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On the global existence of unique solutions of differential equations in a Banach space

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Reprinted from Hokkaido Mathematical Journal
Vol. VII No. 1

SAPPORO, JAPAN

March 1978

Hokkaido Mathematical Journal Vol. 7 (1978) p. 58-73

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(Received July 31, 1976: Revised March 22, 1977)

§1. Introduction and results.

Let E be a (real or complex) Banach space with the dual space E^{*}. The norms in E and E^{*} are denoted by $\| \ \|$. Let D be an open set in E and let F be a closed set in E such that $F \subset D$.

In this paper we consider the Cauchy problem

$$(CP) \qquad x' = f(t, x), \qquad x(t_0) = u_0 \in D, \qquad t_0 \in [0, \infty).$$

Here f is a continuous mapping from $[0, \infty) \times D$ into E. By a solution to (CP) or to $(CP; t_0, u_0)$, we mean a continuously differentiable function u from $[t_0, \infty)$into D such that $u(t_0) = u_0$ ond $u'(t) = f(t, u(t))$ for all $t \in [t_0, \infty)$.

As for the existence of a solution of this kind of problem, various results have been established, for example, see F. E. Browder [3], S. Kato [6, 7], N. Kenmochi and T. Takahashi [8], D. L. Lovelady and R. Martin [10], R. Martin [11, 12] and N. Pavel [14].

We say the set F is flow-invariant for f if $u_0 \in F$ implies that $u(t) \in F$ on $[t_0, \infty)$ for the solution to $(CP; t_0, u_0)$.

J. Bony [1] and H. Brezis [2] gave sufficient conditions for the set F to be flow-invariant for f in case E is a finite dimensional Euclidean space and f is a locally Lipschitz continuous function of D into E. The sufficient conditions proposed by them were generalized into a class of functions satisfying some dissipative type condition by R. M. Redheffer [15], and moreover some results were extended by R. Martin [12] to the case of general Banach space. Recently, N. Kenmochi and T. Takahashi [8] gave some simplications and improvements of results of [12].

The purpose of this paper is to give a criterion for the set F to be flow-invariant for f under more general dissipative type conditions on f.

If we consider [8, 12] from the view-point of the notion of flow-invariant sets, the condition of the present paper is weaker than those of [8, 12]. In §5 we shall give some remarks and examples which connect our results with those of others. Our approach is essentially based on the methods in [5, 6, 7, 8].

Let us consider first the following scalar differential equation

$$(1.1) \qquad w'(t) = g\big(t, w(t)\big), \qquad w(t_0) = w_0 .$$

Here $g(t, \tau)$ is a real-valued function defil.ed on $(0, \infty)\times[0, \infty)$ which is measurable in t for each fixed τ, and continuous nondecreasing in τ for each fixed t. We say w is a solution of (1. 1) on an interval $[t_0, t_0+a]$ if w is an absolutely continuous function defined on $[t_0, t_0+a]$ satisfying (1. 1) almost everywhere on $[t_0, t_0+a]$. We assume furthermore that g satisfies the following conditions:

(i) $g(t, 0)=0$ for $a.e.\, t\in(0, \infty)$, and for each bounded subset B of $(0, \infty)\times[0, \infty)$ there exists a function α_B defied on $(0, \infty)$ such that

$$|g(t, \tau)| \leqq \alpha_B(t) \qquad \text{for all } (t, \tau)\in B$$

and α_B is Lebesgue integrable on (t_1, t_2) for each $t_2>t_1>0$.

(ii) For each $T\in[0, \infty)$, $w\equiv 0$ is the only solution of (1. 1) on $[0, T]$ satisfying the condition $w(0)=(D^+w)(0)=0$, where D^+ denotes the right-sided derivative of w.

From the above conditions (i) and (ii) we see that for each $t_1, t_2\in[0, \infty)$ with $t_2<t_1$, $w\equiv 0$ is the only solution of (1. 1) on $[t_1, t_2]$ satisfying $w(t_1)=(D^+w)(t_1)=0$.

