 4p_1'p_2'p_3'p_4'p_6' + 15p_1'p_2'p_6' \\ &\quad +3p_1'p_6'p_7' + 3p_1'p_2'p_3'p_6'p_7' + p_1'p_6'p_7'p_8' \qquad (43) \end{aligned}$$

$$\begin{aligned} \text{where } & p_1' \in [-0.8, 0.4],\ p_2' \in [-0.8, 0.4], \\ & p_3' \in [-0.8, 0.4],\ p_4' \in [-0.8, 0.4], \\ & p_5' \in [-0.8, 0.4],\ p_6' \in [-6, 3], \\ & p_7' \in [-6, 3],\ p_8' \in [-6, 3]. \end{aligned}$$

And at $p_1' = 0.4$, $p_2' = 0.4$, $p_3' = 0.4$, $p_4' = 0.4$, $p_5' = 0.4$, $p_6' = -6$, $p_7' = 3$, $p_8' = 3$, $H_2'(\boldsymbol{p}') = -112.276$. Hence, the characteristic polynomial is unstable. So the stability conditions with our transformation algorithm also less conservative on this point.

For the parameter intervals $p_1 \in [0, 1]$ and $p_2 \in [0, 1]$, the polynomial (42) is stable. In (42), the number of interval parameters are equal to that of our method. but, if we apply mapping theorem[2),16)] to the stability analysis of (42), we must check the convex hull at each positive ω which is spanned by $F'(j\omega, \boldsymbol{p})$ with the endpoint values of $\mathcal{P}$ in the complex plane.

Table 6 $H_3(p_1)$ and $c_0(p_1)$ at p_1^*

p_1	$H_3(p_1)$	$c_0(p_1)$
$\frac{6}{5}$	$\frac{16128}{25}$	$\frac{49}{5}$
0	-2304	–
$-\frac{6}{5}$	–	$\frac{1}{5}$

For the parameter intervals $p_1 \in [0, 1]$ and $p_2 \in [0, 1]$, $H_2'(\boldsymbol{p}')$ is positive. For (43), we must check the positivity of $H_2'(\boldsymbol{p}')$ at $2^8 = 256$ endpoints. Using our method, we checked only $2^2 = 4$ endpoints of $\boldsymbol{p}$ for the stability and we learn that $F(s, \boldsymbol{p})$ is stable in this example(Table 1).

In case of $p_1 \in [0, 1]$ and $p_2 \in [0, 1]$, we can also check $F(s,\boldsymbol{p})$ is stable, applying Sideris' method to $F(s, \boldsymbol{p})$ or $H_2(\boldsymbol{p})$. But, stability conditions become only sufficient with Sideris' method. On the other hand, for the parameters $p_1 \in [0, 1]$ and $p_2 \in [0, 1]$, our stability conditions are necessary and sufficient.

5.4 Relation between our monotonicity conditions and the edge theorem

The edge theorem can be applied to the polynomial

$$F(s, \boldsymbol{p}) = f_0(s) + p_1 f_1(s) + \cdots + p_m f_m(s) \qquad (44)$$
$$\text{where } p_i \in [\underline{p}_i, \overline{p}_i],\ i = 1, 2, \cdots, m.$$

This polynomial is affine in $\boldsymbol{p}$. The edge theorem requires to check the stability of $F(s, \boldsymbol{p})$ on the exposed edge of $\mathcal{P}$. Our stability conditions also can be applied to the stability analysis of (44). Sometimes, our monotonicity conditions do not hold on the whole parameter region $\mathcal{P}$.

Let us consider the following example.

$$\begin{aligned} F(s, p_1) &= 5 + 4p_1 + (24 + 8p_1)s + 6s^2 \\ &\quad + (24 - 8p_1)s^3 + (5 - 4p_1)s^4 \qquad (45) \\ & -\frac{6}{5} \le p_1 \le \frac{6}{5} \qquad (46) \end{aligned}$$

The original polynomial is given in the reference 3) as an example where the exposed edge determines its stability in z-plane. And an interval parameter p_1 itself becomes the exposed edge. In this paper, we discuss in s-plane, so we apply the bilinear transformation $z = (1 + s)/(1 - s)$. And we have (45) and the following $H_3(p_1)$ and $c_0(p_1)$.

$$H_3(p_1) = 256(-9 + 8p_1^2),\ c_0(p_1) = 5 + 4p_1. \qquad (47)$$

Checking the monotonicity of $H_3(p_1)$ and $c_0(p_1)$,

$$\frac{dH_3(p_1)}{dp_1} = 4096p_1,\ \frac{dc_0(p_1)}{dp_1} = 4, \qquad (48)$$

we can see that $c_0(p_1)$ is monotone in p_1 and $H_3(p_1)$ is not monotone in p_1. The derivative of $H_3(p_1)$ has one zero at $p_1 = 0$ and we divide the interval of p_1 at 0. And in this example, p_1 appears only in the form of p_1^2 in (47), so we can reduce the interval of p_1 to $p_1 \in [0, 6/5]$ for checking the positivity of $H_3(p_1)$ (**Table 6**).

