Twisted cohomology of the complement of theta divisors in an abelian surface

Watanabe, Humihiko

§0 Introduction

Let $X = \boldsymbol{C}^2/(\boldsymbol{Z}^2 + \tau\boldsymbol{Z}^2)$ be a principally polarized abelian surface, where $\boldsymbol{C}^2$ denotes the additive group of complex column vectors ${}^t(z_1, z_2)$ ($z_1, z_2 \in \boldsymbol{C}$, $\boldsymbol{C}$ being the set of complex numbers), $\boldsymbol{Z}^2$ the subgroup of $\boldsymbol{C}^2$ consisting of integral column vectors ${}^t(n_1, n_2)$ ($n_1, n_2 \in \boldsymbol{Z}$, $\boldsymbol{Z}$ being the set of integers), and τ a complex symmetric square matrix of degree 2 of which the imaginary part is positive definite. We assume that X is not a direct product of two elliptic curves (with canonical polarization). Then it is well-known (e.g., [2]) that X is the Jacobian variety of a Riemann surface of genus 2. Let N and N' be natural numbers satisfying the condition $2 \le N \le N'$. Let $a_1, \ldots, a_{2N'}, b_1, \ldots, b_{2N'}$ be $4N'$ real numbers. We assume that the N' theta divisors defined by the equations $\theta \begin{bmatrix} a_{2k-1} \, a_{2k} \\ b_{2k-1} \, b_{2k} \end{bmatrix} (z_1, z_2; \tau) = 0$ ($k = 1, \ldots, N'$) are different from each other, where $\theta \begin{bmatrix} a \, a' \\ b \, b' \end{bmatrix} (z_1, z_2; \tau)$ denotes a theta function (§1). For each k ($1 \le k \le N$), let D_k be the theta divisor corresponding to the theta function $\theta \begin{bmatrix} a_{2k-1} \, a_{2k} \\ b_{2k-1} \, b_{2k} \end{bmatrix} (z_1, z_2; \tau)$. *Throughout this paper we assume that the divisor* $D = \sum_{k=1}^{N} D_k$ *has normal crossings.* We set $M = X - D$. Let $c_1, \ldots, c_N$ be N complex numbers but not integers such that $\sum_{k=1}^{N} c_k = 0$. Let $c_{N+1}, \ldots, c_{N'}$ be non-zero integers such that $\sum_{k=N+1}^{N'} c_k = 0$. We define a multiplicative function $T(z_1, z_2)$ by $T(z_1, z_2) = \prod_{k=1}^{N'} \theta \begin{bmatrix} a_{2k-1} \, a_{2k} \\ b_{2k-1} \, b_{2k} \end{bmatrix} (z_1, z_2; \tau)^{c_k}$. Let $\mathcal{L}$ be a locally constant sheaf of rank one on M defined by a complex one-dimensional representation of the fundamental group $\pi_1(M, *)$ by the multivalued meromorphic function $T(z_1, z_2)^{-1}$ on M. The purpose of this paper is to study the twisted cohomology of the open dense subset M of X, i.e., the cohomology groups $H^k(M, \mathcal{L})$ of M with coefficients in $\mathcal{L}$. The twisted cohomology (and homology) of complement of hyperplanes in a complex projective space is already studied in detail, and it gives us the foundation of the theory of hypergeometric integrals of several variables (e.g., see [1]). Our study in this paper is an analogue of the study for complement of hyperplanes in a complex projective space. In a previous paper [11] (see also [17]), we studied twisted cohomology of a one-dimensional punctured complex torus in connection with an integral representation on a complex torus of hypergeometric function. So it seems very natural to us to proceed to the study for the two-dimensional space M.

The starting point of our study is an isomorphism $H^p(M, \mathcal{L}) \cong \mathbf{H}^p(X, \Omega^{\bullet}_X\langle D\rangle(P), \nabla)$ between cohomology and hypercohomology (Corollary 3.2), where ∇ denotes a covariant differentiation given by $\nabla\psi = d\psi + d(\log T_1) \wedge \psi$ with $T_1(z_1, z_2) = \prod_{k=1}^{N} \theta \begin{bmatrix} a_{2k-1} \, a_{2k} \\ b_{2k-1} \, b_{2k} \end{bmatrix} (z_1, z_2; \tau)^{c_k}$, P a holomorphic line bundle on X associated to the multivalued function $T_2(z_1, z_2)^{-1}$ with $T_2(z_1, z_2) = \prod_{k=N+1}^{N'} \theta \begin{bmatrix} a_{2k-1} \, a_{2k} \\ b_{2k-1} \, b_{2k} \end{bmatrix} (z_1, z_2; \tau)^{c_k}$, and $(\Omega_X^\bullet\langle D\rangle(P), \nabla)$ a complex of sheaves of logarithmic forms over X with logarithmic pole along D with coefficients in P. Note that P belongs to the Picard variety of X. The isomorphism above follows from the fact that the two complexes of sheaves over X, $(\Omega_X^\bullet\langle D\rangle(P), \nabla)$ and $(j_*\mathcal{E}_M^\bullet(P|M), \nabla)$, where j denotes the inclusion $M \hookrightarrow X$, and $\mathcal{E}_M^p(P|M)$ the sheaf of smooth p-forms over M with coefficients in $P|M$ the restriction of P to M, are quasi-isomorphic to each other (Proposition 3.1). This fact is obtained as a corollary of a proposition proved by Deligne ([3], II, Cor. 3.14), while we will give an elementary, direct proof of this fact in Appendix to §3. The vanishing of the cohomology groups $H^p(M, \mathcal{L})$ $(p \neq 2)$ is an immediate consequence of the vanishing of the corresponding hypercohomology groups ([5], §2, Cor. 2.13). In Appendix to §3 we will also give an elementary proof of the vanishing of hypercohomology groups by exploiting the logarithmic Dolbeault complex (Proposition 3.3).

The main task of this paper is to study the structure of the non-vanishing cohomology group $H^2(M, \mathcal{L})$ at length. To this end we consider a spectral sequence with $E_1^{pq} = H^q(X, \Omega_X^p\langle D\rangle(P))$ abutting to the hypercohomology groups $\mathbf{H}^k(X, \Omega_X^\bullet\langle D\rangle(P), \nabla)$: $E_1^{pq} = H^q(X, \Omega_X^p\langle D\rangle(P)) \Longrightarrow \mathbf{H}^{p+q}(X, \Omega_X^\bullet\langle D\rangle(P), \nabla)$. Degeneration of this spectral sequence depends on whether the line bundle P is holomorphically trivial (i.e., $P = \boldsymbol{C}$), or it is topologically trivial (i.e., $c_1(P) = 0$) but not holomorphically. Namely, if P is not holomorphically trivial, the spectral sequence degenerates at E_1 (Proposition 3.5); if $P = \boldsymbol{C}$, it degenerates at E_2 (Proposition 3.6). Moreover, in either case, we can determine explicitly the values of all the terms E_∞^{pq}, which give us information on the structures of $H^k(M, \mathcal{L})$ including $H^2(M, \mathcal{L})$. These facts are proved based on knowledge about the structures of homology groups of M, which we study in detail in §2 (see Propositions 2.4 and 2.8). To obtain information on how to select meromorphic 2-forms realizing a basis of $H^2(M, \mathcal{L})$, we consider two resolutions (4.1) and (4.2) of the sheaves $\Omega_X^1\langle D\rangle(P)$ and $\mathcal{O}_X(P)$ $(= \Omega_X^0\langle D\rangle(P))$ of logarithmic differential forms introduced by Deligne [3], II, §3, Prop. 3.13. Applying Mumford's vanishing theorem (Mumford [12], III, §16) to the long exact sequences of cohomology groups associated with those two resolutions, we have "analytical expressions" of the three groups $H^1(X, \Omega_X^1\langle D\rangle(P))$ and $H^p(X, \mathcal{O}_X(P))$ $(p = 1, 2)$ generated by meromorphic 2-forms with poles of lower order (Propositions 4.1 and 4.2). Combining these results, we have, according as P is holomorphically trivial or not, two direct sum decompositions of the group $H^2(M, \mathcal{L})$, each component of which is generated by meromorphic 2-forms of which the divisors are less than or equal to a prescribed effective divisor D' with support $\mathrm{supp}(D')$ contained in D. This is the main result of this paper (Theorems 4.5 and 4.6).

As we have already mentioned, we need to know the structures of homology groups of M in the process of investigating the degeneration of the spectral sequence (Lemma 3.4). To this end we determine in §2 the structures of homology groups of M with coefficients both in the constant sheaf $\boldsymbol{Z}$ (therefore in $\boldsymbol{C}$) and in the locally constant sheaf $\check{\mathcal{P}}|M$, where $\check{\mathcal{P}}$ denotes the dual of the locally constant sheaf $\mathcal{P}$ over X associated to the holomorphic line bundle P, and $\check{\mathcal{P}}|M$ its restriction to M (Propositions 2.4 and 2.8). An exact homology sequence obtained by combining a usual exact homology sequence of a pair with the Thom isomorphism is useful in studying our homology groups (Proposition 2.1). The construction of homology classes generating our homology groups is based on Pontryagin product, some fundamental properties of which we review in §1.

§1 Preliminaries

Let X be a complex torus of dimension $g(\geq 1)$. Following [2], we first review the definition and some properties of the Pontryagin product of X. Let $\sigma : \Delta_p \to X$ and $\tau : \Delta_q \to X$ be singular $p-$ and $q-$simplices of X, respectively, where Δ_p denotes a standard $p-$simplex. We define a singular $(p+q)-$chain $\sigma * \tau : \Delta_p \times \Delta_q \to X$ by the equation $(\sigma * \tau)(s,t) = \sigma(s) + \tau(t)$ for $s \in \Delta_p$ and $t \in \Delta_q$, where the addition in the right hand side is the one coming from the group structure of X. By extending the operation $*$ by bilinearity, we define a bilinear mapping $* : S_p(X, \boldsymbol{Z}) \times S_q(X, \boldsymbol{Z}) \to S_{p+q}(X, \boldsymbol{Z})$ where $S_p(X, \boldsymbol{Z})$ denotes the group of singular $p-$chains of X, i.e., the abelian group generated over $\boldsymbol{Z}$ by all the singular $p-$simplices of X. We call the operation $*$ the Pontryagin product of singular chains of X. Then we have the following fundamental properties:

Lemma 1.1. Let σ, σ' be singular $p-$chains of X, and τ, υ be singular $q-$ and $r-$chains of X, respectively. Let m, m' be integers. Then the following formulas hold:
(i) $(\sigma * \tau) * \upsilon = \sigma * (\tau * \upsilon)$,
(ii) $(m\sigma + m'\sigma') * \tau = m(\sigma * \tau) + m'(\sigma' * \tau)$,
(iii) $\partial(\sigma * \tau) = (\partial\sigma) * \tau + (-1)^p \sigma * (\partial\tau)$,
(iv) $\sigma * \tau = (-1)^{pq} \tau * \sigma$,
(v) $0 * \sigma = \sigma * 0 = \sigma$, where 0 denotes the unit element of the abelian group X regarded as a $0-$simplex.

Let σ and τ be $p-$ and $q-$cycles respectively. We denote their homology classes by $[\sigma] \in H_p(X, \boldsymbol{Z})$ and $[\tau] \in H_q(X, \boldsymbol{Z})$. We define a new homology class $[\sigma] * [\tau] \in H_{p+q}(X, \boldsymbol{Z})$ by the well-defined equation $[\sigma] * [\tau] = [\sigma * \tau]$. This definition induces a bilinear operation $* : H_p(X, \boldsymbol{Z}) \times H_q(X, \boldsymbol{Z}) \to H_{p+q}(X, \boldsymbol{Z})$, which we call the Pontryagin product of homology classes of X. As is well-known, the Pontryagin product coincides with the composition $H_p(X, \boldsymbol{Z}) \times H_q(X, \boldsymbol{Z}) \to H_{p+q}(X \times X, \boldsymbol{Z}) \to H_{p+q}(X, \boldsymbol{Z})$, where the first arrow represents the cross product and the second represents the operation induced from the addition of the abelian group X. The following formulas are fundamental:

Lemma 1.2. Let σ, σ' be $p-$cycles of X, and τ, υ be $q-$ and $r-$cycles of X, respectively. Let m, m' be integers. Then we have:
(i) $([\sigma] * [\tau]) * [\upsilon] = [\sigma] * ([\tau] * [\upsilon])$,
(ii) $(m[\sigma] + m'[\sigma']) * [\tau] = m([\sigma] * [\tau]) + m'([\sigma'] * [\tau])$,
(iii) $[\sigma] * [\tau] = (-1)^{pq} [\tau] * [\sigma]$,
(iv) $0 * [\sigma] = [\sigma] * 0 = [\sigma]$, where 0 denotes the unit element of X.

Let X be a principally polarized abelian variety of dimension $g(\geq 1)$. Without loss of generality we may set $X = \boldsymbol{C}^g/(\boldsymbol{Z}^g + \tau\boldsymbol{Z}^g)$, where τ denotes a complex symmetric square matrix of degree g of which the imaginary part is positive definite and we regard $\boldsymbol{C}^g$ as the set of column vectors ${}^t(z_1, \ldots, z_g)$ with standard complex coordinates $z_1, \ldots, z_g$. If we set $\tau = (\tau_{ij}), i, j = 1, \ldots, g$, then we have $2g$ vectors $\lambda_1 = {}^t(1, 0, \ldots, 0), \ldots, \lambda_g = {}^t(0, \ldots, 0, 1), \lambda_{g+1} = {}^t(\tau_{11}, \ldots, \tau_{g1}), \ldots, \lambda_{2g} = {}^t(\tau_{1g}, \ldots, \tau_{gg})$ as a symplectic basis of the lattice $\boldsymbol{Z}^g + \tau\boldsymbol{Z}^g$ for the principal polarization. By abuse of notation we regard $\lambda_1, \ldots, \lambda_{2g}$ as a basis of the homology group $H_1(X, \boldsymbol{Z})$. Let $x_1, \ldots, x_{2g}$ be (not necessarily standard) real coordinates on $\boldsymbol{C}^g$, which is regarded as a real vector space of dimension $2g$, corresponding to the basis $\lambda_1, \ldots, \lambda_{2g}$. Then the 1-forms $dx_1, \ldots, dx_{2g}$ form a basis of the cohomology group $H^1(X, \boldsymbol{Z})$, and satisfy the conditions $\int_{\lambda_i} dx_j = \delta_{ij}$. For an ordered subset $I = \{i_1, \ldots, i_p\}$ (where $i_1 < \cdots < i_p$) of the set $\{1, \ldots, 2g\}$, we set $\lambda_I = \lambda_{i_1} * \cdots * \lambda_{i_p}$ and $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. If $I = \emptyset$, we set $\lambda_I = \bullet$ (a point) and $dx_I = 1$. The following result is well-known:

Lemma 1.3. The set $\{dx_I \mid \#I = p\}$ forms a basis of the cohomology group $H^p(X, \boldsymbol{Z})$; the set $\{\lambda_I \mid \#I = p\}$ forms a basis of the homology group $H_p(X, \boldsymbol{Z})$. They are dual bases each other. Moreover, the union $\cup_{p=0}^4 \{\lambda_I \mid \#I = p\}$ gives a cellular decomposition of X.

For an ordered subset $I \subset \{1, \ldots, 2g\}$ let I° be the complementary ordered subset of $\{1, \ldots, 2g\}$ such that $I \cap I^\circ = \emptyset$ and $I \cup I^\circ = \{1, \ldots, 2g\}$. We define the sign $\varepsilon(I)$ of an ordered subset I by the equation $\varepsilon(I)dx_I \wedge dx_{I^\circ} = dx_1 \wedge dx_{g+1} \wedge \cdots \wedge dx_g \wedge dx_{2g}$. We have $\varepsilon(I) = \pm 1$. Let P be the isomorphism of $H_p(X, \boldsymbol{Z})$ onto $H^{2g-p}(X, \boldsymbol{Z})$ induced by Poincaré duality.

Lemma 1.4. ([2]) For an ordered subset I such that $\#I = p$, then $P(\lambda_I) = (-1)^{g+p}\varepsilon(I)dx_{I^\circ}$.

For $u \in H_p(X, \boldsymbol{Z})$ and $v \in H_q(X, \boldsymbol{Z})$, we define the intersection product $u \cdot v \in H_{p+q-2g}(X, \boldsymbol{Z})$ by $u \cdot v = P^{-1}(Pu \wedge Pv)$. If in particular $p + q = 2g$, then $u \cdot v$ represents the intersection number ($\in \boldsymbol{Z}$).

In the rest of this section let us restrict ourselves to the case where $g = 2$. Namely, let X be a principally polarized abelian surface. Then, as is well-known ([2]), X is either a two-dimensional Jacobian variety or the direct product of two complex tori of dimension one. In what follows we assume that X be a two-dimensional Jacobian variety. Let a_1, a_2, b_1, b_2 be real numbers. We set $\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$. The theta function with characteristics ([9,13]) is given by

$$\theta \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} (z_1, z_2; \tau) = \sum_{\boldsymbol{n} \in \boldsymbol{Z}^2} \exp \left[\pi i \; {}^t\left(\boldsymbol{n} + \frac{\boldsymbol{a}}{2}\right) \tau \left(\boldsymbol{n} + \frac{\boldsymbol{a}}{2}\right) + 2\pi i \; {}^t\left(\boldsymbol{n} + \frac{\boldsymbol{a}}{2}\right)\left(\boldsymbol{z} + \frac{\boldsymbol{b}}{2}\right) \right],$$

where $\boldsymbol{n}, \boldsymbol{z}, \boldsymbol{a}, \boldsymbol{b}$ denote column vectors: $\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$, $\boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Then the following formulas hold:

$$\theta \begin{bmatrix} a_1\, a_2 \\ b_1\, b_2 \end{bmatrix} (z_1+1, z_2; \tau) = e^{\pi i a_1} \theta \begin{bmatrix} a_1\, a_2 \\ b_1\, b_2 \end{bmatrix} (z_1, z_2; \tau),$$

$$\theta \begin{bmatrix} a_1\, a_2 \\ b_1\, b_2 \end{bmatrix} (z_1, z_2+1; \tau) = e^{\pi i a_2} \theta \begin{bmatrix} a_1\, a_2 \\ b_1\, b_2 \end{bmatrix} (z_1, z_2; \tau),$$

$$\theta \begin{bmatrix} a_1\, a_2 \\ b_1\, b_2 \end{bmatrix} (z_1+\tau_{11}, z_2+\tau_{21}; \tau) = e^{-2\pi i z_1 - \pi i \tau_{11} - \pi i b_1} \theta \begin{bmatrix} a_1\, a_2 \\ b_1\, b_2 \end{bmatrix} (z_1, z_2; \tau),$$

$$\theta \begin{bmatrix} a_1\, a_2 \\ b_1\, b_2 \end{bmatrix} (z_1+\tau_{12}, z_2+\tau_{22}; \tau) = e^{-2\pi i z_2 - \pi i \tau_{22} - \pi i b_2} \theta \begin{bmatrix} a_1\, a_2 \\ b_1\, b_2 \end{bmatrix} (z_1, z_2; \tau),$$

$$\theta \begin{bmatrix} a_1\, a_2 \\ b_1\, b_2 \end{bmatrix} (-z_1, -z_2; \tau) = e^{\pi i (a_1 b_1 + a_2 b_2)} \theta \begin{bmatrix} a_1\, a_2 \\ b_1\, b_2 \end{bmatrix} (z_1, z_2; \tau),$$

$$\theta \begin{bmatrix} a_1+2m_1\; a_2+2m_2 \\ b_1+2m_1'\; b_2+2m_2' \end{bmatrix} (z_1, z_2; \tau) = e^{\pi i (a_1 m_1' + a_2 m_2')} \theta \begin{bmatrix} a_1\, a_2 \\ b_1\, b_2 \end{bmatrix} (z_1, z_2; \tau),$$

where m_1, m_2, m_1', m_2' denote integers. A theta divisor D is by definition the subset of X defined by an equation $\theta \begin{bmatrix} a_1\, a_2 \\ b_1\, b_2 \end{bmatrix} (z_1, z_2; \tau) = 0$. By the general theory of theta divisors on X, D is a closed algebraic curve in X with no singularity, and therefore it is a compact Riemann surface.

Lemma 1.5. A theta divisor D, regarded as a compact Riemann surface, is of genus 2. The self intersection number D^2 of D is equal to 2. If D' is another theta divisor, the intersection number $D \cdot D' = 2$.

Proof. Let $[D]$ be the line bundle on X corresponding to a theta divisor D. Then $[D]$ is a principal polarization on X, and the Euler number $\chi([D])$ equals one by the general theory of cohomology of line bundles on X. By the Riemann-Roch theorem (e.g., [2,12]), we have $D^2 = 2\chi([D]) = 2$. Since the canonical class K_X of the abelian surface X is equal to zero, we have the genus of D, $g(D) = \frac{1}{2}(K_X \cdot D + D^2) + 1 = 2$ by the genus formula. Since the product of line bundles $[D] \cdot [D']^{-1}$ belongs to the Picard variety of X, the Chern classes $c_1([D])$ and $c_1([D'])$ coincide with each other. By the formula of intersection number, we have $D \cdot D' = \int_X c_1([D]) \wedge c_1([D']) = \int_X c_1([D]) \wedge c_1([D]) = D^2 = 2$.

It is easy to see that any two of theta divisors are related to each other by parallel translation with respect to the real parameters a_1, a_2, b_1, b_2 of the theta function. Therefore any theta divisor defines an identical homology class in $H_2(X, \boldsymbol{Z})$. By abuse of notation, we denote the homology class in $H_2(X, \boldsymbol{Z})$ corresponding to a theta divisor D by the same symbol D. The next lemma is a corollary of a general theorem in [2].

Lemma 1.6. We have $D = -\lambda_1 * \lambda_3 - \lambda_2 * \lambda_4$ as the equality in $H_2(X, \boldsymbol{Z})$.

After simple calculation we have the following formulas:

Lemma 1.7. (i) $D \cdot (\lambda_1 * \lambda_2 * \lambda_3) = \lambda_2$,
(ii) $D \cdot (\lambda_1 * \lambda_2 * \lambda_4) = -\lambda_1$,
(iii) $D \cdot (\lambda_1 * \lambda_3 * \lambda_4) = -\lambda_4$,

(iv) $D \cdot (\lambda_2 * \lambda_3 * \lambda_4) = \lambda_3$,
(v) $D \cdot (\lambda_1 * \lambda_2) = D \cdot (\lambda_2 * \lambda_3) = D \cdot (\lambda_3 * \lambda_4) = D \cdot (\lambda_4 * \lambda_1) = 0$,
(vi) $D \cdot (\lambda_1 * \lambda_3) = D \cdot (\lambda_2 * \lambda_4) = 1$.