We define the functional $[\ ,\ ]: E\times E\to R$ by

$$[x, y] = \lim_{h\to -0} (\|x+hy\| - \|x\|)/h .$$

Now, let f be a mapping from $[0, \infty)\times D$ into E and consider the following conditions:

(K_1) f is continuous from $[0, \infty)\times D$ into E.

(K_2) $\big[x-y, f(t, x)-f(t, y)\big] \leqq g(t, \|x-y\|)$

for all x, y in D and for $a.e.\, t\in(0, \infty)$.
Then we have the following main result.

THEOREM. *Suppose that f satisfies the conditions* (K_1) and (K_2). *Then the set F is flow-invariant for f if and only if*

$$(1.2) \qquad \liminf_{h\to +0} p\big(x+hf(t, x), F\big)/h = 0$$

for all $(t, x)\in[0, \infty)\times F$, where $d(z, F)$ denotes the distance from $z\in E$ to F.

The author would like to express his hearty thanks to Professor T. Shirota for the kind criticism. The author thanks also Mr. N. Kenmochi and Mr. T. Takahashi for usefull suggestions.

§ 2. Some lemmas.

In this section we give some lemmas without proof. For proofs of Lemmas 2. 1–2. 3 see [6]. In Lemmas 2. 1–2. 5 we assume that g satisfies the conditions (i) and (ii) stated in § 1.

LEMMA 2. 1. *Let $t_1, t_2 \in [0, \infty)$ be such that $t_1 < t_2$ and let $\{w_n\}$ be a sequence of functions from $[t_1, t_2]$ to $[0, \infty)$ converging uniformly on $[t_1, t_2]$ to a function w_0. Let $M > 0$ be such that*

$$|w_n(t) - w_n(s)| \leqq M|t-s| \qquad \text{for all } s, t \in [t_1, t_2] \text{ and } n \geqq 1 .$$

Suppose furthermore that for each $n \geqq 1$ and $\sigma_n \geqq 0$ with $\sigma_n \downarrow 0$

$$w_n'(t) \leqq g\big(t, w_n(t)\big) + \sigma_n$$

for $t \in (t_1, t_2)$ such that $w_n'(t)$ exists. Then

$$w_0'(t) \leqq g\big(t, w_0(t)\big) \qquad \text{for a.e. } t \in (t_1, t_2) .$$

LEMMA 2. 2. *Let $t_1, t_2 \in [0, \infty)$ be such that $t_1 < t_2$ and let Φ be a uniformly bounded family of functions from $[t_1, t_2]$ into $[0, \infty)$ with the property that, for each $s, t \in [t_1, t_2]$ and $w \in \Phi$, $|w(t) - w(s)| \leqq M|t-s|$ for some constant $M > 0$.*
Let $w_0 = \sup\{w\,;\, w \in \Phi\}$ and let $\sigma \geqq 0$ be a constant. Suppose furthermore that for each $w \in \Phi$

$$w'(t) \leqq g\big(t, w(t)\big) + \sigma$$

for $t \in (t_1, t_2)$ such that $w'(t)$ exists. Then

$$w_0'(t) \leqq g\big(t, w_0(t)\big) + \sigma \qquad \text{for a.e. } t \in (t_1, t_2) .$$

LEMMA 2. 3. *Let w be an absolutely continuous function from $[t_1, t_2]$ $(0 \leqq t_1 < t_2 < \infty)$ to $[0, \infty)$ such that $w(t_1) = (D^+ w)(t_1) = 0$ and*

$$w'(t) \leqq g\big(t, w(t)\big) \qquad \text{for a.e. } t \in (t_1, t_2) .$$

Then $w \equiv 0$ on $[t_1, t_2]$.

Let $t_0 > 0$. We define a functian g_{t_0} by

$$g_{t_0}(\mathrm{t}, \tau) = \begin{cases} g(t, \tau) & (t \geqq t_0,\ \tau \geqq 0) \\ 0 & (\text{otherwise}) . \end{cases}$$

For each $t_0 > 0$ we consider the following scalar differential equation

$$w'(t) = g_{t_0}\big(t, w(t)\big), \qquad w(t_0) = w_0 . \tag{2. 1}$$

Concerning this equation we give the following two lemmas which are used in the proof of the Theorem.