Table 7 $H_2(\boldsymbol{p})$ at $\boldsymbol{p}^*$

p_1	p_2	$H_2(\boldsymbol{p})$
0.3	0	1.24
0.3	1	0.24
0.3	1.7	0.73
1	0	0.75
1	1	-0.25
1	1.7	0.24
2.5	0	3
2.5	1	2
2.5	1.7	2.49

$H_3(p_1)$ becomes negative at $p_1 = 0$. This means that the polynomial (45) is not stable on the whole parameter region. In the reference 3), Mori and Kokame showed that the polynomial(45) is not stable on the exposed edge, but stable at both endpoints of p_1. In this example, we use only the interval division method to $H_3(p_1)$. So, the stability condition is still necessary and sufficient.

It is known that for the polynomial whose coefficients are multi-linear in interval parameters, the edge theorem is incompetent for analyzing the stability [17]. In the reference 17), the following example is given.

$$\begin{aligned} F(s,\boldsymbol{p}) &= s^3 + (p_1+p_2+1)s^2 + (p_1+p_2+3)s \\ &\quad + 1 + r^2 + 6p_1 + 6p_2 + 2p_1p_2 \\ &\quad\quad p_1 \in [0.3, 2.5],\ p_2 \in [0, 1.7] \end{aligned} \tag{49}$$

Calculating the Hurwitz determinant $H_2(\boldsymbol{p})$, we have

$$H_2(\boldsymbol{p}) = 2 - 2p_1 + p_1^2 - 2p_2 + p_2^2 - r^2, \tag{50}$$

$$\frac{\partial H_2(\boldsymbol{p})}{\partial p_1} = -2 + 2p_1,\ \frac{\partial H_2(\boldsymbol{p})}{\partial p_2} = -2 + 2p_2. \tag{51}$$

In this example, partial derivatives (51) vanish at $p_1 = 1$ and $p_2 = 1$, respectively. Then, we divide the intervals of p_1 and p_2 at 1. Substituting $r = 0.5$, we have **Table 7**. $H_2(\boldsymbol{p})$ becomes negative at $p_1 = p_2 = 1$. This means that (49) is unstable. Since we use only our interval division method to $H_3(\boldsymbol{p})$ with respect to $\boldsymbol{p}$, derived conditions are still necessary and sufficient. In the reference 17), Ackermann et al. showed that the unstable parameter region is a circle whose center is $p_1 = p_2 = 1$ and radius is r in defined parameter region $\mathcal{P}$. And They also showed that $F(s,\boldsymbol{p})$ is stable at all endpoints of $\boldsymbol{p}$ and on all edges of the parameter region $\mathcal{P}$. In this example, the center of unstable region(circle) is the dividing point of interval parameters in the interval division method.

Now we summarize the comparative merits of our method and the edge theorem. The edge theorem is useful only for the affine polynomial with respect to the parameters. And it is known that the calculation of the exposed edge is difficult in general[(1)]. Ackermann et al. showed that the edge theorem is not applicable to the polynomial whose coefficients are multi-linear with respect to interval parameters. On the other hand, our monotonicity method can be applied to the affine or multi-linear polynomials. Even if there exist exposed edges in the parameter region, our monotonicity method is still effective. Conversely, if the monotonicity condition do not hold on the whole parameter region, it may be possible that there exist exposed edges in the parameter region.

(1) For example, the statement is given in the reference 18), p.82.

6. Concluding Remarks

Our stability conditions are based on Frazer-Duncan's theorem and the monotonicity conditions of multivariable polynomials. If all the coefficients are monotone in the sense of multivariable polynomials defined by us, our stability conditions are necessary and sufficient and can be checked at the endpoints of parameter region (Theorem 3). Though our monotonicity conditions are complicated to check and not always satisfied, there are some cases where the conditions can be checked easily. Two typical cases are shown. And in order to derive stability conditions for the cases where the monotonicity conditions do not hold or cannot be verified on the whole parameter region, we "monotonized" the problem by dividing parameter intervals into subintervals (Interval division method) or introducing a new augmented set of parameters (Transformation algorithm). If the former method is applicable, our stability conditions remain necessary and sufficient. In the latter case, the original family of polynomials is embedded into the new family of polynomials. Thus, if it is possible to show that all the polynomials belonging to the new family are positive, the original family is composed of positive polynomials on the whole parameter region. Therefore, we can apply our stability conditions using the transformation algorithm. Our transformation algorithm is the adaptation of Sideris' method and our algorithm terminates in finite steps.

Our methods are illustrated by examples and compared with Šiljak's method and Sideris' method. All three methods are only sufficient conditions, if we use the transformation algorithm in our stability analysis. It is shown that our conditions are less conservative than other two methods by checking the stability of a characteristic polynomial with two different combinations of parameter intervals. In addition, the advantage of our method over the edge theorem is described.

In view of the progress of the information technology, we can check the stability of the characteristic polynomial (1) at very large number of mesh points in the parameter region. But, if the characteristic polynomial is stable at all the mesh points, is it possible to regard it as a stable polynomial ? This is a question to be solved analytically and quantitatively, but not yet. On the other hand, if monotonicity conditions hold or using only interval division method to polynomials, our stability conditions are necessary and sufficient. And our transformation algorithm and Sideris' method can be used safely in the design and analysis of control systems with various safety margins. In this respect, our method has the most narrow safety margin and can be applied to the polynomial type coefficients including the multi-linear type. Nevertheless, the problem of robust stability is not solved completely up to the present.

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