§2 Homology of the complement of theta divisors

Let X be a connected complex manifold of dimension $g(\geq 2)$ which is an open dense subset of a connected compact complex manifold $\bar{X}$ with inclusion $\iota : X \hookrightarrow \bar{X}$. Let Y be a connected complex hypersurface of X of which the closure $\bar{Y}$ in $\bar{X}$ is a connected compact complex hypersurface of $\bar{X}$. We have $\bar{Y} \cap X = Y$. We assume that for every point $y \in \bar{Y} - Y$ there exist an open neighborhood O of y in $\bar{X}$ and a complex local coordinate system $(z_1, \ldots, z_g)$ of y on O such that the point y is expressed by $(z_1, \ldots, z_g) = (0, \ldots, 0)$, $O \cap \bar{Y} = \{p \in O \mid z_g(p) = 0\}$, and $O \cap (\bar{X} - X) = \{p \in O \mid z_1(p) = \cdots = z_{g-1}(p) = 0\}$. Let $\mathcal{S}$ be a locally constant sheaf over $\bar{X}$ with stalk G, where G denotes an abelian group containing the group of integers $\boldsymbol{Z}$ as a subgroup. We denote the restriction of $\mathcal{S}$ to the submanifold X of $\bar{X}$ by the same symbol. Let us consider the exact homology sequence of the pair $(X, X - Y)$ with coefficients in $\mathcal{S}$:

$$\cdots \xrightarrow{\partial} H_p(X - Y; \mathcal{S}|X - Y) \xrightarrow{\varphi} H_p(X; \mathcal{S}) \xrightarrow{\psi} H_p(X, X - Y; \mathcal{S}) \xrightarrow{\partial} H_{p-1}(X - Y; \mathcal{S}|X - Y) \xrightarrow{\varphi} \cdots,$$

where $\mathcal{S}|X - Y$ denotes the restriction of $\mathcal{S}$ to $X - Y$. Let $\bar{T}$ be a tubular neighborhood of $\bar{Y}$ in $\bar{X}$. By definition $\bar{T}$ is a closed subset of $\bar{X}$ containing $\bar{Y}$, and is identified topologically with the total space of a subbundle of the normal bundle of $\bar{Y}$ in $\bar{X}$ having as fibers real two-dimensional closed disks with a common small radius with respect to a Riemannian metric in the normal bundle. We denote by π the projection of $\bar{T}$ onto $\bar{Y}$. We set $\pi^{-1}(Y) = T$. Obviously the closure of T in $\bar{X}$ coincides with $\bar{T}$. By deforming fibers of $\bar{T}$ diffeomorphically with $\bar{Y}$ fixed, we may assume without loss of generality that $\bar{T} \cap X = T$. By excision (e.g., [7,15]), we have an isomorphism $H_p(X, X - Y; \mathcal{S}) \cong H_p(T, T - Y; \mathcal{S}|T)$, where every fiber of the fiber-bundle pair $\pi : (T, T - Y) \to Y$ is identified with (D^2, S^1) topologically, where D^2 denotes a closed disk and $S^1 = \partial D^2$. Note that the complex manifold $\bar{X}$ and its submanifolds (with boundary) are orientable, and that orientations of submanifolds of $\bar{X}$ are induced from restrictions of an orientation of $\bar{X}$. Since $\bar{Y}$ is a retract of $\bar{T}$, we have $H_{2g-2}(\bar{Y}; \boldsymbol{Z}) \cong H_{2g-2}(\bar{T}; \boldsymbol{Z})$. By the Poincaré-Lefschetz duality ([7,15]), we have $H_{2g-2}(\bar{T}; \boldsymbol{Z}) \cong H^2(\bar{T}, \partial\bar{T}; \boldsymbol{Z}) \cong H^2(\bar{T}, \bar{T} - \bar{Y}; \boldsymbol{Z})$. From these isomorphisms it follows that $H_{2g-2}(\bar{Y}; \boldsymbol{Z}) \cong H^2(\bar{T}, \bar{T} - \bar{Y}; \boldsymbol{Z})$. Let U be the cohomology class in $H^2(\bar{T}, \bar{T} - \bar{Y}; \boldsymbol{Z})$ corresponding to the fundamental class $\bar{Y} \in H_{2g-2}(\bar{Y}; \boldsymbol{Z}) \cong \boldsymbol{Z}$ (by abuse of notation). U is called a Thom class of the fiber-bundle pair $\pi : (\bar{T}, \bar{T} - \bar{Y}) \to \bar{Y}$. Then we have the Thom isomorphism $\Phi : H_p(T, T - Y; \mathcal{S}|T) \to H_{p-2}(Y; \mathcal{S}|Y)$, which is given by $\Phi(z) = \pi_*(z \frown \iota^* U)$ for $z \in H_p(T, T - Y; \mathcal{S}|T)$, where $\iota^* U \in H^2(T, T - Y; \boldsymbol{Z})$, π_* is an isomorphism of $H_{p-2}(T; \mathcal{S}|T)$ onto $H_{p-2}(Y; \mathcal{S}|Y)$, and $\frown$ denotes the cap product, i.e., $\frown : H_p(T, T - Y; \mathcal{S}|T) \times H^2(T, T - Y; \boldsymbol{Z}) \to H_{p-2}(T; \mathcal{S}|T)$. Then we have

Proposition 2.1. The following exact sequence holds:

$$\cdots \xrightarrow{\rho} H_p(X - Y; \mathcal{S}|X - Y) \xrightarrow{\varphi} H_p(X; \mathcal{S}) \xrightarrow{\Psi} H_{p-2}(Y; \mathcal{S}|Y) \xrightarrow{\rho} H_{p-1}(X - Y; \mathcal{S}|X - Y) \xrightarrow{\varphi} \cdots,$$

where Ψ and ρ are given by $\Psi = \Phi \circ \psi$ and $\rho = \partial \circ \Phi^{-1}$, respectively.

The exact sequence with $\mathcal{S} = \boldsymbol{C}$ was already introduced by Hodge-Atiyah [8] and Leray [10].

Remark 2.1. By the definition of cap product we see that the following diagrams are commutative:

$$\begin{array}{ccccc} H_p(X, X-Y; \mathcal{S}) & \xrightarrow{\cong} & H_p(T, T-Y; \mathcal{S}|T) & \xrightarrow{\frown \iota^* U} & H_{p-2}(T; \mathcal{S}|T) \\ & & \Big\downarrow \iota_* & & \Big\downarrow \iota_* \\ H_p(\bar{X}, \bar{X}-\bar{Y}; \mathcal{S}) & \xrightarrow{\cong} & H_p(\bar{T}, \bar{T}-\bar{Y}; \mathcal{S}|\bar{T}) & \xrightarrow{\frown U} & H_{p-2}(\bar{T}; \mathcal{S}|\bar{T}) \end{array} \qquad \begin{array}{ccc} z & \longmapsto & z \frown \iota^* U \\ \downarrow & & \downarrow \\ \iota_* z & \longmapsto & \iota_* z \frown U = \iota_*(z \frown \iota^* U) \end{array}$$

Moreover we have $\iota_* z \frown U = \iota_* z \cdot Y$ by a fundamental property of the intersection product, where the product $\iota_* z \cdot Y$ is regarded as the image of the pair $(\iota_* z, Y)$ in $H_p(\bar{T}, \partial\bar{T}; \mathcal{S}|\bar{T}) \times H_{2g-2}(\bar{T}; \boldsymbol{Z})$ by the composition $H_p(\bar{T}, \partial\bar{T}; \mathcal{S}|\bar{T}) \times H_{2g-2}(\bar{T}; \boldsymbol{Z}) \xrightarrow{\text{PD}} H^{2g-p}(\bar{T}; \mathcal{S}|\bar{T}) \times H^2(\bar{T}, \partial\bar{T}; \boldsymbol{Z}) \xrightarrow{\smile} H^{2g-p+2}(\bar{T}, \partial\bar{T}; \mathcal{S}|\bar{T}) \xrightarrow{\text{PD}} H_{p-2}(\bar{T}; \mathcal{S}|\bar{T})$, PD denoting the Poincaré duality, and $\smile$ the cup product (for the detail, see [7]). For this reason, when $X = \bar{X}$, we write by abuse of notation $\Psi(z) = z \cdot Y$ for $z \in H_p(X; \mathcal{S})$.

In the rest of this paper we denote by X a Jacobian variety of dimension two: $X = \boldsymbol{C}^2/(\boldsymbol{Z}^2 + \tau\boldsymbol{Z}^2)$. Let P be a holomorphic line bundle on X with $c_1(P) = 0$. Let $f(z_1, z_2)$ be an arbitrary global meromorphic section over X of the line bundle P. By the Appell-Humbert theorem, we may assume without loss of generality that $f(z_1, z_2)$ satisfies the following transformation formulas:

$$\begin{aligned} f(z_1+1, z_2) &= e^{-2\pi i\alpha} f(z_1, z_2), \\ f(z_1, z_2+1) &= e^{-2\pi i\beta} f(z_1, z_2), \\ f(z_1+\tau_{11}, z_2+\tau_{21}) &= e^{2\pi i\gamma} f(z_1, z_2), \\ f(z_1+\tau_{12}, z_2+\tau_{22}) &= e^{2\pi i\delta} f(z_1, z_2), \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ denote real numbers depending only on the line bundle P, not on global meromorphic sections. Let $\mathcal{O}_X(P)$ denote the sheaf of local sections of P.

Lemma 2.2. Let $\Gamma(X, \mathcal{O}_X(P))$ be the vector space of sections of the sheaf $\mathcal{O}_X(P)$ over X. A necessary and sufficient condition for $\Gamma(X, \mathcal{O}_X(P)) \neq 0$ is that the constants $\alpha, \beta, \gamma, \delta$ are all integers. Moreover, in this case we have $\Gamma(X, \mathcal{O}_X(P)) = \boldsymbol{C}$.

Proof. Let $f(z_1, z_2)$ be a section in $\Gamma(X, \mathcal{O}_X(P))$. We set $g(z_1, z_2) = e^{2\pi i\alpha z_1 + 2\pi i\beta z_2} f(z_1, z_2)$. Then we have

$$g(z_1+1, z_2) = g(z_1, z_2+1) = g(z_1, z_2), \tag{2.1}$$

$$g(z_1+\tau_{11}, z_2+\tau_{21}) = e^{2\pi i\alpha\tau_{11} + 2\pi i\beta\tau_{21} + 2\pi i\gamma} g(z_1, z_2), \tag{2.2}$$

$$g(z_1+\tau_{12}, z_2+\tau_{22}) = e^{2\pi i\alpha\tau_{12} + 2\pi i\beta\tau_{22} + 2\pi i\delta} g(z_1, z_2). \tag{2.3}$$

The periodicity (2.1) allows us to expand $g(z_1, z_2)$ into a Fourier expansion of the form

$$g(z_1, z_2) = \sum_{(\mu,\nu) \in \boldsymbol{Z}^2} c_{\mu\nu} e^{2\mu\pi i z_1 + 2\nu\pi i z_2} \tag{2.4}$$

where $c_{\mu\nu}$'s denote constants. Combining (2.2) with (2.4) we have the relation

$$c_{\mu\nu} e^{2\mu\pi i\tau_{11} + 2\nu\pi i\tau_{21}} = c_{\mu\nu} e^{2\pi i\alpha\tau_{11} + 2\pi i\beta\tau_{21} + 2\pi i\gamma} \tag{2.5}$$

for each $(\mu, \nu) \in \boldsymbol{Z}^2$. Similarly, combining (2.3) with (2.4), we have

$$c_{\mu\nu} e^{2\mu\pi i\tau_{12}+2\nu\pi i\tau_{22}} = c_{\mu\nu} e^{2\pi i\alpha\tau_{12}+2\pi i\beta\tau_{22}+2\pi i\delta} \tag{2.6}$$

for each $(\mu, \nu) \in \boldsymbol{Z}^2$. Then we see that $g(z_1, z_2) \neq 0$ if and only if there exists a pair $(\mu_0, \nu_0) \in \boldsymbol{Z}^2$ such that

$$(\mu_0 - \alpha)\tau_{11} + (\nu_0 - \beta)\tau_{21} - \gamma \in \boldsymbol{Z} \tag{2.7}$$

and

$$(\mu_0 - \alpha)\tau_{12} + (\nu_0 - \beta)\tau_{22} - \delta \in \boldsymbol{Z}. \tag{2.8}$$

Since the imaginary part of the matrix τ is positive definite, it follows from (2.7) and (2.8) that $\mu_0 - \alpha = \nu_0 - \beta = 0$ and $\gamma, \delta \in \boldsymbol{Z}$. Therefore we have by (2.4) $g(z_1, z_2) = c_{\mu_0\nu_0} e^{2\mu_0\pi i z_1 + 2\nu_0\pi i z_2}$ with an arbitrary constant $c_{\mu_0\nu_0}$. This completes the proof of Lemma 2.2.

Let N be an integer such that $N \geq 1$. Let $a_1 \ldots, a_{2N}, b_1, \ldots, b_{2N}$ be real numbers. We denote by D_k the theta divisor corresponding to the theta function $\theta \begin{bmatrix} a_{2k-1} & a_{2k} \\ b_{2k-1} & b_{2k} \end{bmatrix} (z_1, z_2; \tau),\ k = 1, \ldots N$. We assume that the N theta divisors D_k's are different from each other, and that the divisor $D = \sum_{k=1}^N D_k$ has normal crossings. We set $M = X - D$, which is an open dense subset of X.

Lemma 2.3. $\chi(M) = N(N+1)$.

Proof. Since X is homeomorphic to the direct product $S^1 \times S^1 \times S^1 \times S^1$, it follows from $\chi(S^1) = 0$ that $\chi(X) = 0$, and therefore $\chi(M) = \chi(X) - \chi(D) = -\chi(D)$. We set $\overset{\circ}{D}_1 = D_1$. For $k \geq 2$ let $\overset{\circ}{D}_k$ be the open subset of D_k obtained by removing from D_k $2k-2$ intersection points with $D_1 \cup \cdots \cup D_{k-1}$. Since D coincides with the disjoint union $\cup_{k=1}^N \overset{\circ}{D}_k$, we have $\chi(D) = \sum_{k=1}^N \chi(\overset{\circ}{D}_k)$. Since $\chi(\overset{\circ}{D}_k) = -2k$, we have $\chi(M) = -\sum_{k=1}^N \chi(\overset{\circ}{D}_k) = N(N+1)$.

Let $\mathcal{P}$ be the locally constant sheaf over X associated to the holomorphic line bundle P with $c_1(P) = 0$ introduced above. $\mathcal{P}$ has the one-dimensional complex vector space $\boldsymbol{C}$ as stalk. Let $\check{\mathcal{P}}$ be the dual of $\mathcal{P}$: $\check{\mathcal{P}} = \mathrm{Hom}(\mathcal{P}, \boldsymbol{C})$. $\check{\mathcal{P}}$ is also a locally constant sheaf. Let $\check{\mathcal{P}}|M$ be the restriction of $\check{\mathcal{P}}$ to M. Our study of the homology groups $H_p(M, \check{\mathcal{P}}|M)$ consists of two cases. In the first case we study the homology with coefficients in the constant sheaf $\check{\mathcal{P}} = \boldsymbol{C}$. To this end it suffices to consider the case where the coefficients are in $\boldsymbol{Z}$.

Proposition 2.4. $H_4(M, \boldsymbol{Z}) = H_3(M, \boldsymbol{Z}) = 0$, $H_2(M, \boldsymbol{Z}) \cong \boldsymbol{Z}^{N^2+2N+2}$, $H_1(M, \boldsymbol{Z}) \cong \boldsymbol{Z}^{N+3}$, $H_0(M, \boldsymbol{Z}) \cong \boldsymbol{Z}$.

Proof. We proceed by induction on N. Assume that $N = 1$. We have $D = D_1$ and $M = X - D_1$. By Proposition 2.1 we have the following exact sequence:

$$\begin{aligned} 0 &\to H_4(M, \boldsymbol{Z}) \to H_4(X, \boldsymbol{Z}) \to H_2(D, \boldsymbol{Z}) \to H_3(M, \boldsymbol{Z}) \to H_3(X, \boldsymbol{Z}) \to H_1(D, \boldsymbol{Z}) \\ &\to H_2(M, \boldsymbol{Z}) \to H_2(X, \boldsymbol{Z}) \to H_0(D, \boldsymbol{Z}) \to H_1(M, \boldsymbol{Z}) \to H_1(X, \boldsymbol{Z}) \to 0. \end{aligned}$$

Since $H_4(X, \boldsymbol{Z}) \cong H_2(D, \boldsymbol{Z}) \cong \boldsymbol{Z}$, it follows immediately that $H_4(M, \boldsymbol{Z}) = 0$. The mapping $H_3(X, \boldsymbol{Z}) \to H_1(D, \boldsymbol{Z})$ is given by $u \to u \cdot D$ for $u \in H_3(X, \boldsymbol{Z})$, and it is, by Lemma 1.7, an isomorphism. Namely we have $H_3(X,\boldsymbol{Z}) \cong H_1(D,\boldsymbol{Z}) \cong \boldsymbol{Z}^4$, and it follows that $H_3(M,\boldsymbol{Z}) = 0$. Similarly the mapping $H_2(X,\boldsymbol{Z}) \to H_0(D,\boldsymbol{Z})$ is given by $u \to u \cdot D$ for $u \in H_2(X,\boldsymbol{Z})$, and it is, by Lemma 1.7, a surjection. So we have the short exact sequence $0 \to H_2(M,\boldsymbol{Z}) \to H_2(X,\boldsymbol{Z}) \to H_0(D,\boldsymbol{Z}) \to 0$. By Lemma 1.3 we can choose as a basis of $H_2(X,\boldsymbol{Z})$ $\lambda_1 * \lambda_2,\ \lambda_1 * \lambda_3,\ \lambda_1 * \lambda_4,\ \lambda_2 * \lambda_3,\ \lambda_2 * \lambda_4,\ \lambda_3 * \lambda_4$. It follows that $H_2(M,\boldsymbol{Z})$ is of rank 5 and is generated by $\lambda_1 * \lambda_2,\ \lambda_2 * \lambda_3,\ \lambda_3 * \lambda_4,\ \lambda_4 * \lambda_1,\ \lambda_1 * \lambda_3 - \lambda_2 * \lambda_4$. In fact we can construct a 2-cycle defining the homology class $\lambda_1 * \lambda_3 - \lambda_2 * \lambda_4$ (cf. Remark 2.2). Finally we have $H_1(M,\boldsymbol{Z}) \cong H_1(X,\boldsymbol{Z}) \cong \boldsymbol{Z}^4$ and $H_0(M,\boldsymbol{Z}) \cong \boldsymbol{Z}$. The generators $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of $H_1(X,\boldsymbol{Z})$ also generate $H_1(M,\boldsymbol{Z})$. The proposition in the case where $N = 1$ is thus proved.

Remark 2.2. Let us construct a 2-cycle defining the homology class $\lambda_1 * \lambda_3 - \lambda_2 * \lambda_4 \in H_2(M,\boldsymbol{Z})$. By abuse of notation we regard $\lambda_1 * \lambda_3,\ \lambda_2 * \lambda_4 \in Z_2(X,\boldsymbol{Z})$. By homologously translating them appropriately, we may assume that there exists a unique point $p \in X$ such that $\lambda_1 * \lambda_3 \cap \lambda_2 * \lambda_4 \cap D = \{p\}$ where $D = D_1$ is also regarded as a 2-cycle in X. Then we can regard $\lambda_1 * \lambda_3 - \lambda_2 * \lambda_4 = (\lambda_1 * \lambda_3 - \{p\}) - (\lambda_2 * \lambda_4 - \{p\})$ set-theoretically. Let D_{13} be a small open disk centered at p in the torus $\lambda_1 * \lambda_3$, and D_{24} a small open disk centered at p in the torus $\lambda_2 * \lambda_4$. Obviously, D_{13} and D_{24} intersect transversely at p. We set $\lambda_{13} = \partial D_{13}$ and $\lambda_{24} = \partial D_{24}$. Then there exists a cylinder Z such that $\partial Z = \lambda_{13} - \lambda_{24}$. In fact we identify the point p with the origin of $\boldsymbol{C}^2$, and $\lambda_{13}, \lambda_{24}$ with the circles $(e^{i\varphi}, 0),\ (0, e^{i\varphi})$ $(0 \le \varphi \le 2\pi)$ in $\boldsymbol{C}^2$, respectively. As a cylinder Z with the desired property we can take the 2-dimensional real surface with boundary defined by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{i\varphi} \\ 0 \end{pmatrix}$, $0 \le \theta \le \frac{\pi}{2},\ 0 \le \varphi \le 2\pi$. Then the 2-cycle $\lambda_1 * \lambda_3 - \lambda_2 * \lambda_4 = (\lambda_1 * \lambda_3 - \{p\}) - (\lambda_2 * \lambda_4 - \{p\})$ in $Z_2(M,\boldsymbol{Z})$ is homotopically equivalent to the closed surface $(\lambda_1 * \lambda_3 - D_{13}) - Z - (\lambda_2 * \lambda_4 - D_{24})$ of genus 2 in M. In what follows, we denote this surface by S.