LEMMA 2. 4. *Let $t_0>0$ and suppose that the maximal solution $m_{t_0}(\cdot, w_0)$ of* (2. 1) *through (t_0, w_0) exists over an interval $[t_0, t_0+a]$. Then there exists a $\delta>0$ such that* (2. 1) *has a maximal solution $m_{t_0}(\cdot, \sigma)$ for each σ, $w_0\leqq\sigma< w_0+\delta$ on $[t_0, t_0+a]$ with $m_{t_0}(t_0, \sigma)=\sigma$. Moreover, $m_{t_0}(\cdot, \sigma)\to m_{t_0}(\cdot, w_0)$ as $\sigma\to w_0+0$, uniformly on $[t_0, t_0+a]$.*

For a proof see [4, Theorem 2. 4, p. 47].

LEMMA 2. 5. *Suppose that the hypothesis of Lemma 2. 4 are satisfied, and let w be an absolutely continuous function on $[t_0, t_0+a]$. Suppose furthermore that*

$$w'(t)\leqq g_{t_0}\big(t, w(t)\big) \qquad \text{for a.e.}\, t\in[t_0, t_0+a]\,.$$

Then $w(t_0)\leqq w_0$ implies that $w(t)\leqq m_{t_0}(t, w_0)$ on $[t_0, t_0+a]$.

For a proof of the above lemma see [9, Theorem 1. 10. 4, p. 43].

The following lemma on the functional $[\ ,\ ]: E\times E\to R$ is well-known.

LEMMA 2. 6. *Let x, y and z be in E. Then the functional $[\ ,\ ]$ has the following properties:*

(1) $|[x, y]|\leqq\|y\|$.

(2) $[x, y+z]\leqq[x, y]+\|z\|$.

(3) $[x, y]\leqq[x, y-z]+\|z\|$.

(4) *Let u be a function from a real interval I into E such that $u'(t)$ and $\dfrac{d}{dt}\|u(t)\|$ exist for a.e. $t\in I$. Then*

$$\frac{d}{dt}\big\|u(t)\big\|=\big[u(t), u'(t)\big] \qquad \text{for a.e.}\, t\in I\,.$$

§ 3. Local existence.

Assume that conditions (K_1), (K_2) and (1. 2) are satisfied. Then we have the following important

PROPOSITION 3. 1. *Let $(t_0, u_0)\in[0, \infty)\times F$ and let M, r_0 and T_1 be positive numbers such that $S(u_0, 2r_0)\subset D$ and*

$$\big\|f(t, x)\big\|\leqq M \qquad \text{for all } (t, x)\in[t_0, t_0+2T_1]\times S(u_0, 2r_0)\,.$$

Then $(CP;\ t_0, u_0)$ has a unique solution u on $[t_0, t_0+T_0]$ such that $u(t)\in F\cap S(u_0, r_0)$ for all $t\in[t_0, t_0+T_0]$, where $T_0=\text{Min}\,\{r_0/(2M), T_1/2\}$ and $S(u_0, r_0)=\{v;\ \|v-u_0\|\leqq r_0\}$.

In order to prove this proposition, under the same assumptions and notations as in the proposition for each $\varepsilon>0$ sufficiently small we consider the set H_ε of all pairs (z, a) such that $t_0<a\leqq t_0+T_0$ and $z=z(t)$ is a function from $[t_0, a]$ into $S(u_0, 2r_0)$ satisfying the following conditions:

(i) $z(t_0)=u_0$ and $z(a)\in F$;

(ii) $\|z(t)-z(s)\|\leqq 2M|t-s|$ for all $s, t\in[t_0, a]$;

(iii) $\|z'(t)-f(t, z(t))\|\leqq\varepsilon$ for $a.e.\, t\in[t_0, a]$;

(iv) every subinterval of $[t_0, a]$, with length being $\geqq\varepsilon$, contains at least one point τ such that $z(\tau)\in F$.

Also, define an order "$\leqq$" in H_ε by the following manner: $(z_1, a_1)\leqq(z_2, a_2)$ if and only if $a_1\leqq a_2$ and $z_1(t)=z_2(t)$ for all $t\in[t_0, a_1]$. Then H_ε becomes a partially ordered set and we have

LEMMA 3.1. *H_ε is non-empty and inductive with respect to the order "$\leqq$".*

PROOF. For simplicity we may assume that $t_0=0$. Let $(t^0, v_0)\in[0, 2T_0]\times(F\cap S(u_0, r_0))$. Now, take a number δ so that

$$0<\delta<\operatorname{Min}\{r, \varepsilon_0, M\}$$

and

$$\|f(t, x)-f(t^0, v_0)\|\leqq\varepsilon/2 \tag{3.1}$$

whenever $t^0\leqq t\leqq t^0+\delta$ and $\|x-v_0\|\leqq\delta$, and by using (1.2), take a number h_1 with $0<h_1<\operatorname{Min}\{\delta/(\delta+2M), \delta\}$ having the property: for each $h\in(0, h_1]$ there is $v_h\in F$ such that

$$\|(v_h-v_0)/h-f(t^0, v_0)\|\leqq\delta/2\,. \tag{3.2}$$

Then it follows from (3.2) that

$$\begin{aligned}\|v_h-v_0\|/h&\leqq\delta/2+\|f(t^0, v_0)\|\\&\leqq\delta/2+M\leqq\delta/2h\end{aligned} \tag{3.3}$$

for all $h\in(0, h_1]$. Therefore, defining

$$Q(t)=Q(t\,;\,v_0, t^0, h)=v_0+(t-t^0)(v_h-v_0)/h \tag{3.4}$$

for $t\in[t^0, t^0+h]$ with $h\in(0, h_1]$, we have by (3.3)

$$\|Q(t)-v_0\|\leqq\|v_h-v_0\|\leqq\delta/2<r_0$$

and hence $Q(t)\in S(u_0, 2r_0)$ for all $t\in[t^0, t^0+h]$. In particular $Q(t^0)=v_0\in F$ and $Q(t^0+h)=v_h\in F$. Besides it follows from (3.2) and (3.3) that

$$\begin{aligned}\big\|Q(t)-Q(s)\big\| &= |t-s|\,\big\|v_h-v_0\big\|/h \\ &\leqq (\delta/2+M)|t-s| \leqq 2M|t-s|\end{aligned}$$

and

$$\begin{aligned}\Big\|Q'(t)-f\big(t, Q(t)\big)\Big\| &= \Big\|(v_h-v_0)/h-f\big(t, Q(t)\big)\Big\| \\ &\leqq \Big\|(v_h-v_0)/h-f(t^0, v_0)\Big\|+\Big\|f(t^0, v_0)-f\big(t, Q(t)\big)\Big\| \\ &\leqq \delta/2+\varepsilon/2 \leqq \varepsilon\end{aligned}$$

for all $t, s\in[t^0, t^0+h]$. Thus $(Q, h)\in H_\varepsilon$ if we take $t^0=0$ and $v_0=u_0$, so that $H_\varepsilon \neq \phi$.

Next we show that H_ε is inductive. Let $L=\{(z_\lambda, a_\lambda)\,;\ \lambda\in\Lambda\}$ be any totally ordered subset of H_ε, and put

$$\bar{a}=\sup\{a_\lambda\,;\ \lambda\in\Lambda\}\,.$$

If $\bar{a}=a_\lambda$ for some $\lambda\in\Lambda$, then (z_λ, a_λ) is clearly an upper bound for L. In case $a_\lambda<\bar{a}$ for all $\lambda\in\Lambda$, define a function $z:[0, \bar{a})\to S(u_0, 2r_0)$ by putting

$$z(t)=z_\lambda(t) \qquad \text{if } t<a_\lambda\,.$$

Then it is easy to see that z satisfies the properties (ii), (iii) and (iv) on $[0, \bar{a})$. Since $\|z(a_\lambda)-z(a_\gamma)\| \leqq 2M|a_\lambda-a_\gamma|$ for $\lambda, \gamma\in\Lambda$, the limit $z(\bar{a})=\lim_{t\uparrow\bar{a}} z(t)$ exists and $z(\bar{a})\in F$. If we denote again by z the function extended on $[0, \bar{a}]$ by the limit, the pair $(z, \bar{a})$ is clearly an upper bound for L. Thus H_ε is inductive. Q. E. D.