Let us complete the proof of Proposition 2.4. Assume that $N \ge 2$. Inductively we set $X_k = X_{k-1} - \overset{\circ}{D}_k$ for every k such that $1 \le k \le N$, where $X_0 = X$. Assume that the proposition holds for $M = X_k$ where $1 \le k \le N-1$. By Proposition 2.1 we have the following exact sequence:

$$
\begin{aligned}
&0 \to H_4(X_N,\boldsymbol{Z}) \to H_4(X_{N-1},\boldsymbol{Z}) \to H_2(\overset{\circ}{D}_N,\boldsymbol{Z}) \to H_3(X_N,\boldsymbol{Z}) \\
&\to H_3(X_{N-1},\boldsymbol{Z}) \to H_1(\overset{\circ}{D}_N,\boldsymbol{Z}) \to H_2(X_N,\boldsymbol{Z}) \to H_2(X_{N-1},\boldsymbol{Z}) \\
&\to H_0(\overset{\circ}{D}_N,\boldsymbol{Z}) \to H_1(X_N,\boldsymbol{Z}) \to H_1(X_{N-1},\boldsymbol{Z}) \to 0.
\end{aligned}
$$

Since $H_4(X_{N-1},\boldsymbol{Z}) = H_2(\overset{\circ}{D}_N,\boldsymbol{Z}) = H_3(X_{N-1},\boldsymbol{Z}) = 0$, we have $H_4(X_N,\boldsymbol{Z}) = H_3(X_N,\boldsymbol{Z}) = 0$. Since any 2-cycle in X_{N-1} does not intersect with D_N, the mapping $H_2(X_{N-1},\boldsymbol{Z}) \to H_0(\overset{\circ}{D}_N,\boldsymbol{Z})$ is the zero mapping, and so we have the short exact sequence $0 \to H_1(\overset{\circ}{D}_N,\boldsymbol{Z}) \to H_2(X_N,\boldsymbol{Z}) \to H_2(X_{N-1},\boldsymbol{Z}) \to 0$. Since we can regard elements of $H_2(X_{N-1},\boldsymbol{Z})$ as those of $H_2(X_N,\boldsymbol{Z})$, we have an inclusion $H_2(X_{N-1},\boldsymbol{Z}) \hookrightarrow H_2(X_N,\boldsymbol{Z})$, and therefore the short exact sequence above is split. Then we have $H_2(X_N,\boldsymbol{Z}) \cong H_1(\overset{\circ}{D}_N,\boldsymbol{Z}) \oplus H_2(X_{N-1},\boldsymbol{Z}) \cong \boldsymbol{Z}^{2N+1} \oplus \boldsymbol{Z}^{N^2+1} \cong \boldsymbol{Z}^{N^2+2N+2}$. Finally we have the short exact sequence $0 \to H_0(\overset{\circ}{D}_N,\boldsymbol{Z}) \to H_1(X_N,\boldsymbol{Z}) \to H_1(X_{N-1},\boldsymbol{Z}) \to 0$. Since this sequense is also split, we have $H_1(X_N,\boldsymbol{Z}) \cong H_0(\overset{\circ}{D}_N,\boldsymbol{Z}) \oplus H_1(X_{N-1},\boldsymbol{Z}) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}^{N+2} \cong \boldsymbol{Z}^{N+3}$. Thus, Proposition 2.4 is proved completely.

Remark 2.3. It follows from the short exact sequence $0 \to H_1(\overset{\circ}{D}_N, \boldsymbol{Z}) \to H_2(X_N, \boldsymbol{Z}) \to H_2(X_{N-1}, \boldsymbol{Z}) \to 0$ that the union of a basis of $H_2(X_{N-1}, \boldsymbol{Z})$ with the image of a basis of $H_1(\overset{\circ}{D}_N, \boldsymbol{Z})$ by the injection $H_1(\overset{\circ}{D}_N, \boldsymbol{Z}) \to H_2(X_N, \boldsymbol{Z})$ forms a basis of $H_2(X_N, \boldsymbol{Z})$. We set $\overset{\circ}{D}_N = D_N - \{q_{N1}, q'_{N1}, q_{N2}, q'_{N2}, \ldots, q_{N,N-1}, q'_{N,N-1}\}$, where $D_N \cap D_i = \{q_{Ni}, q'_{Ni}\}$. Let γ_{Ni} and γ'_{Ni} be 1-cycles on $\overset{\circ}{D}_N$ turning once around q_{Ni} and q'_{Ni}, respectively, in the counterclockwise direction. Then we have $\gamma_{N1} + \gamma'_{N1} + \gamma_{N2} + \gamma'_{N2} + \cdots + \gamma_{N,N-1} + \gamma'_{N,N-1} = 0$. The homology classes of $H_1(\overset{\circ}{D}_N, \boldsymbol{Z})$ defined by the four global cycles $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $2N-3$ cycles among $\gamma_{N1}, \gamma'_{N1}, \gamma_{N2}, \gamma'_{N2}, \ldots, \gamma_{N,N-1}, \gamma'_{N,N-1}$ form a basis of $H_1(\overset{\circ}{D}_N, \boldsymbol{Z})$. We denote by $\lambda_i^{(N)}$ the image of λ_i by the injection $H_1(\overset{\circ}{D}_N, \boldsymbol{Z}) \to H_2(X_N, \boldsymbol{Z})$, and by T_{Ni} and T'_{Ni} the images of γ_{Ni} and γ'_{Ni} respectively. Since $(\lambda_1 * \lambda_2 * \lambda_4) \cdot D = \lambda_1$ by Lemma 1.7, we may set $\lambda_1^{(N)} = \lambda_1 * \lambda_{24}^{(N)}$, where $\lambda_{24}^{(N)}$ denotes a small circle on $\lambda_2 * \lambda_4$ centered at $p \in \lambda_2 * \lambda_4 \cap D_N$. Similarly we have $\lambda_2^{(N)} = \lambda_2 * \lambda_{13}^{(N)}$, $\lambda_3^{(N)} = \lambda_3 * \lambda_{24}^{(N)}$, $\lambda_4^{(N)} = \lambda_4 * \lambda_{13}^{(N)}$. Note that $\lambda_i^{(N)}$ is a torus, i.e., a real 2-dimensional surface of genus one. Let γ_{iN} and γ'_{iN} be 1-cycles on D_i turning once around q_{Ni} and q'_{Ni}, respectively, in the counterclockwise direction. Then we may set $T_{Ni} = \gamma_{Ni} * \gamma_{iN}$, $T'_{Ni} = \gamma'_{Ni} * \gamma'_{iN}$, which are also tori.

Remark 2.4. Similarly, we may take as a basis of $H_1(X_N, \boldsymbol{Z})$ the union of a basis of $H_1(X_{N-1}, \boldsymbol{Z})$ with the image of a single generator of $H_0(\overset{\circ}{D}_N, \boldsymbol{Z})$ by the injection $H_0(\overset{\circ}{D}_N, \boldsymbol{Z}) \to H_1(X_N, \boldsymbol{Z})$. This image is realized by, for an arbitrary fixed point $p \in \overset{\circ}{D}_N$ (which determines a homology class generating $H_0(\overset{\circ}{D}_N, \boldsymbol{Z})$), the homology class of $H_1(X_N, \boldsymbol{Z})$ which is defined by the boundary of a small disk centered at p and transverse to $\overset{\circ}{D}_N$.

The proof above of Proposition 2.4 depends on ordering of the theta divisors $D_1, \ldots, D_N$. If we adopt new ordering to prove the proposition, then we obtain another new basis of $H_2(X_N, \boldsymbol{Z})$. Let us study the relations between bases of $H_2(X_N, \boldsymbol{Z})$. For $i \neq j$ we set $D_i \cap D_j = \{q_{ij}, q'_{ij}\}$, where we understand that $q_{ij} = q_{ji}$ and $q'_{ij} = q'_{ji}$. Moreover we set $D'_i = D_i - \{q_{i1}, q'_{i1}, q_{i2}, q'_{i2}, \ldots, q_{i,i-1}, q'_{i,i-1}, q_{i,i+1}, q'_{i,i+1}, \ldots, q_{iN}, q'_{iN}\}$. Let γ_{ij} and γ'_{ij} be 1-cycles on D'_i turning once around q_{ij} and q'_{ij}, respectively, in the counterclockwise direction. Then we have $\gamma_{i1} + \gamma'_{i1} + \gamma_{i2} + \gamma'_{i2} + \cdots + \gamma_{i,i-1} + \gamma'_{i,i-1} + \gamma_{i,i+1} + \gamma'_{i,i+1} + \cdots + \gamma_{iN} + \gamma'_{iN} = 0$. The homology classes of $H_1(D'_i, \boldsymbol{Z})$ defined by the four global cycles $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $2N-3$ cycles among $\gamma_{i1}, \gamma'_{i1}, \ldots, \gamma_{i,i-1}, \gamma'_{i,i-1}, \gamma_{i,i+1}, \gamma'_{i,i+1}, \ldots, \gamma_{iN}, \gamma'_{iN}$ form a basis of $H_1(D'_i, \boldsymbol{Z})$. We denote by $\lambda_k^{(i)}$ the image of λ_k by the injection $H_1(D'_i, \boldsymbol{Z}) \to H_2(X_N, \boldsymbol{Z})$, and by T_{ij} and T'_{ij} $(i \neq j)$ the images of γ_{ij} and γ'_{ij} respectively. As is easily seen, the 2-cycles $\lambda_k^{(i)}, T_{ij}, T'_{ij}$ are tori, and we have by construction $T_{ij} = -T_{ji}$ and $T'_{ij} = -T'_{ji}$. Consequently, the group $H_2(M, \boldsymbol{Z})$ is generated by $\lambda_1 * \lambda_2,\ \lambda_2 * \lambda_3,\ \lambda_3 * \lambda_4,\ \lambda_4 * \lambda_1,\ S(= \lambda_1 * \lambda_3 - \lambda_2 * \lambda_4),\ \lambda_k^{(i)},\ T_{ij},\ T'_{ij}$.

Proposition 2.5. The relations in $H_2(M, \boldsymbol{Z})$ satisfied by the generators above are as follows:

$$\sum_{i=1}^{N} \lambda_k^{(i)} = 0 \quad \text{for} \quad k = 1, 2, 3, 4; \tag{2.9}$$

$$\sum_{1 \leq j \leq N,\ j \neq i} (T_{ij} + T'_{ij}) = 0 \quad \text{for} \quad i = 1, \ldots, N. \tag{2.10}$$

Proof. For each i, formula (2.10) follows from the relation $\sum_{1\le j\le N,\ j\ne i}(\gamma_{ij}+\gamma'_{ij})=0$ on D'_i. In order to prove formula (2.9), it suffices to consider the case where $k=1$. In this case we may set $\lambda_1^{(i)}=\lambda_1*\lambda_{24}^{(i)}$, where $\lambda_{24}^{(i)}$ denotes a small circle on $\lambda_2*\lambda_4$ centered at the point $p_i\in\lambda_2*\lambda_4\cap D_i$. Formula (2.9) follows from the relation $\sum_{i=1}^N\lambda_{24}^{(i)}=0$ on the torus $\lambda_2*\lambda_4$.

Let us now proceed to the second case where we study the homology with coefficients in a locally constant sheaf $\check{\mathcal{P}}$ different from constant sheaves. Let $f(z_1,z_2)$ be a global meromorphic section over X of the line bundle associated to $\check{\mathcal{P}}$. By Lemma 2.2 we may assume that there exist four real numbers $\alpha,\beta,\gamma,\delta$, not all of which are integers, depending only on the line bundle of $\check{\mathcal{P}}$, not on global meromorphic sections, such that $f(z_1,z_2)$ satisfies the following formulas:

$$\begin{aligned}
f(z_1+1,z_2)&=e^{2\pi i\alpha}f(z_1,z_2),\\
f(z_1,z_2+1)&=e^{2\pi i\beta}f(z_1,z_2),\\
f(z_1+\tau_{11},z_2+\tau_{21})&=e^{-2\pi i\gamma}f(z_1,z_2),\\
f(z_1+\tau_{12},z_2+\tau_{22})&=e^{-2\pi i\delta}f(z_1,z_2).
\end{aligned}$$

We regard the complex torus X as a cell complex with respect to the cellular decomposition given in Lemma 1.3. Let $C_k(X,\check{\mathcal{P}})$ be the group of cellular k-chains of the cell complex X with coefficients in the locally constant sheaf $\check{\mathcal{P}}$. By the definition of cell-chain, we have an identification $C_k(X,\check{\mathcal{P}})\cong\sum_{\lambda\in\Lambda_k}H_k(D_\lambda^k,S_\lambda^{k-1};\boldsymbol{Z})\otimes\check{\mathcal{P}}_\lambda$, where Λ_k denotes the set of indices parametrizing all the k-cells in X, D_λ^k and S_λ^{k-1} for each $\lambda\in\Lambda_k$ denote copies of the k-dimensional disk D^k and the $(k-1)$-dimensional sphere S^{k-1}, respectively, and $\check{\mathcal{P}}_\lambda$ the restriction of $\check{\mathcal{P}}$ to the k-cell in X indexed by $\lambda\in\Lambda_k$ ([7]). As usual we set $Z_k(X,\check{\mathcal{P}})=\mathrm{Ker}\left[C_k(X,\check{\mathcal{P}})\xrightarrow{\partial}C_{k-1}(X,\check{\mathcal{P}})\right]$ and $B_k(X,\check{\mathcal{P}})=\mathrm{Im}\left[C_{k+1}(X,\check{\mathcal{P}})\xrightarrow{\partial}C_k(X,\check{\mathcal{P}})\right]$. If, by abuse of notation, we regard $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ as elements of $C_1(X,\check{\mathcal{P}})$, we have the following formulas: $\partial\lambda_1=(e^{2\pi i\alpha}-1)\bullet$, $\partial\lambda_2=(e^{2\pi i\beta}-1)\bullet$, $\partial\lambda_3=(e^{-2\pi i\gamma}-1)\bullet$, $\partial\lambda_4=(e^{-2\pi i\delta}-1)\bullet$, where $\bullet$ denotes a unique element generating $C_0(X,\check{\mathcal{P}})$.

Lemma 2.6. For all $p\ge 0$, $H_p(X,\check{\mathcal{P}})=0$.

Proof. Regarding $X=\lambda_1*\lambda_2*\lambda_3*\lambda_4$ as a generator of $C_4(X,\check{\mathcal{P}})$, we have $\partial X\ne 0$, from which it follows that $H_4(X,\check{\mathcal{P}})=0$. The group $C_3(X,\check{\mathcal{P}})$ is generated by the four elements $\lambda_1*\lambda_2*\lambda_3$, $\lambda_1*\lambda_2*\lambda_4$, $\lambda_1*\lambda_3*\lambda_4$, $\lambda_2*\lambda_3*\lambda_4$. By simple calculation we see that a linear combination $x(\lambda_1*\lambda_2*\lambda_3)+y(\lambda_1*\lambda_2*\lambda_4)+z(\lambda_1*\lambda_3*\lambda_4)+w(\lambda_2*\lambda_3*\lambda_4)$ with constants x,y,z,w belongs to $Z_3(X,\check{\mathcal{P}})$ if and only if $x=A(-e^{-2\pi i\delta}+1)$, $y=A(e^{-2\pi i\gamma}-1)$, $z=A(-e^{2\pi i\beta}+1)$, $w=A(e^{2\pi i\alpha}-1)$ with an arbitrary constant A. Since $(-e^{-2\pi i\delta}+1)(\lambda_1*\lambda_2*\lambda_3)+(e^{-2\pi i\gamma}-1)(\lambda_1*\lambda_2*\lambda_4)+(-e^{2\pi i\beta}+1)(\lambda_1*\lambda_3*\lambda_4)+(e^{2\pi i\alpha}-1)(\lambda_2*\lambda_3*\lambda_4)=\partial(\lambda_1*\lambda_2*\lambda_3*\lambda_4)$, we have $H_3(X,\check{\mathcal{P}})=0$. The group $C_2(X,\check{\mathcal{P}})$ is generated by the six elements $\lambda_1*\lambda_2$, $\lambda_1*\lambda_3$, $\lambda_1*\lambda_4$, $\lambda_2*\lambda_3$, $\lambda_2*\lambda_4$, $\lambda_3*\lambda_4$. A linear combination $u(\lambda_1*\lambda_2)+v(\lambda_1*\lambda_3)+w(\lambda_1*\lambda_4)+x(\lambda_2*\lambda_3)+y(\lambda_2*\lambda_4)+z(\lambda_3*\lambda_4)$ with constants u,v,w,x,y,z belongs to $Z_2(X,\check{\mathcal{P}})$ if and only if u,v,w,x,y,z satisfy the following linear equations

$$
\begin{aligned}
(e^{2\pi i\beta}-1)u+(e^{-2\pi i\gamma}-1)v+(e^{-2\pi i\delta}-1)w=0,\\
(e^{2\pi i\alpha}-1)u-(e^{-2\pi i\gamma}-1)x-(e^{-2\pi i\delta}-1)y=0,\\
(e^{2\pi i\alpha}-1)v+(e^{2\pi i\beta}-1)x-(e^{-2\pi i\delta}-1)z=0,\\
(e^{2\pi i\alpha}-1)w+(e^{2\pi i\beta}-1)y+(e^{-2\pi i\gamma}-1)z=0.
\end{aligned}
$$

Since not all of $\alpha,\beta,\gamma,\delta$ are integers, we have by easy calculation $u(\lambda_1*\lambda_2)+v(\lambda_1*\lambda_3)+w(\lambda_1*\lambda_4)+x(\lambda_2*\lambda_3)+y(\lambda_2*\lambda_4)+z(\lambda_3*\lambda_4)\in B_2(X,\check{\mathcal{P}})$ for an arbitrary solution (u,v,w,x,y,z) of the linear equations above. Therefore $H_2(X,\check{\mathcal{P}})=0$. The group $C_1(X,\check{\mathcal{P}})$ is generated by λ_1, λ_2, λ_3, λ_4. A linear combination $x\lambda_1+y\lambda_2+z\lambda_3+w\lambda_4$ with constants x,y,z,w belongs to $Z_1(X,\check{\mathcal{P}})$ if and only if x,y,z,w satisfy a single linear equation $(e^{2\pi i\alpha}-1)x+(e^{2\pi i\beta}-1)y+(e^{-2\pi i\gamma}-1)z+(e^{-2\pi i\delta}-1)w=0$. For an arbitrary solution (x,y,z,w) of this linear equation we have $x\lambda_1+y\lambda_2+z\lambda_3+w\lambda_4\in B_1(X,\check{\mathcal{P}})$, and therefore $H_1(X,\check{\mathcal{P}})=0$. Obviously $H_0(X,\check{\mathcal{P}})=0$.

We regard the theta divisor D_1 as a cellular subcomplex of X. Then we have

Lemma 2.7. *$H_p(D_1,\check{\mathcal{P}}|D_1)=0$ if $p\neq 1$; $H_1(D_1,\check{\mathcal{P}}|D_1)\cong \boldsymbol{C}^2$.*

Proof. The theta divisor $D_1=-\lambda_1*\lambda_3-\lambda_2*\lambda_4$ generates the group $C_2(D_1,\check{\mathcal{P}}|D_1)$. Since $\partial D_1\neq 0$, we have $H_2(D_1,\check{\mathcal{P}}|D_1)=0$. The group $C_1(D_1,\check{\mathcal{P}}|D_1)$ is generated by λ_1, λ_2, λ_3, λ_4. A linear combination $x\lambda_1+y\lambda_2+z\lambda_3+w\lambda_4$ with constants x,y,z,w belongs to $Z_1(D_1,\check{\mathcal{P}}|D_1)$ if and only if x,y,z,w satisfy a linear equation $(e^{2\pi i\alpha}-1)x+(e^{2\pi i\beta}-1)y+(e^{-2\pi i\gamma}-1)z+(e^{-2\pi i\delta}-1)w=0$. Suppose that two of $\alpha,\beta,\gamma,\delta$, for examle γ and δ, are not integers. Then, by using the preceding linear equation, we have

$$
\begin{aligned}
x\lambda_1+y\lambda_2+z\lambda_3+w\lambda_4=&\frac{x}{e^{-2\pi i\delta}-1}\left\{(e^{-2\pi i\delta}-1)\lambda_1-(e^{2\pi i\alpha}-1)\lambda_4\right\}\\
&+\frac{y}{e^{-2\pi i\delta}-1}\left\{(e^{-2\pi i\delta}-1)\lambda_2-(e^{2\pi i\beta}-1)\lambda_4\right\}\\
&+\frac{z}{e^{-2\pi i\delta}-1}\left\{(e^{-2\pi i\delta}-1)\lambda_3-(e^{-2\pi i\gamma}-1)\lambda_4\right\}.
\end{aligned}
$$

Since $(e^{-2\pi i\delta}-1)\lambda_2-(e^{2\pi i\beta}-1)\lambda_4=(e^{2\pi i\alpha}-1)\lambda_3-(e^{-2\pi i\gamma}-1)\lambda_1+\partial D_1$, we have

$$
\begin{aligned}
x\lambda_1+y\lambda_2+z\lambda_3+w\lambda_4\equiv&\frac{x}{e^{-2\pi i\delta}-1}\left\{(e^{-2\pi i\delta}-1)\lambda_1-(e^{2\pi i\alpha}-1)\lambda_4\right\}\\
&+\frac{y}{e^{-2\pi i\delta}-1}\left\{(e^{2\pi i\alpha}-1)\lambda_3-(e^{-2\pi i\gamma}-1)\lambda_1\right\}\\
+&\frac{z}{e^{-2\pi i\delta}-1}\left\{(e^{-2\pi i\delta}-1)\lambda_3-(e^{-2\pi i\gamma}-1)\lambda_4\right\}\quad\mathrm{mod}\quad B_1(D_1,\check{\mathcal{P}}|D_1).
\end{aligned}
$$

Noting that

$$
\begin{aligned}
(e^{-2\pi i\delta}-1)\lambda_1-(e^{2\pi i\alpha}-1)\lambda_4=&\frac{e^{-2\pi i\delta}-1}{e^{-2\pi i\gamma}-1}\{(e^{-2\pi i\gamma}-1)\lambda_1-(e^{2\pi i\alpha}-1)\lambda_3\}\\
&+\frac{e^{2\pi i\alpha}-1}{e^{-2\pi i\gamma}-1}\{(e^{-2\pi i\delta}-1)\lambda_3-(e^{-2\pi i\gamma}-1)\lambda_4\},
\end{aligned}
$$

we see that the 1-cycle $x\lambda_1+y\lambda_2+z\lambda_3+w\lambda_4$ is congruent modulo $B_1(D_1,\check{\mathcal{P}}|D_1)$ with a linear combination of the two 1-cycles $(e^{-2\pi i\gamma}-1)\lambda_1-(e^{2\pi i\alpha}-1)\lambda_3$ and $(e^{-2\pi i\delta}-1)\lambda_3-(e^{-2\pi i\gamma}-1)\lambda_4$

which are not homologous to each other. Therefore we have $H_1(D_1, \check{\mathcal{P}}|D_1) \cong \boldsymbol{C}^2$ in this case. Next, suppose that only δ is not an integer and the others are integers. By the linear combination above satisfied by $\alpha, \beta, \gamma, \delta$, we have $w = 0$. Since $\partial D_1 = (e^{-2\pi i\delta} - 1)\lambda_2$, we see that the 1-cycle $x\lambda_1 + y\lambda_2 + z\lambda_3$ is congruent modulo $B_1(D_1, \check{\mathcal{P}}|D_1)$ with $x\lambda_1 + z\lambda_3$, where λ_1 and λ_3 are not homologous to each other. Therefore we have $H_1(D_1, \check{\mathcal{P}}|D_1) \cong \boldsymbol{C}^2$ in this case, too. Obviously we have $H_0(D_1, \check{\mathcal{P}}|D_1) = 0$.

Using the preceding two lemmas, we shall prove the following

Proposition 2.8. $H_p(M, \check{\mathcal{P}}|M) = 0$ if $p \neq 2$; $H_2(M, \check{\mathcal{P}}|M) \cong \boldsymbol{C}^{N(N+1)}$.