LEMMA 3. 2. *H_ε has a maximal element $(z_\varepsilon, a_\varepsilon)$ such that $a_\varepsilon=t_0+T_0$.*

PROOF. Since H_ε is inductive by Lemma 3. 1, it has at least one maximal element $(z_\varepsilon, a_\varepsilon)$. Moreover $a_\varepsilon=t_0+T_0$. In fact, suppose for contradiction that $a_\varepsilon<t_0+T_0$. Then $z_\varepsilon(a_\varepsilon)\in F\cap S(u_0, r_0)$ by (i) and (ii), and hence we can extend z_ε to the interval $[t_0, a_\varepsilon+h]$ by means of $Q(t)=Q(t\,;\ z_\varepsilon(a_\varepsilon), a_\varepsilon, h)$ on $[a_\varepsilon, a_\varepsilon+h]$, where h is a sufficiently small positive number and $Q(t)$ is the function as constructed in the previous lemma. This contradicts the fact that $(z_\varepsilon, a_\varepsilon)$ is maximal. Q. E. D.

PROOF of PROPOSITION 3. 1. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n\downarrow 0$ as $n\to\infty$ and let (z_n, t_0+T_0) be a maximal element in H_{ε_n} for each n.

We show that the sequence $\{z_n\}$ converges uniformly on $[t_0, t_0+T_0]$. For simplicity we assume again that $t_0=0$. Let $w_{mn}(t)=\|z_m(t)-z_n(t)\|$ for $t\in[0, T_0]$ and $m>n\geqq 1$, and remark first that $w'_{mn}(t)$ exists for $a.e.\, t\in[0, T_0]$ since

$$(3.5)\qquad \left|w_{mn}(t)-w_{mn}(s)\right|\leqq 4M|t-s| \qquad \text{for all } s, t\in[0, T_0].$$

Thus we have by Lemma 2.6 and the condition (K_2)

$$\begin{aligned}(3.6)\qquad w'_{mn}(t) &= \left[z_m(t)-z_n(t), z'_m(t)-z'_n(t)\right]\\ &\leqq g\left(t, \left\|z_m(t)-z_n(t)\right\|\right)+\left\|z'_m(t)-f\left(t, z_m(t)\right)\right\|\\ &\qquad +\left\|z'_n(t)-f\left(t, z_n(t)\right)\right\|\\ &\leqq g\left(t, w_{mn}(t)\right)+2\varepsilon_n\end{aligned}$$

for $a.e.\, t\in(0, T_0]$ and $m>n\geqq 1$.
Let $w_n(t)=\sup_{m>n}\{w_{mn}(t)\}$ for $t\in[0, T_0]$. Then $w_n(0)=0$ for all $n\geqq 1$. It thus follows from (3.5), (3.6) and Lemma 2.2 that

$$(3.7)\qquad \left|w_n(t)-w_n(s)\right|\leqq 4M|t-s| \qquad \text{for all } s, t\in[0, T_0]$$

and

$$(3.8)\qquad w'_n(t)\leqq g\left(t, w_n(t)\right)+2\varepsilon_n \qquad \text{for } a.e.\, t\in(0, T_0].$$

Since $0\leqq w_n(t)\leqq w_n(0)+4Mt\leqq 4MT_0$ for $t\in[0, T_0]$ and $n\geqq 1$, the sequence $\{w_n\}$ is equicontinuous and uniformly bounded, and hence it has a subsequence converging uniformly on $[0, T_0]$ to a function $w=w(t)$, and obviously $w(0)=0$. From (3.8) and Lemma 2.1 we have

$$w'(t)\leqq g\left(t, w(t)\right) \qquad \text{for all } a.e.\, t\in(0, T_0].$$

We show next that $(D^+w)(0)=0$. For each $\varepsilon>0$ we can fined a $\delta>0$ such that

$$\left\|f(t, x)-f(0, u_0)\right\|<\varepsilon \qquad \text{for all } (t, x)\in[0, \delta]\times S(u_0, \delta).$$

Let $\delta_0=\mathrm{Min}\,\{\delta, \delta/2M\}$. Since $\|z_n(t)-u_0\|\leqq 2Mt\leqq\delta$ by (ii),

$$\left\|f\left(t, z_m(t)\right)-f\left(t, z_n(t)\right)\right\|<2\varepsilon$$

whenever $m>n\geqq 1$ and $t\in[0, \delta_0]$. From Lemma 2.6 we have

$$\begin{aligned} w'_{mn}(t) &= \left[z_m(t)-z_n(t), z'_m(t)-z'_n(t)\right]\\ &\leqq \left\|z'_m(t)-f\left(t, z_m(t)\right)\right\|+\left\|z'_n(t)-f\left(t, z_n(t)\right)\right\|\\ &\qquad +\left\|f\left(t, z_m(t)\right)-f\left(t, z_n(t)\right)\right\|\\ &<2(\varepsilon+\varepsilon_n)\end{aligned}$$

for $a.e. t \in [0, \delta_0]$, and hence, by integrating the above inequality, $0 \leqq w_{mn}(t) \leqq 2(\varepsilon+\varepsilon_n)t$, whence $(D^+ w)(0)=0$. Consequently, from Lemma 2.3 we deduce now that $w \equiv 0$, and this implies that the sequencs $\{z_n\}$ is uniformly convsrgent on $[0, T_0]$. The limit $z=z(t)$ of of this sequence satisfies

$$z(t) = u_0 + \int_0^t f\big(s, z(s)\big)\, ds \qquad \text{for } t \in [0, T_0].$$

Thus $z=z(t)$ is a solution to $(CP;\ 0, u_0)$ and $z(t) \in F \cap S(u_0, r_0)$ on $[0, T_0]$. Since the uniqueness of a solution to $(CP;\ 0, u_0)$ is well-known (cf. [6, Theorem 1], the proof of Proposition 3.1 is complete.

§4. Proof of Theorem.

Before proving Theorem, we prepare the following two lemmas.

LEMMA 4.1. *Let b be any positive number and let $u_0 \in F$. Then there exists a $\delta > 0$ for which $(CP;\ s, u_0)$ has a solution u on $[s, s+\delta]$ for each $s \in [0, b]$ such that $u(t) \in F$ for all $t \in [s, s+\delta]$.*

PROOF. We first see from the continuity of f on $[0, \infty) \times D$ that there exist positive constants r_0 and M such that

$$\big\|f(t, x)\big\| \leqq M \qquad \text{for all } (t, x) \in [0, 4b] \times S(u_0, 2r_0).$$

Let $\delta = \operatorname{Min}\{3b/4, r_0/2M\}$. Then, by Proposition (3.1), $(CP;\ s, u_0)$ has a unique solution u on $[s, s+\delta]$ for each $s \in [0, b]$ such that

$$u(t) \in F \text{ for all } t \in [s, s+\delta]. \qquad \text{Q.E.D.}$$

LEMMA 4.2. *Let $t_0 > 0$ and $u_0 \in F$. Suppose that T is a positive number such that $(CP;\ t_0, u_0)$ has a solution u such that $u(t) \in F$ for all $t \in [t_0, t_0+T]$. Then there exists a positive number r having the property: for each $v_0 \in F \cap S(u_0, r)$, $(CP;\ t_0, v_0)$ has a solution v such that $v(t) \in F$ for all $t \in [t_0, t_0+T]$.*