Proof. Induction on N. Assume that $N = 1$. We have $D = D_1$ and $M = X - D_1$. By Proposition 2.1 we have the following exact sequence:

$$
\begin{aligned}
&0 \to H_4(M, \check{\mathcal{P}}|M) \to H_4(X, \check{\mathcal{P}}) \to H_2(D, \check{\mathcal{P}}|D) \to H_3(M, \check{\mathcal{P}}|M) \to H_3(X, \check{\mathcal{P}}) \\
&\to H_1(D, \check{\mathcal{P}}|D) \to H_2(M, \check{\mathcal{P}}|M) \to H_2(X, \check{\mathcal{P}}) \to H_0(D, \check{\mathcal{P}}|D) \to H_1(M, \check{\mathcal{P}}|M) \\
&\to H_1(X, \check{\mathcal{P}}) \to 0.
\end{aligned}
$$

By Lemmas 2.6 and 2.7 we have $H_4(M, \check{\mathcal{P}}|M) = H_3(M, \check{\mathcal{P}}|M) = H_1(M, \check{\mathcal{P}}|M) = 0$ and $H_2(M, \check{\mathcal{P}}|M) \cong H_1(D, \check{\mathcal{P}}|D) \cong \boldsymbol{C}^2$. In addition obviously we have $H_0(M, \check{\mathcal{P}}|M) = 0$.

Remark 2.5. Suppose that γ and δ are not integers. As is seen in the proof of Lemma 2.7, the homology classes defined by $(e^{-2\pi i\gamma} - 1)\lambda_1 - (e^{2\pi i\alpha} - 1)\lambda_3$ and $(e^{-2\pi i\delta} - 1)\lambda_3 - (e^{-2\pi i\gamma} - 1)\lambda_4$ form a basis of $H_1(D, \check{\mathcal{P}}|D)$. The image of $(e^{-2\pi i\gamma} - 1)\lambda_1 - (e^{2\pi i\alpha} - 1)\lambda_3$ by the isomorphism $H_1(D, \check{\mathcal{P}}|D) \to H_2(M, \check{\mathcal{P}}|M)$ is obviously given by $(e^{-2\pi i\gamma} - 1)\lambda_1^{(1)} - (e^{2\pi i\alpha} - 1)\lambda_3^{(1)}$, where $\lambda_1^{(1)} = \lambda_1 * \lambda_{24}^{(1)}$ and $\lambda_3^{(1)} = \lambda_3 * \lambda_{24}^{(1)}$. To construct the image of $(e^{-2\pi i\delta} - 1)\lambda_3 - (e^{-2\pi i\gamma} - 1)\lambda_4$, let us recall the cylinder Z defined in Remark 2.2 satisfying $\partial Z = \lambda_{13}^{(1)} - \lambda_{24}^{(1)}$. Then the image of $(e^{-2\pi i\delta} - 1)\lambda_3 - (e^{-2\pi i\gamma} - 1)\lambda_4$ by the isomorphism $H_1(D, \check{\mathcal{P}}|D) \to H_2(M, \check{\mathcal{P}}|M)$ is given by $(e^{-2\pi i\delta} - 1)\lambda_3^{(1)} - (e^{-2\pi i\gamma} - 1)\lambda_4^{(1)} + (e^{-2\pi i\gamma} - 1)(e^{-2\pi i\delta} - 1)Z$. We can also deal with the other cases for $\alpha, \beta, \gamma, \delta$ similarly.

In order to continue proving Proposition 2.8 we need one more lemma:

Lemma 2.9. For $N \geq 2$, we have $H_2(\overset{\circ}{D}_N, \check{\mathcal{P}}|\overset{\circ}{D}_N) = H_0(\overset{\circ}{D}_N, \check{\mathcal{P}}|\overset{\circ}{D}_N) = 0$ and $H_1(\overset{\circ}{D}_N, \check{\mathcal{P}}|\overset{\circ}{D}_N) \cong \boldsymbol{C}^{2N}$.

Proof. The equality $H_2(\overset{\circ}{D}_N, \check{\mathcal{P}}|\overset{\circ}{D}_N) = H_0(\overset{\circ}{D}_N, \check{\mathcal{P}}|\overset{\circ}{D}_N) = 0$ is obvious. Let Δ be the closed subset of D_N which is obtained by deleting from D_N mutually disjoint $2N-2$ small open disks centered at the points of $\{q_1, q_1', \cdots, q_{N-1}, q_{N-1}'\}$. Then we have $H_1(\overset{\circ}{D}_N, \check{\mathcal{P}}|\overset{\circ}{D}_N) = H_1(\Delta, \check{\mathcal{P}}|\Delta)$. Let U be a connected and simply connected open subset of D_N with a smooth boundary which contains the closure of the union of those $2N-2$ open disks above, and V be the subset of U which is obtained by deleting from U those $2N-2$ open disks above. Moreover we set $\Delta_1 = \Delta - V$ and $\Delta_2 = \bar{V}$ (the closure of V). Then we have $\Delta = \Delta_1 \cup \Delta_2$, and $\Delta_1 \cap \Delta_2$ is homeomorphic to the one-dimensional circle S^1. Let us consider the Mayer-Vietoris exact sequence:

$$
\begin{aligned}
&0 \to H_1(\Delta_1 \cap \Delta_2, \check{\mathcal{P}}|\Delta_1 \cap \Delta_2) \to H_1(\Delta_1, \check{\mathcal{P}}|\Delta_1) \oplus H_1(\Delta_2, \check{\mathcal{P}}|\Delta_2) \to H_1(\Delta, \check{\mathcal{P}}|\Delta) \\
&\to H_0(\Delta_1 \cap \Delta_2, \check{\mathcal{P}}|\Delta_1 \cap \Delta_2) \to H_0(\Delta_1, \check{\mathcal{P}}|\Delta_1) \oplus H_0(\Delta_2, \check{\mathcal{P}}|\Delta_2) \to 0.
\end{aligned}
$$

Obviously we have $H_1(\Delta_1\cap\Delta_2,\check{\mathcal{P}}|\Delta_1\cap\Delta_2)\cong \boldsymbol{C}$. Since we may regard $D_N=-\lambda_1*\lambda_3-\lambda_2*\lambda_4$, we have $\partial D_N=(e^{-2\pi i\gamma}-1)\lambda_1-(e^{2\pi i\alpha}-1)\lambda_3+(e^{-2\pi i\delta}-1)\lambda_2-(e^{2\pi i\beta}-1)\lambda_4$. It follows that $\partial\Delta_1=\partial D_N-(\Delta_1\cap\Delta_2)=(e^{-2\pi i\gamma}-1)\lambda_1-(e^{2\pi i\alpha}-1)\lambda_3+(e^{-2\pi i\delta}-1)\lambda_2-(e^{2\pi i\beta}-1)\lambda_4-(\Delta_1\cap\Delta_2)$, where by abuse of notation $\Delta_1\cap\Delta_2$ denotes a 1-cycle which defines a generator of $H_1(\Delta_1\cap\Delta_2,\check{\mathcal{P}}|\Delta_1\cap\Delta_2)$ with a suitable orientation. Then we see that the group $C_1(\Delta_1,\check{\mathcal{P}}|\Delta_1)$ is generated by $\lambda_1,\lambda_2,\lambda_3,\lambda_4$. A linear combination $x\lambda_1+y\lambda_2+z\lambda_3+w\lambda_4$ with constants x,y,z,w belongs to $Z_1(\Delta_1,\check{\mathcal{P}}|\Delta_1)$ if and only if x,y,z,w satisfy a linear equation $(e^{2\pi i\alpha}-1)x+(e^{2\pi i\beta}-1)y+(e^{-2\pi i\gamma}-1)z+(e^{-2\pi i\delta}-1)w=0$. Suppose that δ is not an integer. Then, by using the preceding linear relation, we have

$$\begin{aligned}x\lambda_1+y\lambda_2+z\lambda_3+w\lambda_4 =&\frac{x}{e^{-2\pi i\delta}-1}\left\{(e^{-2\pi i\delta}-1)\lambda_1-(e^{2\pi i\alpha}-1)\lambda_4\right\}\\ &+\frac{y}{e^{-2\pi i\delta}-1}\left\{(e^{-2\pi i\delta}-1)\lambda_2-(e^{2\pi i\beta}-1)\lambda_4\right\}\\ &+\frac{z}{e^{-2\pi i\delta}-1}\left\{(e^{-2\pi i\delta}-1)\lambda_3-(e^{-2\pi i\gamma}-1)\lambda_4\right\},\end{aligned}$$

from which it follows that $H_1(\Delta_1,\check{\mathcal{P}}|\Delta_1)\cong\boldsymbol{C}^3$. We can deal with the other cases for $\alpha,\beta,\gamma,\delta$ similarly. Since the $2N-2$ 1-cycles $\gamma_{N1},\gamma'_{N1},\dots,\gamma_{N,N-1},\gamma'_{N,N-1}$ generate the group $H_1(\Delta_2,\check{\mathcal{P}}|\Delta_2)$, we have $H_1(\Delta_2,\check{\mathcal{P}}|\Delta_2)\cong\boldsymbol{C}^{2N-2}$. Obviously we have $H_0(\Delta_1\cap\Delta_2,\check{\mathcal{P}}|\Delta_1\cap\Delta_2)\cong\boldsymbol{C}$, $H_0(\Delta_1,\check{\mathcal{P}}|\Delta_1)=0$, $H_0(\Delta_2,\check{\mathcal{P}}|\Delta_2)\cong\boldsymbol{C}$. Then we have an isomorphism $H_0(\Delta_1\cap\Delta_2,\check{\mathcal{P}}|\Delta_1\cap\Delta_2)\cong H_0(\Delta_1,\check{\mathcal{P}}|\Delta_1)\oplus H_0(\Delta_2,\check{\mathcal{P}}|\Delta_2)$, and by Mayer-Vietoris sequence the short exact sequence $0\to H_1(\Delta_1\cap\Delta_2,\check{\mathcal{P}}|\Delta_1\cap\Delta_2)\to H_1(\Delta_1,\check{\mathcal{P}}|\Delta_1)\oplus H_1(\Delta_2,\check{\mathcal{P}}|\Delta_2)\to H_1(\Delta,\check{\mathcal{P}}|\Delta)\to 0$, from which it follows immediately that $H_1(\overset{\circ}{D}_N,\check{\mathcal{P}}|\overset{\circ}{D}_N)\cong H_1(\Delta,\check{\mathcal{P}}|\Delta)\cong\boldsymbol{C}^{2N}$. Lemma 2.9 is proved.

Let us complete the proof of Proposition 2.8. Assume that $N\geq 2$ and the proposition holds for $M=X_k$ where $1\leq k\leq N-1$. By Proposition 2.1 we have the following exact sequence:

$$\begin{aligned}&0\to H_4(X_N,\check{\mathcal{P}}|X_N)\to H_4(X_{N-1},\check{\mathcal{P}}|X_{N-1})\to H_2(\overset{\circ}{D}_N,\check{\mathcal{P}}|\overset{\circ}{D}_N)\\ &\to H_3(X_N,\check{\mathcal{P}}|X_N)\to H_3(X_{N-1},\check{\mathcal{P}}|X_{N-1})\to H_1(\overset{\circ}{D}_N,\check{\mathcal{P}}|\overset{\circ}{D}_N)\\ &\to H_2(X_N,\check{\mathcal{P}}|X_N)\to H_2(X_{N-1},\check{\mathcal{P}}|X_{N-1})\to H_0(\overset{\circ}{D}_N,\check{\mathcal{P}}|\overset{\circ}{D}_N)\\ &\to H_1(X_N,\check{\mathcal{P}}|X_N)\to H_1(X_{N-1},\check{\mathcal{P}}|X_{N-1})\to 0.\end{aligned}$$

By hypothesis we have $H_4(X_{N-1},\check{\mathcal{P}}|X_{N-1})=H_3(X_{N-1},\check{\mathcal{P}}|X_{N-1})=H_1(X_{N-1},\check{\mathcal{P}}|X_{N-1})=0$. By Lemma 2.9 we have $H_2(\overset{\circ}{D}_N,\check{\mathcal{P}}|\overset{\circ}{D}_N)=H_0(\overset{\circ}{D}_N,\check{\mathcal{P}}|\overset{\circ}{D}_N)=0$. Then it follows from the exact sequence that $H_4(X_N,\check{\mathcal{P}}|X_N)=H_3(X_N,\check{\mathcal{P}}|X_N)=H_1(X_N,\check{\mathcal{P}}|X_N)=0$, and the short exact sequence $0\to H_1(\overset{\circ}{D}_N,\check{\mathcal{P}}|\overset{\circ}{D}_N)\to H_2(X_N,\check{\mathcal{P}}|X_N)\to H_2(X_{N-1},\check{\mathcal{P}}|X_{N-1})\to 0$ holds. By Lemma 2.9 we have $H_1(\overset{\circ}{D}_N,\check{\mathcal{P}}|\overset{\circ}{D}_N)\cong\boldsymbol{C}^{2N}$. Moreover we may regard the group $H_2(X_{N-1},\check{\mathcal{P}}|X_{N-1})\cong\boldsymbol{C}^{(N-1)N}$ as a subgroup of $H_2(X_N,\check{\mathcal{P}}|X_N)$. Therefore it follows by the exact sequence that $H_2(X_N,\check{\mathcal{P}}|X_N)\cong H_1(\overset{\circ}{D}_N,\check{\mathcal{P}}|\overset{\circ}{D}_N)\oplus H_2(X_{N-1},\check{\mathcal{P}}|X_{N-1})\cong\boldsymbol{C}^{N(N+1)}$. Obviously we have $H_0(X_N,\check{\mathcal{P}}|X_N)=0$, and Proposition 2.8 is proved completely.

Remark 2.6. It follows from the short exact sequence $0\to H_1(\overset{\circ}{D}_N,\check{\mathcal{P}}|\overset{\circ}{D}_N)\to H_2(X_N,\check{\mathcal{P}}|X_N)\to H_2(X_{N-1},\check{\mathcal{P}}|X_{N-1})\to 0$ that the union of a basis of $H_2(X_{N-1},\check{\mathcal{P}}|X_{N-1})$ with the image of a basis of $H_1(\overset{\circ}{D}_N,\check{\mathcal{P}}|\overset{\circ}{D}_N)$ by the injection $H_1(\overset{\circ}{D}_N,\check{\mathcal{P}}|\overset{\circ}{D}_N)\to H_2(X_N,\check{\mathcal{P}}|X_N)$ forms a basis of $H_2(X_N,\check{\mathcal{P}}|X_N)$. For simplicity we assume that δ is not an integer. The other cases would be also treated similarly. Let $\Lambda_{14}, \Lambda_{24}, \Lambda_{34}$ be the homology classes of $H_1(\overset{\circ}{D}_N, \check{\mathcal{P}}|\ \overset{\circ}{D}_N)$ defined respectively by the three 1-cycles $(e^{-2\pi i\delta}-1)\lambda_1-(e^{2\pi i\alpha}-1)\lambda_4,\ (e^{-2\pi i\delta}-1)\lambda_2-(e^{2\pi i\beta}-1)\lambda_4,\ (e^{-2\pi i\delta}-1)\lambda_3-(e^{-2\pi i\gamma}-1)\lambda_4$. By abuse of notation we denote the homology classes of $H_1(\overset{\circ}{D}_N, \check{\mathcal{P}}|\ \overset{\circ}{D}_N)$ defined by $\gamma_{N1}, \gamma'_{N1}, \ldots, \gamma_{N,N-1}, \gamma'_{N,N-1}$ by the same symbols. Then the $2N+1$ homology classes $\Lambda_{14}, \Lambda_{24}, \Lambda_{34}, \gamma_{N1}, \gamma'_{N1}, \ldots, \gamma_{N,N-1}, \gamma'_{N,N-1}$ generate $H_1(\overset{\circ}{D}_N, \check{\mathcal{P}}|\ \overset{\circ}{D}_N)$, and satisfy a single equality

$$\frac{e^{-2\pi i\gamma}-1}{e^{-2\pi i\delta}-1}\Lambda_{14}-\frac{e^{2\pi i\alpha}-1}{e^{-2\pi i\delta}-1}\Lambda_{34}+\Lambda_{24}+\sum_{i=1}^{N-1}(\gamma_{Ni}+\gamma'_{Ni})=0.$$

Therefore $2N$ homology classes among those $2N+1$ homology classes form a basis of $H_1(\overset{\circ}{D}_N, \check{\mathcal{P}}|\ \overset{\circ}{D}_N)$. The images of $\Lambda_{14}, \Lambda_{24}, \Lambda_{34}$ by the injection $H_1(\overset{\circ}{D}_N, \check{\mathcal{P}}|\ \overset{\circ}{D}_N) \to H_2(X_N, \check{\mathcal{P}}|X_N)$ are constructed in the same way as in Remark 2.5. The images of γ_{Ni} and γ'_{Ni} coincide respectively with T_{Ni} and T'_{Ni} given in Remark 2.3.

The proof above of Proposition 2.8 depends on ordering of the theta divisors $D_1, \ldots, D_N$. If we adopt new ordering to prove the proposition, then we obtain another new basis of $H_2(X_N, \check{\mathcal{P}}|X_N)$. Let us study the relations between bases of $H_2(X_N, \check{\mathcal{P}}|X_N)$. For simplicity we assume that δ is not an integer. For i such that $1 \le i \le N$, the group $H_1(D'_i, \check{\mathcal{P}}|D'_i)$ is generated by $2N+1$ homology classes Λ_{14}, Λ_{24}, Λ_{34}, γ_{i1}, $\gamma'_{i1}, \ldots, \gamma_{i,i-1}$, $\gamma'_{i,i-1}$, $\gamma_{i,i+1}$, $\gamma'_{i,i+1}, \ldots, \gamma_{i,N}$, $\gamma'_{i,N}$, which satisfy a single equation

$$\frac{e^{-2\pi i\gamma}-1}{e^{-2\pi i\delta}-1}\Lambda_{14}-\frac{e^{2\pi i\alpha}-1}{e^{-2\pi i\delta}-1}\Lambda_{34}+\Lambda_{24}+\sum_{1\le j\le N, j\ne i}(\gamma_{ij}+\gamma'_{ij})=0.$$

We denote by $\Lambda^{(i)}_{14}, \Lambda^{(i)}_{24}, \Lambda^{(i)}_{34}$ the images of $\Lambda_{14}, \Lambda_{24}, \Lambda_{34}$, respectively, by the injection $H_1(D'_i, \check{\mathcal{P}}|D'_i) \to H_2(X_N, \check{\mathcal{P}}|X_N)$. We denote by T_{ij} and T'_{ij} $(i \ne j)$ the images of γ_{ij} and γ'_{ij}, respectively, by the injection $H_1(D'_i, \check{\mathcal{P}}|D'_i) \to H_2(X_N, \check{\mathcal{P}}|X_N)$. Then the group $H_2(X_N, \check{\mathcal{P}}|X_N)$ is generated by the $2N^2+N$ homology classes $\Lambda^{(i)}_{14}, \Lambda^{(i)}_{24}, \Lambda^{(i)}_{34}$ $(i=1,\ldots,N)$, T_{ij}, T'_{ij} $(i \ne j)$. Obviously we have the following

Proposition 2.10. Assume that δ is not an integer. Then the preceding $2N^2+N$ generators satisfy the following N^2 equalities:

$$T_{ij}=-T_{ji},\quad T'_{ij}=-T'_{ji}\quad (i\ne j);$$
$$\frac{e^{-2\pi i\gamma}-1}{e^{-2\pi i\delta}-1}\Lambda^{(i)}_{14}-\frac{e^{2\pi i\alpha}-1}{e^{-2\pi i\delta}-1}\Lambda^{(i)}_{34}+\Lambda^{(i)}_{24}+\sum_{1\le j\le N, j\ne i}(T_{ij}+T'_{ij})=0\quad (i=1,\ldots,N).$$

§3 Twisted cohomology of the complement of theta divisors

Let N, N' be integers such that $2 \le N \le N'$. Let $a_1, \ldots, a_{2N'}, b_1, \ldots, b_{2N'}$ be real numbers. We assume that the N' theta divisors defined by the equations $\theta\begin{bmatrix} a_{2k-1} & a_{2k} \\ b_{2k-1} & b_{2k}\end{bmatrix}(z_1, z_2; \tau) = 0,$

$k = 1, \ldots, N'$, are different from each other. For each k such that $1 \le k \le N$, let D_k be the theta divisor corresponding to the theta function $\theta \begin{bmatrix} a_{2k-1}\, a_{2k} \\ b_{2k-1}\, b_{2k} \end{bmatrix} (z_1, z_2; \tau)$. We assume that the divisor $D = \sum_{k=1}^{N} D_k$ has normal crossings. Let $c_1, \ldots, c_N$ be complex numbers but not integers such that

$$\sum_{k=1}^{N} c_k = 0. \tag{3.1}$$

Let $c_{N+1}, \ldots, c_{N'}$ be non-zero integers such that

$$\sum_{k=N+1}^{N'} c_k = 0. \tag{3.2}$$

We set $T_1(z_1, z_2) = \prod_{k=1}^{N} \theta \begin{bmatrix} a_{2k-1}\, a_{2k} \\ b_{2k-1}\, b_{2k} \end{bmatrix} (z_1, z_2; \tau)^{c_k}$, $T_2(z_1, z_2) = \prod_{k=N+1}^{N'} \theta \begin{bmatrix} a_{2k-1}\, a_{2k} \\ b_{2k-1}\, b_{2k} \end{bmatrix} (z_1, z_2; \tau)^{c_k}$, and $T(z_1, z_2) = T_1(z_1, z_2)T_2(z_1, z_2)$. Then we have

$$\begin{aligned} T(z_1 + 1, z_2) &= e^{\pi i \sum_{k=1}^{N'} a_{2k-1} c_k} T(z_1, z_2), \\ T(z_1, z_2 + 1) &= e^{\pi i \sum_{k=1}^{N'} a_{2k} c_k} T(z_1, z_2), \\ T(z_1 + \tau_{11}, z_2 + \tau_{21}) &= e^{-\pi i \sum_{k=1}^{N'} b_{2k-1} c_k} T(z_1, z_2), \\ T(z_1 + \tau_{12}, z_2 + \tau_{22}) &= e^{-\pi i \sum_{k=1}^{N'} b_{2k} c_k} T(z_1, z_2). \end{aligned}$$

Analogous formulas hold also for $T_1(z_1, z_2)$ and $T_2(z_1, z_2)$. Let $\mathcal{L}$ be the locally constant sheaf of rank one on $M = X - D$ defined by the one-dimensional representation of the fundamental group $\pi_1(M, *)$ by the multivalued function $T(z_1, z_2)^{-1}$. Namely, for a sufficiently small open subset $U \subset M$ we have $\Gamma(U, \mathcal{L}) = \boldsymbol{C}(T|U)^{-1}$, where $T|U$ denotes a branch of $T(z_1, z_2)$ over U. Let $\mathcal{U}_M = \{U_\mu\}$ be an open covering of M. For every μ let h_μ be a fixed branch of $T(z_1, z_2)^{-1}$ over U_μ. Obviously, there is a constant $g_{\mu\nu}$ such that $h_\nu = g_{\nu\mu} h_\mu$ on $U_\mu \cap U_\nu$, and we see that the family $\{g_{\nu\mu}\}$ satisfies the cocycle condition. For $(p, \xi_\mu) \in U_\mu \times \boldsymbol{C}$ and $(q, \xi_\nu) \in U_\nu \times \boldsymbol{C}$ we define an equivalence relation $(p, \xi_\mu) \sim (q, \xi_\nu)$ by the equations $p = q$ and $\xi_\nu g_{\nu\mu} = \xi_\mu$. Thus we have a line bundle L on M associated to the multivalued function $T(z_1, z_2)^{-1}$. By similar construction we have a line bundle L_1 on M associated to $T_1(z_1, z_2)^{-1}$, and a line bundle P on X associated to $T_2(z_1, z_2)^{-1}$. Note that P belongs to $\mathrm{Pic}^0(X)$, i.e., P is a line bundle with $c_1(P) = 0$. If $N = N'$, we regard P as the trivial line bundle $\boldsymbol{C}$. If $N < N'$, a global meromorphic section $f(z_1, z_2)$ over X of the line bundle P satisfies

$$\begin{aligned} f(z_1 + 1, z_2) &= e^{\pi i \sum_{k=N+1}^{N'} a_{2k-1} c_k} f(z_1, z_2), \\ f(z_1, z_2 + 1) &= e^{\pi i \sum_{k=N+1}^{N'} a_{2k} c_k} f(z_1, z_2), \\ f(z_1 + \tau_{11}, z_2 + \tau_{21}) &= e^{-\pi i \sum_{k=N+1}^{N'} b_{2k-1} c_k} f(z_1, z_2), \\ f(z_1 + \tau_{12}, z_2 + \tau_{22}) &= e^{-\pi i \sum_{k=N+1}^{N'} b_{2k} c_k} f(z_1, z_2). \end{aligned}$$

We denote by $P|M$ the restriction of the line bundle P to M. Then we have $L = L_1 \otimes P|M$. Let $\mathcal{O}_M(L)$ be the sheaf of modules over the structure sheaf $\mathcal{O}_M$ generated by local sections of L. By definition, a local section $\varphi \in \Gamma(U, \mathcal{O}_M(L))$ is identified with a family $(\varphi_\nu)_\nu$ where $\varphi_\nu \in \Gamma(U \cap U_\nu, \mathcal{O}_M)$ with $\varphi_\nu g_{\nu\mu} = \varphi_\mu$. We define a sheaf homomorphism of $\mathcal{O}_M(L)$ to $\mathcal{O}_M \otimes_{\boldsymbol{C}} \mathcal{L}$ by the correspondence $\Gamma(U_\nu, \mathcal{O}_M) \ni \varphi_\nu \to \varphi_\nu \xi_\nu^{-1} \otimes \xi_\nu h_\nu \in \Gamma(U_\nu, \mathcal{O}_M \otimes \mathcal{L})$ where $\varphi = (\varphi_\nu)$ is a section of L. Then the homomorphism thus defined gives an isomorphism $\mathcal{O}_M(L) \cong \mathcal{O}_M \otimes_{\boldsymbol{C}} \mathcal{L}$. Let us consider the exact sequence of sheaves on M:

$$0 \longrightarrow \boldsymbol{C} \longrightarrow \Omega_M^0 \xrightarrow{d} \Omega_M^1 \xrightarrow{d} \Omega_M^2 \xrightarrow{d} 0,$$

where Ω_M^p denotes the sheaf of holomorphic p-forms on M and $\Omega_M^0 = \mathcal{O}_M$. Tensoring $\mathcal{L}$ from the right on this sequence keeps the exactness, and so we have

$$0 \longrightarrow \mathcal{L} \longrightarrow \Omega_M^0 \otimes_{\boldsymbol{C}} \mathcal{L} \xrightarrow{d\otimes 1} \Omega_M^1 \otimes_{\boldsymbol{C}} \mathcal{L} \xrightarrow{d\otimes 1} \Omega_M^2 \otimes_{\boldsymbol{C}} \mathcal{L} \xrightarrow{d\otimes 1} 0. \tag{3.3}$$

Here the operator $d \otimes 1$ is a canonical connection in the sense of [3], I, Prop. 2.16. We set $\Omega_M^p(L) = \Omega_M^p \otimes_{\mathcal{O}_M} \mathcal{O}_M(L)$. Then the following diagrams are commutative:

$$\begin{array}{ccc} \Omega_M^p \otimes_{\boldsymbol{C}} \mathcal{L} & \xrightarrow{d\otimes 1} & \Omega_M^{p+1} \otimes_{\boldsymbol{C}} \mathcal{L} \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ \Omega_M^p(L) & \xrightarrow[d]{} & \Omega_M^{p+1}(L) \end{array} \qquad \begin{array}{ccc} \varphi\xi^{-1} \otimes \xi h & \overset{d\otimes 1}{\longmapsto} & d(\varphi\xi^{-1}) \otimes \xi h \\ \updownarrow & & \updownarrow \\ \varphi & \underset{d}{\longmapsto} & d\varphi = d(\varphi\xi^{-1})\xi \end{array}$$

Therefore (3.3) is equivalent to the following exact sequence:

$$0 \longrightarrow \mathcal{L} \longrightarrow \Omega_M^0(L) \xrightarrow{d} \Omega_M^1(L) \xrightarrow{d} \Omega_M^2(L) \xrightarrow{d} 0. \tag{3.4}$$

For an open set $U \subset M$ and a section $\varphi \in \Gamma(U, \Omega_M^p(L))$, there is a section $\psi \in \Gamma(U, \Omega_M^p(P|M))$ such that $\varphi = (T_1|U) \cdot \psi$, where $T_1|U$ is a branch of T_1 defined on U. Then we have a sheaf isomorphism

$$\Omega_M^p(P|M) \xrightarrow{\sim} \Omega_M^p(L) \qquad\qquad \psi \longmapsto \varphi = T_1 \cdot \psi$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \Omega_M^p(P|M) & \xrightarrow{\nabla} & \Omega_M^{p+1}(P|M) \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ \Omega_M^p(L) & \xrightarrow[d]{} & \Omega_M^{p+1}(L) \end{array}$$

where $\nabla\psi = d\psi + d(\log T_1) \wedge \psi$ and $\nabla^2 = 0$. Note that $\nabla h_1 = 0$ for any branch h_1 of T_1^{-1}. Then the exact sequence (3.4) is equivalent to the following one:

$$0 \longrightarrow \mathcal{L} \longrightarrow \Omega_M^0(P|M) \xrightarrow{\nabla} \Omega_M^1(P|M) \xrightarrow{\nabla} \Omega_M^2(P|M) \xrightarrow{\nabla} 0.$$

Since $\Omega_M^p(P|M)$ is a locally free and therefore coherent sheaf on an affine algebraic manifold M, we have by Serre's theorem $H^q(M, \Omega_M^p(P|M)) = 0$ for $p \geq 0$ and $q > 0$. Therefore it follows by the standard argument of the de Rham theory ([18]) that $H^p(M, \mathcal{L}) \cong H^p_{\mathrm{DR}}(\Omega_M^\bullet(P|M), \nabla)$. Obviously $H^p(M, \mathcal{L}) = 0$ if $p > 2$. We denote the sheaf of complex-valued C^∞ differential forms of type (p, q) on M by $\mathcal{E}_M^{pq}$. We set $\mathcal{E}_M^k = \sum_{p+q=k} \mathcal{E}_M^{pq}$, the sheaf of complex-valued C^∞ differential forms of total degree k on M. Then replacement of the sheaf Ω_M^p by $\mathcal{E}_M^p$ in the argument above is valid, and gives us the following exact sequence of sheaves:

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E}_M^0(P|M) \xrightarrow{\nabla} \mathcal{E}_M^1(P|M) \xrightarrow{\nabla} \mathcal{E}_M^2(P|M) \xrightarrow{\nabla} 0,$$

where $\mathcal{E}^p_M(P|M) = \mathcal{E}^p_M \otimes_{\mathcal{E}^0_M} \mathcal{E}^0_M(P|M)$, and $\mathcal{E}^0_M(P|M)$ denotes the sheaf of modules over the sheaf of complex-valued C^∞ functions $\mathcal{E}^0_M$ generated by local sections of the line bundle $P|M$ over M. Since $\mathcal{E}^p_M(P|M)$ is a soft sheaf over M, we have $H^q(M, \mathcal{E}^p_M(P|M)) = 0$ for $p \geq 0$ and $q > 0$ ([18], Chap. II). Therefore we have $H^p(M, \mathcal{L}) \cong H^p_{\mathrm{DR}}(\mathcal{E}^\bullet_M(P|M), \nabla)$. Let $\Omega^p_X\langle D\rangle$ be the sheaf of p-forms over X with logarithmic pole along D. We have inclusion of sheaves over X: $\Omega^p_X\langle D\rangle \subset j_*\Omega^p_M \subset j_*\mathcal{E}^p_M$, where j denotes a natural inclusion mapping of M into X. We set $\Omega^p_X\langle D\rangle(P) = \Omega^p_X\langle D\rangle \otimes_{\mathcal{O}_X} \mathcal{O}_X(P)$. Since $j_*\Omega^p_M \otimes_{\mathcal{O}_X} \mathcal{O}_X(P) \cong j_*\Omega^p_M(P|M)$ and $j_*\mathcal{E}^p_M \otimes_{\mathcal{E}^0_X} \mathcal{E}^0_X(P) \cong j_*\mathcal{E}^p_M(P|M)$, we have $\Omega^p_X\langle D\rangle(P) \subset j_*\Omega^p_M(P|M) \subset j_*\mathcal{E}^p_M(P|M)$. Let us consider a complex of sheaves of logarithmic forms:

$$(\Omega^\bullet_X\langle D\rangle(P), \nabla) \quad : \quad \Omega^0_X\langle D\rangle(P) \xrightarrow{\nabla} \Omega^1_X\langle D\rangle(P) \xrightarrow{\nabla} \Omega^2_X\langle D\rangle(P) \longrightarrow 0.$$

The following proposition is an immediate consequence of a result proved by Deligne ([3], II, Cor. 3.14).

Proposition 3.1. Two complexes of sheaves over X, $(\Omega^\bullet_X\langle D\rangle(P), \nabla)$ and $(j_*\mathcal{E}^\bullet_M(P|M), \nabla)$, are quasi-isomorphic to each other.

In Appendix to §3, we will give an elementary, direct proof of this proposition.

Corollary 3.2. $H^p(M, \mathcal{L}) \cong \mathbf{H}^p(X, \Omega^\bullet_X\langle D\rangle(P), \nabla)$ for all $p \geq 0$.

Proof. Since the complexes of sheaves $(\Omega^\bullet_X\langle D\rangle(P), \nabla)$ and $(j_*\mathcal{E}^\bullet_M(P|M), \nabla)$ are quasi-isomorphic to each other, their p-th derived objects for a left-exact functor F for each p are isomorphic to each other ([16]). Especially, if F is the functor of global sections (i.e., $F(\bullet) = \Gamma(X, \bullet)$), then derived objects are turned into hypercohomologies. Namely we have

$$\mathbf{H}^p(X, j_*\mathcal{E}^\bullet_M(P|M), \nabla) \cong \mathbf{H}^p(X, \Omega^\bullet_X\langle D\rangle(P), \nabla).$$

Since $H^q(X, j_*\mathcal{E}^p_M(P|M)) = H^q(M, \mathcal{E}^p_M(P|M)) = 0$ for $p \geq 0$ and $q > 0$, we have

$$\mathbf{H}^p(X, j_*\mathcal{E}^\bullet_M(P|M), \nabla) \cong H^p_{\mathrm{DR}}(X, j_*\mathcal{E}^\bullet_M(P|M), \nabla) = H^p_{\mathrm{DR}}(M, \mathcal{E}^\bullet_M(P|M), \nabla) \cong H^p(M, \mathcal{L}),$$

which proves Corollary 3.2.

Remark 3.1. This corollary is slightly different from Deligne's Corollary 6.10 in [3], II, §6 because our locally constant sheaf is not a restriction to M of the locally constant sheaf associated to the line bundle P.

Proposition 3.3. $H^p(M, \mathcal{L}) = 0$ if $p \neq 2$.

This is an immediate consequence of a vanishing theorem of hypercohomology ([5], §2, Cor. 2.13). In Appendix to §3, we will give another elementary proof of this proposition by using the logarithmic Dolbeault complex.

To study the structure of the non-vanishing cohomology group $H^2(M, \mathcal{L})$, let us consider the following logarithmic Čech-de Rham complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
C^2(\mathcal{U},\Omega^0_X\langle D\rangle(P)) & \xrightarrow{\nabla} & C^2(\mathcal{U},\Omega^1_X\langle D\rangle(P)) & \xrightarrow{\nabla} & C^2(\mathcal{U},\Omega^2_X\langle D\rangle(P)) & \longrightarrow & 0 \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
C^1(\mathcal{U},\Omega^0_X\langle D\rangle(P)) & \xrightarrow{\nabla} & C^1(\mathcal{U},\Omega^1_X\langle D\rangle(P)) & \xrightarrow{\nabla} & C^1(\mathcal{U},\Omega^2_X\langle D\rangle(P)) & \longrightarrow & 0 \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
C^0(\mathcal{U},\Omega^0_X\langle D\rangle(P)) & \xrightarrow{\nabla} & C^0(\mathcal{U},\Omega^1_X\langle D\rangle(P)) & \xrightarrow{\nabla} & C^0(\mathcal{U},\Omega^2_X\langle D\rangle(P)) & \longrightarrow & 0,
\end{array}$$

where $\mathcal{U} = \{U_i\}$ denotes an open covering of X, and δ the coboundary operator satisfying $\delta\nabla + \nabla\delta = 0$. The total differentiation operator Δ is defined to be $\Delta = \delta + \nabla$. We set $K^0 = C^0(\mathcal{U},\Omega^0_X\langle D\rangle(P))$, $K^1 = C^1(\mathcal{U},\Omega^0_X\langle D\rangle(P)) \oplus C^0(\mathcal{U},\Omega^1_X\langle D\rangle(P))$, and $K^n = C^n(\mathcal{U},\Omega^0_X\langle D\rangle(P)) \oplus C^{n-1}(\mathcal{U},\Omega^1_X\langle D\rangle(P)) \oplus C^{n-2}(\mathcal{U},\Omega^2_X\langle D\rangle(P))$ for $n \geq 2$. Then we see that $\Delta^2 = 0$ and $\Delta(K^n) \subset K^{n+1}$. Therefore the pair $(K(\mathcal{U}),\Delta)$, where $K(\mathcal{U}) = \oplus_{n=0}^{\infty} K^n$, is a complex. By the definition of hypercohomology we have

$$\mathbf{H}^p = \mathbf{H}^p(X,\Omega^{\bullet}_X\langle D\rangle(P),\nabla) = \varinjlim_{\mathcal{U}} H^p(K(\mathcal{U}),\Delta),$$

where $H^p(K(\mathcal{U}),\Delta) = \mathrm{Ker}[K^p \xrightarrow{\Delta} K^{p+1}]/\mathrm{Im}[K^{p-1} \xrightarrow{\Delta} K^p]$ and $\varinjlim_{\mathcal{U}}$ means the inductive limit taken with respect to refinements of open coverings of X ([6]). By setting $K_0 = K(\mathcal{U})$ and $K_1 = \oplus_{n=0}^{\infty} C^n(\mathcal{U},\Omega^1_X\langle D\rangle(P)) \oplus C^n(\mathcal{U},\Omega^2_X\langle D\rangle(P))$, $K_2 = \oplus_{n=0}^{\infty} C^n(\mathcal{U},\Omega^2_X\langle D\rangle(P))$, we introduce a filtration of $K(\mathcal{U})$: $K(\mathcal{U}) = K_0 \supset K_1 \supset K_2 \supset 0$. The spectral sequence $E_r(\mathcal{U})$ associated to the filtered modul $K(\mathcal{U})$ is given as follows: $E_1^p(\mathcal{U}) = H(K_p/K_{p+1}) = H_\delta(\oplus_{q=0}^{\infty} C^q(\mathcal{U},\Omega^p_X\langle D\rangle(P)) = \oplus_{q=0}^{\infty} H^q(\mathcal{U},\Omega^p_X\langle D\rangle(P))$ and the other $E_r^p(\mathcal{U})$ $(r \geq 2)$ are given inductively. The limit $E_r = \varinjlim_{\mathcal{U}} E_r(\mathcal{U})$ is also a spectral sequence and abuts to the hypercohomology $\mathbf{H}^p$. We have $E_1^p = \oplus_{q=0}^{\infty} H^q(X,\Omega^p_X\langle D\rangle(P))$ and $E_1^{pq} = H^q(X,\Omega^p_X\langle D\rangle(P))$.

Lemma 3.4. (i) Suppose that either $N = N'$, or $N < N'$ and all of the four quantities $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k}c_k$ are integers. Then $E_1^{pq} = 0$ if $p+q > 2$.
(ii) Suppose that $N < N'$ and not all of the four quantities $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k}c_k$ are integers. Then $E_1^{pq} = 0$ if $p+q \neq 2$.

Proof. It is well-known that the complexes of sheaves $(\Omega^{\bullet}_X\langle D\rangle(P), d)$ and $(j_*\mathcal{E}^{\bullet}_M(P|M), d)$ are quasi-isomorphic to each other. By the same argument as in the proof of Corollary 3.2, we have $H^k(M,\mathcal{P}|M) \cong \mathbf{H}^k(X,\Omega^{\bullet}_X\langle D\rangle(P), d)$, where $\mathcal{P}$ denotes the locally constant sheaf over X defined by the line bundle P, and $\mathcal{P}|M$ the restriction of $\mathcal{P}$ to M. By Deligne's theorem [4], we have the decomposition of hypercohomology $\mathbf{H}^k(X,\Omega^{\bullet}_X\langle D\rangle(P), d) \cong \oplus_{p+q=k} H^q(X,\Omega^p_X\langle D\rangle(P))$. Since $H^k(M,\mathcal{P}|M)$ is the dual space of the vector space $H_k(M,\check{\mathcal{P}}|M)$ introduced in §2, Lemma 3.4 follows immediately from Propositions 2.4 and 2.8.

The first result about the degeneration of the spectral sequence is as follows:

Proposition 3.5. Assume that $N < N'$ and not all of the four quantities $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k}c_k$ are integers. Then the spectral sequence E_r, $r \geq 1$ degenerates at E_1 (i.e., $E_1 = E_\infty$). Namely we have $E_\infty^{20} = H^0(X, \Omega_X^2(D)(P))$, $E_\infty^{11} = H^1(X, \Omega_X^1\langle D\rangle(P))$, and $E_\infty^{pq} = 0$ if $(p,q) \neq (2,0),(1,1)$. Moreover we have $\dim E_\infty^{20} = N^2$ and $\dim E_\infty^{11} = N$.

Proof. From Lemma 3.4, (ii), it follows immediately that $E_\infty^{20} = H^0(X, \Omega_X^2(D)(P))$, $E_\infty^{11} = H^1(X, \Omega_X^1\langle D\rangle(P))$, $E_\infty^{02} = H^2(X, \mathcal{O}_X(P))$, and $E_\infty^{pq} = 0$ if $p+q \neq 2$. By Lemma 2.1 we have $H^0(X, \mathcal{O}_X(P)) = 0$, and it follows by [2], Lemma 3.5.1 that $E_\infty^{02} = 0$. Therefore we have $H^2(M, \mathcal{L}) \cong E_\infty^{20} \oplus E_\infty^{11}$. Proposition 3.3 and Lemma 2.2 imply that $\dim H^2(M, \mathcal{L}) = \chi(M) = N(N+1)$. Since $\dim E_\infty^{20} = N^2$ by the general theory of cohomology of line bundles on X, we have $\dim E_\infty^{11} = N$.

Next, let us consider the spectral sequence for the case where either $N = N'$, or $N < N'$ and all of $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k}c_k$ are integers. In this case we may regard the line bundle P as the holomorphically trivial one, and so we may set $P = \boldsymbol{C}$. For each q let us consider the complex $E_1^{0q} \xrightarrow{\nabla} E_1^{1q} \xrightarrow{\nabla} E_1^{2q} \longrightarrow 0$. By definition we have $E_2^{pq} = H^p(E_1^{*q}, \nabla)$. In the rest of this section we prove the following

Proposition 3.6. Assume that either $N = N'$, or $N < N'$ and all of $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k}c_k$ are integers. Then the spectral sequence E_r, $r \geq 1$ degenerates at E_2 (i.e., $E_2 = E_\infty$). Namely we have $E_\infty^{20} = H^0(X, \Omega_X^2(D))/\nabla H^0(X, \Omega_X^1\langle D\rangle)$, $E_\infty^{11} = H^1(X, \Omega_X^1\langle D\rangle)/\nabla H^1(X, \mathcal{O}_X)$, $E_\infty^{02} = H^2(X, \mathcal{O}_X)$, and $E_\infty^{pq} = 0$ if $p+q \neq 2$. Moreover we have $\dim E_\infty^{20} = N^2 - N$, $\dim E_\infty^{11} = 2N - 1$, $\dim E_\infty^{02} = 1$.

Since $E_1^{pq} = 0$ for $p+q > 2$ by Lemma 3.4, (i), it follows immediately that $E_2^{pq} = 0$ for $p+q > 2$. Therefore, to prove Proposition 3.6, it suffices to prove the following

Lemma 3.7. $E_2^{01} = 0$.

If this lemma is established, then the degeneration of E_r, $r \geq 1$ at E_2 follows immediately, the vanishing of E_∞^{00}, E_∞^{10}, E_∞^{01} follows from Proposition 3.3, and by the definition of E_2^{pq} we can easily determine the values of E_∞^{20}, E_∞^{11}, E_∞^{02}. Since $\dim H^0(X, \Omega_X^1\langle D\rangle) = N + 1$ and $d\log T_1 = \nabla(1)$, we have $\dim \nabla H^0(X, \Omega_X^1\langle D\rangle) = N$. Moreover we have $\dim H^0(X, \Omega_X^2(D)) = N^2$. So we have $\dim E_\infty^{20} = N^2 - N$. Since $H^2(M, \mathcal{L}) \cong E_\infty^{20} \oplus E_\infty^{11} \oplus E_\infty^{02}$ and $\dim H^2(M, \mathcal{L}) = N^2 + N$, we have $\dim E_\infty^{11} = 2N - 1$ ($\dim E_\infty^{02} = \dim H^2(X, \mathcal{O}_X) = 1$ is obvious).