PROOF. By the condition (ii) in §1, $w \equiv 0$ is a maximal solution on $[t_0, t_0+T]$ of (2.1) with $w(t_0)=(D^+ w)(t_0)=0$. It thus follows from Lemma 2.4 that there exists a positive number δ such that (2.1) has a maximal solution $m_{t_0}(\cdot, \sigma)$ for each σ, $0 \leqq \sigma < \delta$ on $[t_0, t_0+T]$ with $m_{t_0}(t_0, \sigma)=\sigma$. Moreover, $m_{t_0}(\cdot, \sigma)$ converges to 0 uniformly on $[t_0, t_0+T]$ as $\sigma \to +0$. Since the set $\{(t, u(t));\ t \in [t_0, t_0+T]\}$ is compact in $[t_0, t_0+T] \times D$, there exist positive constants ρ and M such that

$$\text{(4.1)} \qquad \big\|f(t, x)\big\| \leqq M \qquad \text{for all } t \in [t_0, t_0+T] \text{ and } x \in S\big(u(t), \rho\big).$$

Here we may choose ρ such that $S(u(t), \rho) \subset D$ for all $t \in [t_0, t_0+T]$. Consequently, we can choose a positive number r such that $0 < r < \operatorname{Min}\{\delta, \rho\}$ and

$$(4.2)\qquad \left|m_{t_0}\left(t, \|v_0-u_0\|\right)\right|<\rho$$

for all $(t, v_0)\in[t_0, t_0+T]\times(F\cap S(u_0, r))$.
By virtue of Proposition 3. 1, $(CP; t_0, v_0)$ has a unique local solution v with $v(t)\in F$ on some interval $[t_0, t_0+T(v_0))$ for each $v_0\in F\cap S(u_0, r)$. Assume that $T(v_0)\leqq T$ and $[t_0, t_0+T(v_0))$ is a maximal interval of existence of v with the property that $v(t)\in F$ on $[t_0, t_0+T(v_0))$.
Since $\|v(t)-u(t)\|$ is absolutely continuous on each closed interval $[t_0, t_0+T(v_0))$ we have

$$\frac{d}{dt}\left\|v(t)-u(t)\right\|=\left[v(t)-u(t), f\left(t, v(t)\right)-f\left(t, u(t)\right)\right]$$
$$\leqq g\left(t, \left\|v(t)-u(t)\right\|\right)$$

for $a.e.\, t\in[t_0, t_0+T(v_0))$. Hence we have by Lemma 2. 5

$$\left\|v(t)-u(t)\right\|\leqq m_{t_0}\left(t, \left\|v_0-u_0\right\|\right)\qquad \text{for all } t\in\left[t_0, t_0+T(v_0)\right).$$

It thus follows from (4. 1) and (4. 2) that

$$\left\|f\left(t, v(t)\right)\right\|\leqq M\qquad \text{for all } t\in\left[t_0, t_0+T(v_0)\right),$$

and this implies that $\lim_{t\to T(v_0)} v(t)$ exists in F. Applying Proposition 3. 1 once again we have a contradiction. Thus $T<T(v_0)$ and the proof is complete.

PROOF of the THEOREM. The method of the following proof is essentially based on that of [8].

Let $(t_0, u_0)\in[0, \infty)\times F$. Then, by Proposition 3. 1, $(CP; t_0, u_0)$ has a unique local solution u on some interval $[t_0, t_1]$ such that $u(t)\in F$ for all $t\in[t_0, t_1]$. We note that $t_1>0$ and $u(t_1)\in F$. Let b be any positive number such that $b>t_1$. Then, by Lemma 4. 1, there exists a positive constant δ such that $(CP; s, u(t_1))$ has a solution v with $v(t)\in F$ on $[s, s+\delta]$ for each $s\in(0, b]$. We note here that if $s=0$, then we can not apply Lemmas 2. 4, 2. 5 and 4. 2 in the following discussion. Therefore, we omit the case $s=0$.