To prove Lemma 3.7 let us introduce the logarithmic Dolbeault complex. We denote by $\mathcal{E}_X^{pq}$ the sheaf of C^∞ differential forms of type (p,q) on X. Let us consider the following resolution of the sheaf $\mathcal{O}_X$:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}_X^{00} \xrightarrow{\bar\partial} \mathcal{E}_X^{01} \xrightarrow{\bar\partial} \mathcal{E}_X^{02} \xrightarrow{\bar\partial} 0.$$

Since $\Omega_X^p\langle D\rangle$ is a locally free $\mathcal{O}_X$-module ([14]), tensoring $\Omega_X^p\langle D\rangle$ from the right on this sequence keeps the exactness. Setting $\mathcal{E}_X^{0q} \otimes_{\mathcal{O}_X} \Omega_X^p\langle D\rangle = \mathcal{E}_X^{pq}\langle D\rangle$, we have the following resolution of $\Omega_X^p\langle D\rangle$:

$$0 \longrightarrow \Omega_X^p\langle D\rangle \longrightarrow \mathcal{E}_X^{p0}\langle D\rangle \xrightarrow{\bar\partial} \mathcal{E}_X^{p1}\langle D\rangle \xrightarrow{\bar\partial} \mathcal{E}_X^{p2}\langle D\rangle \xrightarrow{\bar\partial} 0,$$

which we call the logarithmic Dolbeault complex. Since $\mathcal{E}_X^{pq}\langle D\rangle$ is a locally free $\mathcal{E}_X^{pq}$-module, we have $H^r(X,\mathcal{E}_X^{pq}\langle D\rangle)=0$ for $r>0$, $p\geq 0$, $q\geq 0$. By the standard argument we have $H^q(X,\Omega_X^p\langle D\rangle)\cong H^q_{\bar\partial}(X,\mathcal{E}_X^{p*}\langle D\rangle)$. If we set $\nabla_0=\partial+d\log T_1\wedge$, then we have $\nabla=\nabla_0+\bar\partial$ and $\bar\partial\nabla_0+\nabla_0\bar\partial=0$. Recall the complex $E_1^{01}\xrightarrow{\nabla}E_1^{11}\xrightarrow{\nabla}0$, which is equivalent to $H^1(X,\mathcal{O}_X)\xrightarrow{\nabla}H^1(X,\Omega_X^1\langle D\rangle)\xrightarrow{\nabla}0$. Then we have $E_2^{01}=\operatorname{Ker}\left[H^1(X,\mathcal{O}_X)\xrightarrow{\nabla}H^1(X,\Omega_X^1\langle D\rangle)\right]$. Here the operator ∇ is given as follows. Note that

$$H^1(X,\mathcal{O}_X)\cong\operatorname{Ker}\left[\Gamma(X,\mathcal{E}_X^{01})\xrightarrow{\bar\partial}\Gamma(X,\mathcal{E}_X^{02})\right]/\operatorname{Im}\left[\Gamma(X,\mathcal{E}_X^{00})\xrightarrow{\bar\partial}\Gamma(X,\mathcal{E}_X^{01})\right]$$

and

$$H^1(X,\Omega_X^1\langle D\rangle)\cong\operatorname{Ker}\left[\Gamma(X,\mathcal{E}_X^{11}\langle D\rangle)\xrightarrow{\bar\partial}\Gamma(X,\mathcal{E}_X^{12}\langle D\rangle)\right]/\operatorname{Im}\left[\Gamma(X,\mathcal{E}_X^{10}\langle D\rangle)\xrightarrow{\bar\partial}\Gamma(X,\mathcal{E}_X^{11}\langle D\rangle)\right].$$

Let α be an element in $\Gamma(X,\mathcal{E}_X^{01})$ such that $\bar\partial\alpha=0$. Then we have $\nabla\alpha=\nabla_0\alpha\in\Gamma(X,\mathcal{E}_X^{11}\langle D\rangle)$. For $[\alpha]\in H^1(X,\mathcal{O}_X)$ we set $\nabla[\alpha]=[\nabla\alpha]$. Then the operator $\nabla:H^1(X,\mathcal{O}_X)\to H^1(X,\Omega_X^1\langle D\rangle)$ thus defined is well-defined. We are now in a position to prove Lemma 3.7.

Proof of Lemma 3.7. It suffices to prove that the operator $\nabla:H^1(X,\mathcal{O}_X)\to H^1(X,\Omega_X^1\langle D\rangle)$ is injective. Let α be an element in $\Gamma(X,\mathcal{E}_X^{01})$ such that $\bar\partial\alpha=\nabla_0\alpha+\bar\partial\beta=0$ for some $\beta\in\Gamma(X,\mathcal{E}_X^{10}\langle D\rangle)$. Since $\Gamma(X,\mathcal{E}_X^{10}\langle D\rangle)$ is generated by dz_1, dz_2, $d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}$, $k=1,\dots,N-1$, we can set

$$\beta=\beta_1+\sum_{k=1}^{N-1}\beta_{0k}\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right),$$

where $\beta_1\in\Gamma(X,\mathcal{E}_X^{10})$ and $\beta_{0k}\in\Gamma(X,\mathcal{E}_X^{00})$. We have

$$\bar\partial\beta=\bar\partial\beta_1+\sum_{k=1}^{N-1}\bar\partial\beta_{0k}\wedge\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right).$$

Then the equation $\nabla_0\alpha+\bar\partial\beta=0$ is decomposed into the following two equations:

$$\partial\alpha+\bar\partial\beta_1=0,$$
$$d\log T_1\wedge\alpha+\sum_{k=1}^{N-1}\bar\partial\beta_{0k}\wedge\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right)=0.$$

Since

$$\alpha\wedge d\log T_1=\sum_{k=1}^{N-1}(c_1+\cdots+c_k)\alpha\wedge\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right),$$

the second equation is turned to

$$\sum_{k=1}^{N-1}\left(\bar\partial\beta_{0k}-(c_1+\cdots+c_k)\alpha\right)\wedge\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right)=0,$$

from which it follows that $\bar\partial\beta_{0k}=(c_1+\cdots+c_k)\alpha$, $k=1,\dots,N-1$. Since $c_1\neq 0$, we have $\alpha=\bar\partial\beta_{01}/c_1$, which means that $\operatorname{Ker}\left[\nabla:H^1(X,\mathcal{O}_X)\to H^1(X,\Omega_X^1\langle D\rangle)\right]=0$. Lemma 3.7 is proved.

Appendix to §3: Proofs of Propositions 3.1 and 3.3

Let us first prove Proposition 3.1.

Proof of Proposition 3.1. By definition it suffices to prove that their cohomology sheaves are isomorphic: $\mathcal{H}^p(\Omega_X^\bullet\langle D\rangle(P),\nabla)\cong\mathcal{H}^p(j_*\mathcal{E}_M^p(P|M),\nabla)$ for each p. First of all let x be a point of M, and U be an open set of M containing the point x which is biholomorphic to a polydisk $\{(z_1,z_2)\in\boldsymbol{C}^2\mid |z_1|<\varepsilon,\ |z_2|<\varepsilon\}$ $(\varepsilon>0)$ with the origin $(z_1,z_2)=(0,0)$ corresponding to the point x. Since $\Omega_X^\bullet\langle D\rangle(P)|U\cong\Omega_U^\bullet(P|U)$ and $j_*\mathcal{E}_M^\bullet(P|M)|U\cong\mathcal{E}_U^\bullet(P|U)$, it suffices to prove that $H^p_{\mathrm{DR}}(\Omega_U^\bullet(P|U),\nabla)\cong H^p_{\mathrm{DR}}(\mathcal{E}_U^\bullet(P|U),\nabla)$. Note that $H^q(U,\mathcal{E}_U^p(P|U))=H^q(U,\Omega_U^p(P|U))=0$ for $p\geq 0$ and $q>0$. According to the standard argument in the de Rham theory, the following two resolutions of $\mathcal{L}$ over U

$$0\longrightarrow\mathcal{L}\longrightarrow\mathcal{E}_U^0(P|U)\xrightarrow{\nabla}\mathcal{E}_U^1(P|U)\xrightarrow{\nabla}\mathcal{E}_U^2(P|U)\xrightarrow{\nabla}0$$

and

$$0\longrightarrow\mathcal{L}\longrightarrow\Omega_U^0(P|U)\xrightarrow{\nabla}\Omega_U^1(P|U)\xrightarrow{\nabla}\Omega_U^2(P|U)\xrightarrow{\nabla}0$$

gives us the isomorphisms $H^p_{\mathrm{DR}}(\mathcal{E}_U^\bullet(P|U),\nabla)\cong H^p(U,\mathcal{L})\cong H^p_{\mathrm{DR}}(\Omega_U^\bullet(P|U),\nabla)$.

Next, let x be a point of D_i for some i such that $x\notin D_j$ for any $j(\neq i)$. Let (z_1,z_2) be a local coordinate system around the point x such that the origin $(0,0)$ corresponds to the point x and the local equation of D_i around x is written as $z_1=0$. Let U and $\bar U$ be open sets of X which are biholomorphic to $\{(z_1,z_2)\in\boldsymbol{C}^2\mid 0<|z_1|<\varepsilon,\ |z_2|<\varepsilon\}$ and $\{(z_1,z_2)\in\boldsymbol{C}^2\mid |z_1|<\varepsilon,\ |z_2|<\varepsilon\}$ respectively with the origin $(z_1,z_2)=(0,0)$ corresponding to the point x. Since $\Omega_X^\bullet\langle D\rangle(P)|\bar U\cong\Omega_{\bar U}^\bullet\langle D\rangle(P|\bar U)$ and $j_*\mathcal{E}_M^\bullet(P|M)|\bar U\cong\mathcal{E}_U^\bullet(P|U)$, it suffices to prove that $H^p_{\mathrm{DR}}(\Omega_{\bar U}^\bullet\langle D\rangle(P|\bar U),\nabla)\cong H^p_{\mathrm{DR}}(\mathcal{E}_U^\bullet(P|U),\nabla)$. Obviously we have $H^p_{\mathrm{DR}}(\mathcal{E}_U^\bullet(P|U),\nabla)\cong H^p(U,\mathcal{L})=0$. So it suffices to prove that $H^p_{\mathrm{DR}}(\Omega_{\bar U}^\bullet\langle D\rangle(P|\bar U),\nabla)=0$ for all p. It is obvious that $H^3_{\mathrm{DR}}(\Omega_{\bar U}^\bullet\langle D\rangle(P|\bar U),\nabla)=H^4_{\mathrm{DR}}(\Omega_{\bar U}^\bullet\langle D\rangle(P|\bar U),\nabla)=0$. Let f be an element of $\Gamma(\bar U,\Omega_{\bar U}^0\langle D\rangle(P|\bar U))=\Gamma(\bar U,\mathcal{O}_{\bar U}(P|\bar U))$ such that $\nabla f=0$. Then we have locally $f=cT_1^{-1}$ with some constant c. Since f is holomorphic along $z_1=0$, we have $f=0$, which proves $H^0_{\mathrm{DR}}(\Omega_{\bar U}^\bullet\langle D\rangle(P|\bar U),\nabla)=0$. Let ω be an element of $\Gamma(\bar U,\Omega_{\bar U}^1\langle D\rangle(P|\bar U))$ such that $\nabla\omega=0$. We set $\omega=adz_1+bdz_2$, where a denotes a meromorphic P-valued section with pole along $z_1=0$, and b a holomorphic P-valued section. If we set $\Omega=T_1\omega$, then we have $d\Omega=T_1\nabla\omega$. So $\nabla\omega=0$ if and only if $d\Omega=0$. Moreover $d\Omega=0$ is equivalent to an integrability condition $\dfrac{\partial(aT_1)}{\partial z_2}=\dfrac{\partial(bT_1)}{\partial z_1}$. Let us consider the equation $dF=\Omega$ for a $\mathcal{L}$-valued unknown F. This is equivalent to a system of partial differential equations $\dfrac{\partial F}{\partial z_1}=aT_1$, $\dfrac{\partial F}{\partial z_2}=bT_1$. Locally around the point $(z_1,z_2)=(0,0)$ we can write $T_1=z_1^cG(z_1,z_2)$ with a constant c and a holomorphic function G. So we have $aT_1=z_1^{c-1}G_1(z_1,z_2)$ and $bT_1=z_1^cG_2(z_1,z_2)$ with holomorphic functions G_1 and G_2. Then we see by an elementary theory of partial differential equations that there exists a unique solution F of the form $z_1^cH(z_1,z_2)$ with a holomorphic function H which satisfies the preceding system of partial differential equations. Therefore F is decomposed locally as $F=\gamma T_1$ with a holomorphic P-valued section γ defined on $\bar U$. So we have $\nabla\gamma=\omega$, which proves $H^1_{\mathrm{DR}}(\Omega_{\bar U}^\bullet\langle D\rangle(P|\bar U),\nabla)=0$. Let ω be an element of $\Gamma(\bar U,\Omega_{\bar U}^2\langle D\rangle(P|\bar U))$. Automatically we have $\nabla\omega=0$. We set $\omega=fdz_1\wedge dz_2$ where f denotes a meromorphic P-valued section with pole along $z_1=0$. We set $\Omega=T_1\omega$. Let us consider the equation $d\varphi=\Omega$ for a

$\mathcal{L}$-valued unknown 1-form φ. We set $\varphi = Adz_1 + Bdz_2$ where A and B are holomorphic $\mathcal{L}$-valued sections on U. Then the equation $d\varphi = \Omega$ is equivalent to a differential relation $\dfrac{\partial B}{\partial z_1} - \dfrac{\partial A}{\partial z_2} = T_1 f$. Locally around the point $(z_1, z_2) = (0, 0)$ we can write $T_1 f = z_1^{c-1} G(z_1, z_2)$ with a constant c and a holomorphic function G. Let $\tilde{G}$ be a holomorphic function such that $\dfrac{\partial \tilde{G}}{\partial z_2} = G$. Setting $A = -z_1^{c-1}\tilde{G}$ and $B = 0$, we see that the 1-form $\varphi = Adz_1$ satisfies the equation $d\varphi = \Omega$. Therefore A is decomposed locally as $A = \gamma T_1$ with a P-valued section γ with pole along $z_1 = 0$. So we have $\nabla(\gamma dz_1) = \omega$, which proves $H^2_{\mathrm{DR}}(\Omega^\bullet_{\bar{U}}\langle D\rangle(P|\bar{U}), \nabla) = 0$.

Finally let x be a point of $D_i \cap D_j$ for some i and j. Let (z_1, z_2) be a local coordinate system around the point x such that the origin $(0, 0)$ corresponds to the point x and the local equations of D_i and D_j around x are written as $z_1 = 0$ and $z_2 = 0$ respectively. Let U and $\bar{U}$ be open sets of X which are biholomorphic to $\{(z_1, z_2) \in \boldsymbol{C}^2 \mid 0 < |z_1| < \varepsilon,\ 0 < |z_2| < \varepsilon\}$ and $\{(z_1, z_2) \in \boldsymbol{C}^2 \mid |z_1| < \varepsilon,\ |z_2| < \varepsilon\}$ respectively with the origin $(z_1, z_2) = (0, 0)$ corresponding to the point x. Since $\Omega^\bullet_X\langle D\rangle(P)|\bar{U} \cong \Omega^\bullet_{\bar{U}}\langle D\rangle(P|\bar{U})$ and $j_*\mathcal{E}^\bullet_M(P|M)|\bar{U} \cong \mathcal{E}^\bullet_U(P|U)$, it suffices to prove that $H^p_{\mathrm{DR}}(\Omega^\bullet_{\bar{U}}\langle D\rangle(P|\bar{U}), \nabla) \cong H^p_{\mathrm{DR}}(\mathcal{E}^\bullet_U(P|U), \nabla)$. Obviously we have $H^p_{\mathrm{DR}}(\mathcal{E}^\bullet_U(P|U), \nabla) \cong H^p(U, \mathcal{L}) = 0$ for any p. So it suffices to prove that $H^p_{\mathrm{DR}}(\Omega^\bullet_{\bar{U}}\langle D\rangle(P|\bar{U}), \nabla) = 0$ for all p. It is obvious that $H^3_{\mathrm{DR}}(\Omega^\bullet_{\bar{U}}\langle D\rangle(P|\bar{U}), \nabla) = H^4_{\mathrm{DR}}(\Omega^\bullet_{\bar{U}}\langle D\rangle(P|\bar{U}), \nabla) = 0$. Let f be an element of $\Gamma(\bar{U}, \Omega^0_{\bar{U}}\langle D\rangle(P|\bar{U})) = \Gamma(\bar{U}, \mathcal{O}_{\bar{U}}(P|\bar{U}))$ such that $\nabla f = 0$. Then we have locally $f = cT_1^{-1}$ with some constant c. Since f is holomorphic along $z_1 = 0$ and $z_2 = 0$, we have $f = 0$, which proves $H^0_{\mathrm{DR}}(\Omega^\bullet_{\bar{U}}\langle D\rangle(P|\bar{U}), \nabla) = 0$. Let ω be an element of $\Gamma(\bar{U}, \Omega^1_{\bar{U}}\langle D\rangle(P|\bar{U}))$ such that $\nabla\omega = 0$. We set $\omega = adz_1 + bdz_2$, where a denotes a meromorphic P-valued section with pole along $z_1 = 0$, and b a meromorphic P-valued section with pole along $z_2 = 0$. If we set $\Omega = T_1\omega$, then we have $d\Omega = T_1\nabla\omega$. So $\nabla\omega = 0$ if and only if $d\Omega = 0$. Moreover $d\Omega = 0$ is equivalent to an integrability condition $\dfrac{\partial(aT_1)}{\partial z_2} = \dfrac{\partial(bT_1)}{\partial z_1}$. Let us consider the equation $dF = \Omega$ for a $\mathcal{L}$-valued unknown F. This is equivalent to a system of partial differential equations $\dfrac{\partial F}{\partial z_1} = aT_1$, $\dfrac{\partial F}{\partial z_2} = bT_1$. Locally around the point $(z_1, z_2) = (0, 0)$ we can write $T_1 = z_1^{c_1} z_2^{c_2} G(z_1, z_2)$ with constants c_1, c_2 and a holomorphic function G. So we have $aT_1 = z_1^{c-1} z_2^{c_2} G_1(z_1, z_2)$ and $bT_1 = z_1^{c_1} z_2^{c_2-1} G_2(z_1, z_2)$ with holomorphic functions G_1 and G_2. Then we see by an elementary theory of partial differential equations that there exists a unique solution F of the form $z_1^{c_1} z_2^{c_2} H(z_1, z_2)$ with a holomorphic function H which satisfies the preceding system of partial differential equations. Therefore F is decomposed locally as $F = \gamma T_1$ with a holomorphic P-valued section γ defined on $\bar{U}$. So we have $\nabla\gamma = \omega$, which proves $H^1_{\mathrm{DR}}(\Omega^\bullet_{\bar{U}}\langle D\rangle(P|\bar{U}), \nabla) = 0$. Let ω be an element of $\Gamma(\bar{U}, \Omega^2_{\bar{U}}\langle D\rangle(P|\bar{U}))$. Automatically we have $\nabla\omega = 0$. We set $\omega = f dz_1 \wedge dz_2$ where f denotes a meromorphic P-valued section with pole along $z_1 = 0$ and $z_2 = 0$. We set $\Omega = T_1\omega$. Let us consider the equation $d\varphi = \Omega$ for a $\mathcal{L}$-valued unknown 1-form φ. We set $\varphi = Adz_1 + Bdz_2$ where A and B are holomorphic $\mathcal{L}$-valued sections on U. Then the equation $d\varphi = \Omega$ is equivalent to a differential relation $\dfrac{\partial B}{\partial z_1} - \dfrac{\partial A}{\partial z_2} = T_1 f$. Locally around the point $(z_1, z_2) = (0, 0)$ we can write $T_1 f = z_1^{c-1} z_2^{c_2-1} G(z_1, z_2)$ with constants c_1, c_2 and a holomorphic function G. If we set $A = z_1^{c-1} z_2^{c_2-1} A'$ and $B = z_1^{c-1} z_2^{c_2-1} B'$ and substitute them into the preceding equation, then we have $\dfrac{c_1 - 1}{z_1} B' + \dfrac{\partial B'}{\partial z_1} - \dfrac{c_2 - 1}{z_2} A' - \dfrac{\partial A'}{\partial z_2} = G$. Obviously, we can find holomorphic functions A' and B' around the point $(z_1, z_2) = (0, 0)$ satisfying the preceding differential relation. Therefore

A is decomposed locally as $A = \gamma T_1$ with a P-valued section γ with pole along $z_1 = 0$, and B is decomposed locally as $B = \delta T_1$ with a P-valued section δ with pole along $z_2 = 0$. So we have $\nabla(\gamma dz_1 + \delta dz_2) = \omega$, which proves $H^2_{\mathrm{DR}}(\Omega^\bullet_{\bar U}\langle D\rangle(P|\bar U), \nabla) = 0$. Proposition 3.1 is proved.

Next, we give a proof of Proposition 3.3 by exploiting the logarithmic Dolbeault complex. As will be seen below, our proof contains that of Lemma 3.7.

Let us again consider the logarthmic Dolbeault complex:

$$0 \longrightarrow \Omega^p_X\langle D\rangle \longrightarrow \mathcal{E}^{p0}_X\langle D\rangle \xrightarrow{\bar\partial} \mathcal{E}^{p1}_X\langle D\rangle \xrightarrow{\bar\partial} \mathcal{E}^{p2}_X\langle D\rangle \xrightarrow{\bar\partial} 0.$$

Since $\mathcal{O}_X(P)$ is a locally free $\mathcal{O}_X$-module, setting $\mathcal{E}^{pq}_X\langle D\rangle(P) = \mathcal{E}^{pq}_X\langle D\rangle \otimes_{\mathcal{O}_X} \mathcal{O}_X(P)$, we have

$$0 \longrightarrow \Omega^p_X\langle D\rangle(P) \longrightarrow \mathcal{E}^{p0}_X\langle D\rangle(P) \xrightarrow{\bar\partial} \mathcal{E}^{p1}_X\langle D\rangle(P) \xrightarrow{\bar\partial} \mathcal{E}^{p2}_X\langle D\rangle(P) \xrightarrow{\bar\partial} 0.$$

Since $\mathcal{E}^{pq}_X\langle D\rangle(P)$ is a locally free $\mathcal{E}^{pq}_X$-module, we have $H^r(X, \mathcal{E}^{pq}_X\langle D\rangle(P)) = 0$ for $r > 0,\ p \geq 0,\ q \geq 0$. By the standard argument we have $H^q(X, \Omega^p_X\langle D\rangle(P)) \cong H^q_{\bar\partial}(X, \mathcal{E}^{p*}_X\langle D\rangle(P))$. Consider the following double complex:

$$\begin{array}{ccccccc}
\Gamma(X, \mathcal{E}^{00}_X(P)) & \xrightarrow{\bar\partial} & \Gamma(X, \mathcal{E}^{01}_X(P)) & \xrightarrow{\bar\partial} & \Gamma(X, \mathcal{E}^{02}_X(P)) & \xrightarrow{\bar\partial} & 0 \\
\downarrow{\scriptstyle \nabla_0} & & \downarrow{\scriptstyle \nabla_0} & & \downarrow{\scriptstyle \nabla_0} & & \\
\Gamma(X, \mathcal{E}^{10}_X\langle D\rangle(P)) & \xrightarrow{\bar\partial} & \Gamma(X, \mathcal{E}^{11}_X\langle D\rangle(P)) & \xrightarrow{\bar\partial} & \Gamma(X, \mathcal{E}^{12}_X\langle D\rangle(P)) & \xrightarrow{\bar\partial} & 0 \\
\downarrow{\scriptstyle \nabla_0} & & \downarrow{\scriptstyle \nabla_0} & & \downarrow{\scriptstyle \nabla_0} & & \\
\Gamma(X, \mathcal{E}^{20}_X(D)(P)) & \xrightarrow{\bar\partial} & \Gamma(X, \mathcal{E}^{21}_X(D)(P)) & \xrightarrow{\bar\partial} & \Gamma(X, \mathcal{E}^{22}_X(D)(P)) & \xrightarrow{\bar\partial} & 0 \\
\downarrow{\scriptstyle \nabla_0} & & \downarrow{\scriptstyle \nabla_0} & & \downarrow{\scriptstyle \nabla_0} & & \\
0 & & 0 & & 0 & &
\end{array}$$

where $\nabla_0 = \partial + d\log T_1 \wedge$. We set $L^0 = \Gamma(X, \mathcal{E}^{00}_X(P))$, $L^1 = \Gamma(X, \mathcal{E}^{10}_X\langle D\rangle(P)) \oplus \Gamma(X, \mathcal{E}^{01}_X(P))$, $L^2 = \Gamma(X, \mathcal{E}^{20}_X(D)(P)) \oplus \Gamma(X, \mathcal{E}^{11}_X\langle D\rangle(P)) \oplus \Gamma(X, \mathcal{E}^{02}_X(P))$, $L^3 = \Gamma(X, \mathcal{E}^{21}_X(D)(P)) \oplus \Gamma(X, \mathcal{E}^{12}_X\langle D\rangle(P))$, $L^4 = \Gamma(X, \mathcal{E}^{22}_X(D)(P))$. The operator ∇ maps L^k into L^{k+1}, and the pair (L, ∇), where $L = \oplus_{k=0}^4 L^k$, is a complex. Let us introduce a filtration of L. Namely, if we set $L_0 = L$, $L_1 = \oplus_{q=0}^2 \Gamma(X, \mathcal{E}^{1q}_X\langle D\rangle(P)) \oplus \Gamma(X, \mathcal{E}^{2q}_X(D)(P))$, $L_2 = \oplus_{q=0}^2 \Gamma(X, \mathcal{E}^{2q}_X(D)(P))$, then we have $L = L_0 \supset L_1 \supset L_2 \supset 0$. The spectral sequence E'_r associated to the filtered module L is given as follows: ${E'}^p_1 = H(L_p/L_{p+1}) = H(\oplus_{q=0}^2 \Gamma(X, \mathcal{E}^{pq}_X\langle D\rangle(P)) = H^0_{\bar\partial}(X, \mathcal{E}^{p*}_X\langle D\rangle(P)) \oplus H^1_{\bar\partial}(X, \mathcal{E}^{p*}_X\langle D\rangle(P)) \oplus H^2_{\bar\partial}(X, \mathcal{E}^{p*}_X\langle D\rangle(P)) \cong H^0(X, \Omega^p_X\langle D\rangle(P)) \oplus H^1(X, \Omega^p_X\langle D\rangle(P)) \oplus H^2(X, \Omega^p_X\langle D\rangle(P))$ and ${E'}^{pq}_1 = H^q(X, \Omega^p_X\langle D\rangle(P))$. So we have $E'_r = E_r$ for every $r \geq 1$, where the latter sequence E_r is the one introduced in §3. Therefore it follows from Corollary 3.2 that

$$H^p(M, \mathcal{L}) \cong \mathbf{H}^p(X, \Omega^\bullet_X\langle D\rangle(P), \nabla) \cong H^p(L, \nabla).$$

To prove Proposition 3.3 it suffices to prove the following

Lemma 3.8. *$H^p(L, \nabla) = 0$ for $p \neq 2$.*

Proof. It suffices to prove that $H^0(L, \nabla) = H^1(L, \nabla) = 0$. Let f be an element in $\Gamma(X, \mathcal{E}^{00}_X)$ such that $\bar\partial f = \nabla_0 f = 0$. This is equivalent to the condition that $f \in \Gamma(X, \mathcal{O}_X(P))$ and $\nabla f = 0$. If

$N < N'$ and not all of the quantities $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k}c_k$ are integers, then it follows immediately that $\Gamma(X, \mathcal{O}_X(P)) = 0$ and $H^0(L, \nabla) = 0$. If not, it follows that $\Gamma(X, \mathcal{O}_X(P)) = \boldsymbol{C}$ and f is a constant. Therefore the equation $\nabla f = 0$ implies $f = 0$, and so $H^0(L, \nabla) = 0$. Next, let (β, α) be an element in $\Gamma(X, \mathcal{E}_X^{10}\langle D\rangle(P)) \oplus \Gamma(X, \mathcal{E}_X^{01}(P))$ such that $\nabla_0\beta = \nabla_0\alpha + \bar\partial\beta = \bar\partial\alpha = 0$. Since $\Gamma(X, \mathcal{E}_X^{10}\langle D\rangle)$ is generated by dz_1, dz_2, $d\log\theta\begin{bmatrix} a_{2k-1}\, a_{2k} \\ b_{2k-1}\, b_{2k}\end{bmatrix} - d\log\theta\begin{bmatrix} a_{2k+1}\, a_{2k+2} \\ b_{2k+1}\, b_{2k+2}\end{bmatrix}$, $k = 1, \ldots, N-1$, we can set

$$\beta = \beta_1 + \sum_{k=1}^{N-1} \beta_{0k}\left(d\log\theta\begin{bmatrix} a_{2k-1}\, a_{2k} \\ b_{2k-1}\, b_{2k}\end{bmatrix} - d\log\theta\begin{bmatrix} a_{2k+1}\, a_{2k+2} \\ b_{2k+1}\, b_{2k+2}\end{bmatrix}\right),$$

where $\beta_1 \in \Gamma(X, \mathcal{E}_X^{10}(P))$ and $\beta_{0k} \in \Gamma(X, \mathcal{E}_X^{00}(P))$. We have

$$\bar\partial\beta = \bar\partial\beta_1 + \sum_{k=1}^{N-1} \bar\partial\beta_{0k} \wedge \left(d\log\theta\begin{bmatrix} a_{2k-1}\, a_{2k} \\ b_{2k-1}\, b_{2k}\end{bmatrix} - d\log\theta\begin{bmatrix} a_{2k+1}\, a_{2k+2} \\ b_{2k+1}\, b_{2k+2}\end{bmatrix}\right).$$

Then the equation $\nabla_0\alpha + \bar\partial\beta = 0$ is decomposed into the following two equations:

$$\partial\alpha + \bar\partial\beta_1 = 0, \tag{3.5}$$

$$d\log T_1 \wedge \alpha + \sum_{k=1}^{N-1} \bar\partial\beta_{0k} \wedge \left(d\log\theta\begin{bmatrix} a_{2k-1}\, a_{2k} \\ b_{2k-1}\, b_{2k}\end{bmatrix} - d\log\theta\begin{bmatrix} a_{2k+1}\, a_{2k+2} \\ b_{2k+1}\, b_{2k+2}\end{bmatrix}\right) = 0. \tag{3.6}$$

Since

$$\alpha \wedge d\log T_1 = \sum_{k=1}^{N-1} (c_1 + \cdots + c_k)\alpha \wedge \left(d\log\theta\begin{bmatrix} a_{2k-1}\, a_{2k} \\ b_{2k-1}\, b_{2k}\end{bmatrix} - d\log\theta\begin{bmatrix} a_{2k+1}\, a_{2k+2} \\ b_{2k+1}\, b_{2k+2}\end{bmatrix}\right),$$

the equation (3.6) is turned to

$$\sum_{k=1}^{N-1} \left(\bar\partial\beta_{0k} - (c_1 + \cdots + c_k)\alpha\right) \wedge \left(d\log\theta\begin{bmatrix} a_{2k-1}\, a_{2k} \\ b_{2k-1}\, b_{2k}\end{bmatrix} - d\log\theta\begin{bmatrix} a_{2k+1}\, a_{2k+2} \\ b_{2k+1}\, b_{2k+2}\end{bmatrix}\right) = 0,$$

from which it follows that $\bar\partial\beta_{0k} = (c_1 + \cdots + c_k)\alpha$, $k = 1, \ldots, N-1$. Suppose now that there exists a sequence of integers $k_1, \ldots, k_l$ satisfying the following four conditions : (i) $1 < k_1 < \cdots < k_l < N-1$; (ii) $k_{\nu+1} - k_\nu > 1$ ($\nu = 1, \ldots, l-1$); (iii) $c_1 + \cdots + c_k = 0$ if $k \in \{k_1, \ldots, k_l\}$; (iv) $c_1 + \cdots + c_k \neq 0$ if $k \notin \{k_1, \ldots, k_l, N\}$. We set $I = \{k_1, \ldots, k_l\}$ and $J = \{1, \ldots, N-1\} - I$. Then we have

$$\alpha = \frac{\bar\partial\beta_{0k}}{c_1 + \cdots + c_k} \quad \text{if} \quad k \in J, \tag{3.7}$$

and $\bar\partial\beta_{0k} = 0$, that is, β_{0k} is a constant if $k \in I$. Note that the differences $\omega_{ij} = \dfrac{\beta_{0i}}{c_1 + \cdots + c_i} -$

$\dfrac{\beta_{0j}}{c_1+\cdots+c_j}$, $i,j\in J$ belong to $\Gamma(X,\mathcal{O}_X(P))$. Now we have

$$
\begin{aligned}
\beta-\nabla_0\frac{\beta_{01}}{c_1}=&\beta_1+\sum_{k\in J}\frac{\beta_{0k}}{c_1+\cdots+c_k}(c_1+\cdots+c_k)\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right)\\
&+\sum_{k\in I}\beta_{0k}\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right)-\frac{\partial\beta_{01}}{c_1}-\frac{\beta_{01}}{c_1}d\log T_1\\
=&\beta_1+\sum_{k\in J}\left(\frac{\beta_{01}}{c_1}+\omega_{k1}\right)(c_1+\cdots+c_k)\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right)\\
&+\sum_{k\in I}\beta_{0k}\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right)-\frac{\partial\beta_{01}}{c_1}-\frac{\beta_{01}}{c_1}d\log T_1.
\end{aligned}
\tag{3.8}
$$

Since $d\log T_1=\sum_{k\in J}(c_1+\cdots+c_k)\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right)$, (3.8) is turned to

$$
\begin{aligned}
\beta-\nabla_0\frac{\beta_{01}}{c_1}=&\beta_1-\frac{\partial\beta_{01}}{c_1}+\sum_{k\in J}\omega_{k1}(c_1+\cdots+c_k)\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right)\\
&+\sum_{k\in I}\beta_{0k}\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right).
\end{aligned}
\tag{3.9}
$$

Note that $\beta_1-\dfrac{\partial\beta_{01}}{c_1}\in\Gamma(X,\Omega^1_X(P))$ by (3.5) and (3.7). If $N<N'$ and not all of $\frac{1}{2}\sum_{k=N+1}^{N'}a_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'}a_{2k}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'}b_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'}b_{2k}c_k$ are integers, then we have $\Gamma(X,\mathcal{O}_X(P))=0$, which implies $\beta_1-\dfrac{\partial\beta_{01}}{c_1}=0$, $\omega_{k1}=0$ $(k\in J)$, $\beta_{0k}=0$ $(k\in I)$. Therefore we have $\beta=\nabla_0\dfrac{\partial\beta_{01}}{c_1}$. Since $(\nabla_0+\bar\partial)\left(\dfrac{\beta_{01}}{c_1}\right)=(\beta,\alpha)$, it follows that $H^1(L,\nabla)=0$. Next, suppose that either $N=N'$, or $N<N'$ and all of $\frac{1}{2}\sum_{k=N+1}^{N'}a_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'}a_{2k}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'}b_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'}b_{2k}c_k$ are integers. Then we have $P=\boldsymbol{C}$ and $\Gamma(X,\mathcal{O}_X)=\boldsymbol{C}$. Therefore ω_{k1}'s $(k\in J)$ and β_{0k}'s $(k\in I)$ are constants. Moreover we can set $\beta_1-\dfrac{\partial\beta_{01}}{c_1}=Adz_1+Bdz_2$ with constants A,B. Namely, (3.9) is turned to

$$
\begin{aligned}
\beta-\nabla_0\frac{\beta_{01}}{c_1}=&Adz_1+Bdz_2+\sum_{k\in J}\omega_{k1}(c_1+\cdots+c_k)\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right)\\
&+\sum_{k\in I}\beta_{0k}\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right).
\end{aligned}
\tag{3.10}
$$

Applying ∇ on this equation, we have

$$
\begin{aligned}
\nabla\left(\beta-\nabla_0\frac{\beta_{01}}{c_1}\right)=&d\log T_1\wedge\Bigg[Adz_1+Bdz_2+\sum_{k\in J}\omega_{k1}(c_1+\cdots+c_k)\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right)\\
&+\sum_{k\in I}\beta_{0k}\left(d\log\theta\begin{bmatrix}a_{2k-1}\,a_{2k}\\ b_{2k-1}\,b_{2k}\end{bmatrix}-d\log\theta\begin{bmatrix}a_{2k+1}\,a_{2k+2}\\ b_{2k+1}\,b_{2k+2}\end{bmatrix}\right)\Bigg].
\end{aligned}
$$

The equations $\nabla_0\beta = \bar{\partial}\left(\beta - \nabla_0\dfrac{\beta_{01}}{c_1}\right) = 0$ imply $\nabla\left(\beta - \nabla_0\dfrac{\beta_{01}}{c_1}\right) = 0$. So we have $A = B = \beta_{0k} = 0$ $(k \in I)$ and $\omega_{k1} = \omega_{k'1}$ $(k, k' \in J)$. We set $\omega_{k1} = C$ $(k \in J)$. It follows from (3.10) that

$$\begin{aligned}\beta - \nabla_0\frac{\beta_{01}}{c_1} =& C\sum_{k\in J}(c_1 + \cdots + c_k)\left(d\log\theta\begin{bmatrix}a_{2k-1}\, a_{2k}\\ b_{2k-1}\, b_{2k}\end{bmatrix} - d\log\theta\begin{bmatrix}a_{2k+1}\, a_{2k+2}\\ b_{2k+1}\, b_{2k+2}\end{bmatrix}\right)\\ =&\nabla_0(C).\end{aligned}$$

Consequently, we have $\beta = \nabla_0\left(\dfrac{\beta_{01}}{c_1} + C\right)$ and $\alpha = \bar{\partial}\left(\dfrac{\beta_{01}}{c_1} + C\right)$, and therefore $H^1(L, \nabla) = 0$ also in this case. The above argument is effective also when $I = \emptyset$. Lemma 3.8 is proved.

§4 Structure of the non-vanishing cohomology group

Propositions 3.5 and 3.6 give us information on the structure of the non-vanishing cohomology group $H^2(M, \mathcal{L}) \cong \mathbf{H}^2(X, \Omega^\bullet_X\langle D\rangle(P), \nabla)$. Namely, there exists a filtration of $\mathbf{H}^2 = \mathbf{H}^2(X, \Omega^\bullet_X\langle D\rangle(P), \nabla)$: $\mathbf{H}^2 = \mathbf{H}^2_0 \supset \mathbf{H}^2_1 \supset \mathbf{H}^2_2 \supset 0$ such that $\mathbf{H}^2_2 \cong E^{20}_\infty$, $\mathbf{H}^2_1/\mathbf{H}^2_2 \cong E^{11}_\infty$, $\mathbf{H}^2_0/\mathbf{H}^2_1 \cong E^{02}_\infty$. To obtain information on how to select meromorphic 2-forms realizing a basis of $H^2(M, \mathcal{L})$, we need to study the analytical structures of E^{pq}_∞ with $p + q = 2$ further. To this end let us consider the following sequences of sheaves over X:

$$0 \longrightarrow \Omega^1_X\langle D\rangle(P) \longrightarrow \Omega^1_X(D)(P) \xrightarrow{\nabla'} \frac{\sum_{\sum k_\nu = 1}\Omega^2_X\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)}{\Omega^2_X(D)(P)} \xrightarrow{\nabla'} 0, \tag{4.1}$$

and

$$\begin{aligned}&0 \longrightarrow \mathcal{O}_X(P) \longrightarrow \mathcal{O}_X(D)(P) \xrightarrow{\nabla'} \frac{\sum_{\sum k_\nu = 1}\Omega^1_X\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)}{\Omega^1_X(D)(P)}\\ &\xrightarrow{\nabla'} \frac{\sum_{\sum k_\nu = 2}\Omega^2_X\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)}{\sum_{\sum k_\nu = 1}\Omega^2_X\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)} \xrightarrow{\nabla'} 0,\end{aligned} \tag{4.2}$$

where ∇' denotes the operator induced from ∇, and for $l = 1, 2$ the symbol $\sum_{\sum k_\nu = l}$ represents the abbreviation for the symbol $\sum_{k_1\ge 0,\dots,k_N\ge 0, k_1+\cdots+k_N = l}$. As is well-known (Deligne [3], II, §3, Prop.3.13), since D has normal crossings, the sequences (4.1) and (4.2) are exact, i.e., they give resolutions of the sheaves of logarithmic differential forms $\Omega^1_X\langle D\rangle(P)$ and $\mathcal{O}_X(P) (= \Omega^0_X\langle D\rangle(P))$. In what follows, for a sheaf $\mathcal{F}$ over X let the symbol $H^p(\mathcal{F})$ represent the cohomology group $H^p(X, \mathcal{F})$.

Proposition 4.1. $H^1(\Omega^1_X\langle D\rangle(P)) \cong \dfrac{H^0\left(\sum_{\sum k_\nu = 1}\Omega^2_X\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)\right)}{\nabla H^0(\Omega^1_X(D)(P)) + H^0(\Omega^2_X(D)(P))}.$

Proof. From the short exact sequence of sheaves (4.1), we have the following long exact sequence of cohomology groups:

$$\begin{aligned}0 &\longrightarrow H^0(\Omega^1_X\langle D\rangle(P)) \longrightarrow H^0(\Omega^1_X(D)(P)) \xrightarrow{\nabla'} H^0\left(\frac{\sum_{\sum k_\nu = 1}\Omega^2_X\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)}{\Omega^2_X(D)(P)}\right)\\ &\longrightarrow H^1(\Omega^1_X\langle D\rangle(P)) \longrightarrow H^1(\Omega^1_X(D)(P)) \longrightarrow \cdots.\end{aligned} \tag{4.3}$$

Note that the sheaf of modules $\Omega_X^1(D)(P)$ is isomorphic to a direct sum of two copies of $\mathcal{O}_X(D)(P)$. There exists a positive definite line bundle L on X of type $(1,1)$ such that $H^1(\mathcal{O}_X(D)(P)) \cong H^1(\mathcal{O}_X(L^N))$, where $\mathcal{O}_X(L^N)$ denotes the sheaf of local sections of the line bundle L^N over X. L^N is a positive definite line bundle on X of type (N,N), and has no negative eigenvalues. Then Mumford's vanishing theorem (Mumford [12], III, §16) implies that $H^1(\mathcal{O}_X(L^N)) = 0$, and therefore $H^1(\Omega_X^1(D)(P)) = 0$. Then (4.3) is turned to the following exact sequence:

$$0 \longrightarrow H^0(\Omega_X^1\langle D\rangle(P)) \longrightarrow H^0(\Omega_X^1(D)(P)) \xrightarrow{\nabla'} H^0\left(\frac{\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)}{\Omega_X^2(D)(P)}\right)$$
$$\longrightarrow H^1(\Omega_X^1\langle D\rangle(P)) \longrightarrow 0,$$

from which it follows that

$$H^1(\Omega_X^1\langle D\rangle(P)) \cong \frac{H^0\left(\frac{\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)}{\Omega_X^2(D)(P)}\right)}{\nabla' H^0(\Omega_X^1(D)(P))}. \tag{4.4}$$

Since $H^1(\Omega_X^2(D)(P)) = 0$, we have
(4.5)

$$H^0\left(\frac{\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)}{\Omega_X^2(D)(P)}\right) \cong \frac{H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)\right)}{H^0(\Omega_X^2(D)(P))}.$$

Moreover, by this isomorphism, we may identify the mapping

$$\nabla' : H^0(\Omega_X^1(D)(P)) \longrightarrow H^0\left(\frac{\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)}{\Omega_X^2(D)(P)}\right)$$

with the composition

$$H^0(\Omega_X^1(D)(P)) \xrightarrow{\nabla} H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)\right)$$
$$\longrightarrow \frac{H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)\right)}{H^0(\Omega_X^2(D)(P))}.$$

Then we have

$$\begin{aligned}\nabla' H^0(\Omega_X^1(D)(P)) &\cong \frac{\nabla H^0(\Omega_X^1(D)(P))}{\nabla H^0(\Omega_X^1(D)(P)) \cap H^0(\Omega_X^2(D)(P))} \\ &\cong \frac{\nabla H^0(\Omega_X^1(D)(P)) + H^0(\Omega_X^2(D)(P))}{H^0(\Omega_X^2(D)(P))}.\end{aligned} \tag{4.6}$$

Substitution of (4.5) and (4.6) into (4.4) gives us the desired formula of Proposition 4.1.

Proposition 4.2.

$$H^2(\mathcal{O}_X(P)) \cong \frac{H^0\left(\sum_{\sum k_\nu=2}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)\right)}{\nabla H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)\right) + H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)(P)\right)},$$

$H^1(\mathcal{O}_X(P))$

$$\cong \frac{\nabla^{-1}\left\{\nabla H^0\left(\sum_{\sum k_\nu=1}\Omega^1_X\Big(\sum_{\nu=1}^N(k_\nu+1)D_\nu\Big)(P)\right)\cap H^0\left(\sum_{\sum k_\nu=1}\Omega^2_X\Big(\sum_{\nu=1}^N(k_\nu+1)D_\nu\Big)(P)\right)\right\}}{\nabla H^0(\mathcal{O}_X(D)(P))+H^0(\Omega^1_X(D)(P))}.$$

To prove Proposition 4.2 we need the following

Lemma 4.3. $H^1\left(\sum_{\sum k_\nu=1}\Omega^p_X\Big(\sum_{\nu=1}^N(k_\nu+1)D_\nu\Big)(P)\right)=0$, $p=1,2$.

Proof. Let us consider the following short exact sequence of sheaves over X:

$$0\longrightarrow\sum_{\sum k_\nu=1}\Omega^p_X\Big(\sum_{\nu=1}^N(k_\nu+1)D_\nu\Big)(P)\longrightarrow\Omega^p_X(2D)(P)\longrightarrow\mathcal{S}^p\longrightarrow 0, \tag{4.7}$$

where $\mathcal{S}^p=\dfrac{\Omega^p_X(2D)(P)}{\sum_{\sum k_\nu=1}\Omega^p_X\Big(\sum_{\nu=1}^N(k_\nu+1)D_\nu\Big)(P)}$. Let $\mathcal{S}^p_x$ denote the stalk of the sheaf $\mathcal{S}^p$ at a point $x\in X$. Then we see that $\mathcal{S}^p_x\cong\boldsymbol{C}^{3-p}$ if $x\in D_i\cap D_j$ $(i\neq j)$; otherwise $\mathcal{S}^p_x=0$. In fact, for $x\in D_i\cap D_j$ $(i\neq j)$ the stalk $\mathcal{S}^p_x$ is identified with the vector space $\left\{\dfrac{c_1}{z^2w^2}dz+\dfrac{c_2}{z^2w^2}dw\mid c_1,c_2\in\boldsymbol{C}\right\}$ if $p=1$, and with the vector space $\left\{\dfrac{c}{z^2w^2}dz\wedge dw\mid c\in\boldsymbol{C}\right\}$ if $p=2$, where z,w denote local coordinates around x such that the local equations of D_i and D_j around x are given by $z=0$ and $w=0$ respectively. From (4.7) we have the long exact sequence

$$\begin{aligned}&0\longrightarrow H^0\left(\sum_{\sum k_\nu=1}\Omega^p_X\Big(\sum_{\nu=1}^N(k_\nu+1)D_\nu\Big)(P)\right)\longrightarrow H^0(\Omega^p_X(2D)(P))\longrightarrow H^0(\mathcal{S}^p)\\&\longrightarrow H^1\left(\sum_{\sum k_\nu=1}\Omega^p_X\Big(\sum_{\nu=1}^N(k_\nu+1)D_\nu\Big)(P)\right)\longrightarrow H^1(\Omega^p_X(2D)(P))\longrightarrow\cdots.\end{aligned}\tag{4.8}$$

Since the sheaf $\Omega^p_X(2D)(P)$ is isomorphic to a direct sum of copies of $\mathcal{O}_X(2D)(P)$, Mumford's vanishing theorem implies that $H^1(\Omega^p_X(2D)(P))=0$. Therefore from (4.8) we have

$$\begin{aligned}&0\longrightarrow H^0\left(\sum_{\sum k_\nu=1}\Omega^p_X\Big(\sum_{\nu=1}^N(k_\nu+1)D_\nu\Big)(P)\right)\longrightarrow H^0(\Omega^p_X(2D)(P))\longrightarrow H^0(\mathcal{S}^p)\\&\longrightarrow H^1\left(\sum_{\sum k_\nu=1}\Omega^p_X\Big(\sum_{\nu=1}^N(k_\nu+1)D_\nu\Big)(P)\right)\longrightarrow 0.\end{aligned}\tag{4.9}$$

Note that $H^0(\mathcal{S}^p)$ is isomorphic to the direct sum of $N(N-1)$ copies of the abelian group $\boldsymbol{C}^{3-p}$. Moreover we have $\Omega^1_X(2D)(P)\cong\mathcal{O}_X(2D)(P)\oplus\mathcal{O}_X(2D)(P)$ and $\Omega^2_X(2D)(P)\cong\mathcal{O}_X(2D)(P)$. We denote by $[2D]$ the line bundle over X defined by the divisor $2D$. Then we have $\mathcal{O}_X(2D)(P)\cong\mathcal{O}_X([2D]\otimes P)$, where the right hand side denotes the sheaf of local sections of the line bundle $[2D]\otimes P$. Since $c_1([2D]\otimes P)=c_1([2D])$, we have $\dim H^0(\mathcal{O}_X(2D)(P))=4N^2$. Let $\omega_1,\ldots,\omega_{4N^2}$ be a basis of the vector space $H^0(\mathcal{O}_X(2D)(P))$, and $p_1,\ldots,p_{N(N-1)}$ be the points on X, each of which is an intersection of two theta divisors D_i,D_j $(i\neq j)$, i.e., for any k there exist i

and j, $i \neq j$, such that $p_k \in D_i \cap D_j$. For $(c_{p_1}, \ldots, c_{p_{N(N-1)}}) \in H^0(\mathcal{S}^2) \cong \boldsymbol{C}^{N(N-1)}$, let us consider the following system of $N(N-1)$ linear equations with $4N^2$ unknowns $A_1, \ldots, A_{4N^2}$: for $k = 1, \ldots, N(N-1)$

$$A_1(\omega_1)_{p_k} + \cdots + A_{4N^2}(\omega_{4N^2})_{p_k} \equiv \frac{c_{p_k}}{z^2w^2} dz \wedge dw \qquad \mathrm{mod} \left(\frac{dz \wedge dw}{zw^2}, \frac{dz \wedge dw}{z^2w} \right), \tag{4.10}$$

where $(\omega_l)_{p_k}$ denotes the localization of ω_l at the point p_k, and z, w denote local coordinates around p_k such that the local equations of D_i and Dj around p_k are given by $z = 0$ and $w = 0$ respectively. Obviously the system (4.10) has a non-trivial solution $(A_1, \ldots, A_{4N^2}) \in \boldsymbol{C}^{4N^2}$, which implies that the mapping $H^0(\Omega^2_X(2D)(P)) \longrightarrow H^0(\mathcal{S}^2)$ is surjective. Also in the case where $p = 1$, by the same argument as above, we see that the mapping $H^0(\Omega^1_X(2D)(P)) \longrightarrow H^0(\mathcal{S}^1)$ is surjective. Lemma 4.3 is thus proved.

Corollary 4.4. $H^1 \left(\dfrac{\sum_{\sum k_\nu = 1} \Omega^1_X \left(\sum_{\nu=1}^N (k_\nu + 1) D_\nu \right)(P)}{\Omega^1_X(D)(P)} \right) = 0.$

Proof. Let us consider the following short exact sequence:

$$\begin{aligned} 0 \longrightarrow \Omega^1_X(D)(P) \longrightarrow \sum_{\sum k_\nu = 1} \Omega^1_X \left(\sum_{\nu=1}^N (k_\nu + 1) D_\nu \right)(P) \\ \longrightarrow \frac{\sum_{\sum k_\nu = 1} \Omega^1_X \left(\sum_{\nu=1}^N (k_\nu + 1) D_\nu \right)(P)}{\Omega^1_X(D)(P)} \longrightarrow 0. \end{aligned}$$

Then the corollary follows immediately from the long exact sequence of cohomology groups with the vanishing of cohomology groups: $H^1 \left(\sum_{\sum k_\nu = 1} \Omega^1_X \Big(\sum_{\nu=1}^N (k_\nu + 1) D_\nu \Big)(P) \right) = 0$ by Lemma 4.3 and $H^2(X, \Omega^1_X(D)(P)) = 0$ by Mumford's vanishing theorem.

Proof of Proposition 4.2. Note that the exact sequence (4.2) is decomposed into the following two short exact sequences:

$$0 \longrightarrow \mathcal{O}_X(P) \longrightarrow \mathcal{O}_X(D)(P) \xrightarrow{\nabla'} \nabla' \mathcal{O}_X(D)(P) \longrightarrow 0, \tag{4.11}$$

$$\begin{aligned} 0 \longrightarrow \nabla' \mathcal{O}_X(D)(P) \longrightarrow \frac{\sum_{\sum k_\nu = 1} \Omega^1_X \left(\sum_{\nu=1}^N (k_\nu + 1) D_\nu \right)(P)}{\Omega^1_X(D)(P)} \\ \xrightarrow{\nabla'} \frac{\sum_{\sum k_\nu = 2} \Omega^2_X \left(\sum_{\nu=1}^N (k_\nu + 1) D_\nu \right)(P)}{\sum_{\sum k_\nu = 1} \Omega^2_X \left(\sum_{\nu=1}^N (k_\nu + 1) D_\nu \right)(P)} \xrightarrow{\nabla'} 0. \end{aligned} \tag{4.12}$$

From (4.11) we have the following long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\mathcal{O}_X(P)) \longrightarrow H^0(\mathcal{O}_X(D)(P)) \xrightarrow{\nabla'} H^0(\nabla' \mathcal{O}_X(D)(P)) \longrightarrow H^1(\mathcal{O}_X(P)) \\ \longrightarrow H^1(\mathcal{O}_X(D)(P)) \longrightarrow H^1(\nabla' \mathcal{O}_X(D)(P)) \longrightarrow H^2(\mathcal{O}_X(P)) \longrightarrow H^2(\mathcal{O}_X(D)(P)) \longrightarrow \cdots . \end{aligned} \tag{4.13}$$

Mumford's vanishing theorem implies that $H^1(\mathcal{O}_X(D)(P)) = H^2(\mathcal{O}_X(D)(P)) = 0$. So we have from (4.13)

$$0 \longrightarrow \nabla' H^0(\mathcal{O}_X(D)(P)) \longrightarrow H^0(\nabla' \mathcal{O}_X(D)(P)) \longrightarrow H^1(\mathcal{O}_X(P)) \longrightarrow 0, \tag{4.14}$$

$$H^1(\nabla' \mathcal{O}_X(D)(P)) \cong H^2(\mathcal{O}_X(P)). \tag{4.15}$$

From (4.14) it follows immediately that

$$H^1(\mathcal{O}_X(P)) \cong \frac{H^0(\nabla'\mathcal{O}_X(D)(P))}{\nabla' H^0(\mathcal{O}_X(D)(P))}. \tag{4.16}$$

On the other hand, from (4.12) we have the following long exact sequence

$$\begin{aligned} 0 \longrightarrow & H^0(\nabla'\mathcal{O}_X(D)(P)) \longrightarrow H^0\left(\frac{\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}{\Omega_X^1(D)(P)}\right) \\ & \xrightarrow{\nabla'} H^0\left(\frac{\sum_{\sum k_\nu=2}\Omega_X^2\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}{\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}\right) \longrightarrow H^1(\nabla'\mathcal{O}_X(D)(P)) \\ & \longrightarrow H^1\left(\frac{\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}{\Omega_X^1(D)(P)}\right) \longrightarrow \cdots . \end{aligned} \tag{4.17}$$

Since $H^1\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right) = 0$ by Lemma 4.3 and $H^1(\Omega_X^1(D)(P)) = 0$ by Mumford's vanishing theorem, the mapping ∇' in (4.17) is identified with

$$\nabla' : \frac{H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)}{H^0\left(\Omega_X^1(D)(P)\right)} \longrightarrow \frac{H^0\left(\sum_{\sum k_\nu=2}\Omega_X^2\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)}{H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)}. \tag{4.18}$$

So we have

$$\begin{aligned} & H^0(\nabla'\mathcal{O}_X(D)(P)) \cong \mathrm{Ker}\nabla' \\ & \cong \frac{\nabla^{-1}\left[\nabla H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right) \cap H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)\right]}{H^0(\Omega_X^1(D)(P))}, \end{aligned} \tag{4.19}$$

where ∇ is regarded as the mapping of $H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)$ into $H^0\left(\sum_{\sum k_\nu=2}\Omega_X^2\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)$. From the sheaf mapping $\nabla' : \mathcal{O}_X(D)(P) \longrightarrow \frac{\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}{\Omega_X^1(D)(P)}$ in (4.2), we have the induced mapping $\nabla' : H^0(\mathcal{O}_X(D)(P)) \longrightarrow H^0\left(\frac{\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}{\Omega_X^1(D)(P)}\right)$. Since $H^1(X, \Omega_X^1(D)(P)) = 0$, we have

$$H^0\left(\frac{\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}{\Omega_X^1(D)(P)}\right) \cong \frac{H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)}{H^0\left(\Omega_X^1(D)(P)\right)},$$

and by this isomorphism we may identify the above induced mapping ∇' with the composition

$$\begin{aligned} H^0(\mathcal{O}_X(D)(P)) \xrightarrow{\nabla} & H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right) \\ & \longrightarrow \frac{H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)}{H^0\left(\Omega_X^1(D)(P)\right)}. \end{aligned}$$

Then we have

$$
\begin{aligned}
\nabla' H^0(\mathcal{O}_X(D)(P)) &\cong \frac{\nabla H^0(\mathcal{O}_X(D)(P))}{\nabla H^0(\mathcal{O}_X(D)(P)) \cap H^0(\Omega^1_X(D)(P))} \\
&\cong \frac{\nabla H^0(\mathcal{O}_X(D)(P)) + H^0(\Omega^1_X(D)(P))}{H^0(\Omega^1_X(D)(P))}.
\end{aligned} \tag{4.20}
$$

Substitution of (4.19) and (4.20) into (4.16) gives us the desired formula for $H^1(X, \mathcal{O}_X(P))$ of Proposition 4.2.

Next, applying Corollary 4.4 to (4.17) we have

$$
\begin{aligned}
0 \longrightarrow & H^0(\nabla' \mathcal{O}_X(D)(P)) \longrightarrow H^0\left(\frac{\sum_{\sum k_\nu=1} \Omega^1_X\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}{\Omega^1_X(D)(P)}\right) \\
&\xrightarrow{\nabla'} H^0\left(\frac{\sum_{\sum k_\nu=2} \Omega^2_X\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}{\sum_{\sum k_\nu=1} \Omega^2_X\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}\right) \longrightarrow H^1(\nabla' \mathcal{O}_X(D)(P)) \longrightarrow 0.
\end{aligned} \tag{4.21}
$$

Since the mapping ∇' in (4.21) is identified with (4.18), we have

$$
\begin{aligned}
\mathrm{Im}\nabla' &\cong \frac{\nabla H^0\left(\sum_{\sum k_\nu=1} \Omega^1_X\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)}{\nabla H^0\left(\sum\limits_{\sum k_\nu=1} \Omega^1_X\left(\sum\limits_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right) \cap H^0\left(\sum\limits_{\sum k_\nu=1} \Omega^2_X\left(\sum\limits_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)} \\
&\cong \frac{\nabla H^0\left(\sum\limits_{\sum k_\nu=1} \Omega^1_X\left(\sum\limits_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right) + H^0\left(\sum\limits_{\sum k_\nu=1} \Omega^2_X\left(\sum\limits_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)}{H^0\left(\sum_{\sum k_\nu=1} \Omega^2_X\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)}.
\end{aligned} \tag{4.22}
$$

Combining (4.21) with (4.15), we have

$$
0 \longrightarrow \mathrm{Im}\nabla' \longrightarrow H^0\left(\frac{\sum_{\sum k_\nu=2} \Omega^2_X\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}{\sum_{\sum k_\nu=1} \Omega^2_X\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}\right) \longrightarrow H^2(\mathcal{O}_X(P)) \longrightarrow 0. \tag{4.23}
$$

Applying (4.22) to (4.23), we have

$$
\begin{aligned}
H^2(X, \mathcal{O}_X(P)) &\cong \frac{H^0\left(\frac{\sum_{\sum k_\nu=2} \Omega^2_X\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}{\sum_{\sum k_\nu=1} \Omega^2_X\left(\sum_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)}\right)}{\mathrm{Im}\nabla'} \\
&\cong \frac{H^0\left(\sum\limits_{\sum k_\nu=2} \Omega^2_X\left(\sum\limits_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)}{\nabla H^0\left(\sum\limits_{\sum k_\nu=1} \Omega^1_X\left(\sum\limits_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right) + H^0\left(\sum\limits_{\sum k_\nu=1} \Omega^2_X\left(\sum\limits_{\nu=1}^N (k_\nu+1)D_\nu\right)(P)\right)},
\end{aligned}
$$

which is the desired formula for $H^2(\mathcal{O}_X(P))$, and Proposition 4.2 is proved completely.

Our main theorems in this paper are as follows:

Theorem 4.5. Assume that $N < N'$ and not all of the four quantities $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k}c_k$ are integers. Then we have $H^2(M,\mathcal{L}) \cong E_\infty^{20} \oplus E_\infty^{11}$, where

$$E_\infty^{20} \cong H^0(X, \Omega_X^2(D)(P)),$$
$$E_\infty^{11} \cong \frac{H^0\left(X, \sum_{\sum k_\nu = 1} \Omega_X^2\left(\sum_{\nu=1}^{N}(k_\nu+1)D_\nu\right)(P)\right)}{\nabla H^0(X, \Omega_X^1(D)(P)) + H^0(X, \Omega_X^2(D)(P))}.$$

Moreover, $\dim E_\infty^{20} = N^2$ and $\dim E_\infty^{11} = N$.

Proof. The theorem follows immediately from Propositions 3.5 and 4.1.

Theorem 4.6. Assume that either $N = N'$, or $N < N'$ and all of the four quantities $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} a_{2k}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k-1}c_k$, $\frac{1}{2}\sum_{k=N+1}^{N'} b_{2k}c_k$ are integers. Then we have $H^2(M,\mathcal{L}) \cong E_\infty^{20} \oplus E_\infty^{11} \oplus E_\infty^{02}$, where

$$E_\infty^{20} \cong \frac{H^0(X, \Omega_X^2(D))}{\nabla H^0(X, \Omega_X^1\langle D\rangle)},$$
$$E_\infty^{11} \cong$$
$$\frac{H^0\left(X, \sum_{\sum k_\nu = 1} \Omega_X^2\left(\sum_{\nu=1}^{N}(k_\nu+1)D_\nu\right)\right)}{\nabla H^0\left(X, \sum_{\sum k_\nu = 1} \Omega_X^1\left(\sum_{\nu=1}^{N}(k_\nu+1)D_\nu\right)\right) \cap H^0\left(X, \sum_{\sum k_\nu = 1} \Omega_X^2\left(\sum_{\nu=1}^{N}(k_\nu+1)D_\nu\right)\right) + H^0(X, \Omega_X^2(D))},$$
$$E_\infty^{02} \cong \frac{H^0\left(X, \sum_{\sum k_\nu = 2} \Omega_X^2\left(\sum_{\nu=1}^{N}(k_\nu+1)D_\nu\right)\right)}{\nabla H^0\left(X, \sum_{\sum k_\nu = 1} \Omega_X^1\left(\sum_{\nu=1}^{N}(k_\nu+1)D_\nu\right)\right) + H^0\left(X, \sum_{\sum k_\nu = 1} \Omega_X^2\left(\sum_{\nu=1}^{N}(k_\nu+1)D_\nu\right)\right)}.$$

Moreover, $\dim E_\infty^{20} = N^2 - N$, $\dim E_\infty^{11} = 2N - 1$, and $\dim E_\infty^{02} = 1$.

Proof. It suffices to prove the formula for E_∞^{11}. The others are immediate consequences from Propositions 3.6 and 4.2. Since $E_2^{01} = 0$ by Lemma 3.7, the mapping $\nabla : H^1(\mathcal{O}_X) \to H^1(\Omega_X^1\langle D\rangle)$ is injective. Let us consider the composition

$$\nabla^{-1}\left\{\nabla H^0\left(\sum_{\sum k_\nu = 1} \Omega_X^1\left(\sum_{\nu=1}^{N}(k_\nu+1)D_\nu\right)\right) \cap H^0\left(\sum_{\sum k_\nu = 1} \Omega_X^2\left(\sum_{\nu=1}^{N}(k_\nu+1)D_\nu\right)\right)\right\}$$
$$\xrightarrow{\nabla} H^0\left(\sum_{\sum k_\nu = 1} \Omega_X^2\left(\sum_{\nu=1}^{N}(k_\nu+1)D_\nu\right)\right) \longrightarrow \frac{H^0\left(\sum_{\sum k_\nu = 1} \Omega_X^2\left(\sum_{\nu=1}^{N}(k_\nu+1)D_\nu\right)\right)}{\nabla H^0(\Omega_X^1(D)) + H^0(X, \Omega_X^2(D))}. \tag{4.24}$$

Here we need the following

Lemma 4.7.

$$\left\{\nabla H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)\cap H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)\right\}$$
$$\cap\{\nabla H^0(\Omega_X^1(D))+H^0(X,\Omega_X^2(D))\}=\nabla H^0(\Omega_X^1(D)).$$

Proof. The inclusion $\supset$ is obvious. Let us show the converse. Let a be an element in $\nabla H^0(\Omega_X^1(D))$, and b be an element in $H^0(\Omega_X^2(D))$. Then we have $a+b\in H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)$ automatically. Assume that $a+b\in\nabla H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)$. Since $a\in\nabla H^0(\Omega_X^1(D))$, we have $b\in\nabla H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)$. Then we can set $b=\nabla c$ for some $c\in H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)$. By comparing the orders of poles of b and ∇c, we see that $c\in H^0(\Omega_X^1(D))$, which proves Lemma 4.7.

Let us complete the proof of Theorem 4.6. By (4.24) and Lemma 4.7 we have

$$\begin{aligned}
&\nabla H^1(X,\mathcal{O}_X)\\
\cong&\frac{\nabla H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)\cap H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)}{\nabla H^0(X,\Omega_X^1(D))}\\
=&\frac{\nabla H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)\cap H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)}{\nabla H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)\cap H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)\cap\{\nabla H^0(\Omega_X^1(D))+H^0(\Omega_X^2(D))\}}\\
\cong&\frac{\nabla H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)\cap H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)+H^0(\Omega_X^2(D))}{\nabla H^0(\Omega_X^1(D))+H^0(\Omega_X^2(D))}.
\end{aligned}$$

Combining the preceding formula with Proposition 4.1, we have

$$\begin{aligned}
E_\infty^{11}\cong&\frac{H^1(\Omega_X^1\langle D\rangle)}{\nabla H^1(\mathcal{O}_X)}\\
\cong&\frac{H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)}{\nabla H^0\left(\sum_{\sum k_\nu=1}\Omega_X^1\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)\cap H^0\left(\sum_{\sum k_\nu=1}\Omega_X^2\left(\sum_{\nu=1}^N(k_\nu+1)D_\nu\right)\right)+H^0(\Omega_X^2(D))}.
\end{aligned}$$

Theorem 4.6 is proved completely.

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