Now, let C be a connected component in F containing $u(t_1)$ and let

$$G_s=\left\{x\in C; (CP; s, x) \text{ has a solution } v \text{ such that } v(t)\in F \text{ for } t\in[s, s+\delta]\right\}\qquad \text{for each } s\in(0, b].$$

Then G_s is not empty since $u(t_1)\in G_s$ for each $s\in(0, b]$ by Lemma 4. 1. Moreover, G_s is relatively open in C for each fixed $s\in(0, b]$ by Lemma 4. 2. We show that G_s is also relatively closed in C. For this, let $\{x_n\}$ be any sequence in G_s which converges to $x \in C$ and let v_n be a solution to $(CP;$ $s, x_n)$ on $[s, s+\delta]$. Then

$$\frac{d}{dt}\left\|v_n(t)-v_m(t)\right\| = \left[v_n(t)-v_m(t), f\left(t, v_n(t)\right)-f\left(t, v_m(t)\right)\right]$$
$$\leqq g\left(t, \left\|v_n(t)-v_m(t)\right\|\right)$$

for $a.e. t \in [s, s+\delta]$. Thus we have by Lemma 2. 5

$$\left\|v_n(t)-v_m(t)\right\| \leqq m_s\left(t, \left\|x_n-x_m\right\|\right)$$

for all $t \in [s, s+\delta]$ and for sufficiently large positive integers n and m. Since $\lim_{n,m\to\infty} \|x_n-x_m\|=0$, the sequence $\{v_n\}$ converges uniformly on $[s, s+\delta]$ to a function v by Lemma 2. 4, and clearly v is a solulion to $(CP; s, x)$ on $[s, s+\delta]$ and hence $x \in G_s$. Consequently, $G_s = C$ for all $s \in (0, b]$. In particular, $u(t_1) \in G_{t_1} = C$ and hence $(CP; t_1, u(t_1))$ has a solution v on $[t_1, t_1+\delta]$ such that $v(t) \in F$ for $t \in [t_1, t_1+\delta]$. If $t_1+\delta < b$, then $(CP; t_1+\delta, v(t_1+\delta))$ has a solution w on $[t_1+\delta, t_1+2\delta]$ such that $w(t) \in F$ for $t \in [t_1+\delta, t_1+2\delta]$, because $v(t_1+\delta) \in G_{t_1+\delta} = C$. Obviously

$$\hat{v}(t) = \begin{cases} u(t) & (t_0 \leqq t \leqq t_1) \\ v(t) & (t_1 \leqq t \leqq t_1+\delta) \\ w(t) & (t_1+\delta \leqq t \leqq t_1+2\delta) \end{cases}$$

is a solution to $(CP; t_0, u_0)$ on $[t_0, t_1+2\delta]$. Repeating this argument we see that $(CP; t_0, u_0)$ has a solution on $[t_0, b]$. Since b was arbitrary number such that $b > t_1$, it is proved that $(CP; t_0, u_0)$ has a solution u^* on $[t_0, \infty)$ such that $u^*(t) \in F$ for all $t \in [t_0, \infty)$. Thus the sufficiency is proved.

Conversely, suppose that the set F is flow-invariant for f and let u be a solution to $(CP; t_0, u_0)$ on $[t_0, \infty)$ such that $u(t) \in F$ for all $t \in [t_0, \infty)$. Then

$$d\left(u_0+hf(t_0, u_0), F\right)/h \leqq \left\|\left(u(t_0+h)-u(t_0)\right)/h - f(t_0, u_0)\right\|$$

and

$$\left\|\left(u(t_0+h)-u(t_0)\right)/h - f(t_0, u_0)\right\| \to 0 \text{ as } h \to +0\,.$$

Hence the necessity follows. Q. E. D.

§ 5. Remarks and examples.

In this section we give some remarks and examples which connect our results with those of others.

REMARK 1. In the previous paper [6] we used the functional

$$\langle x, y\rangle = \big([x, y]-[x, -y]\big)/2 .$$

But it can be easily seen that $[x, y]\leqq\langle x, y\rangle$ for each x, y in E. Hence the Theorem of the present paper gives an improvement of Theorem 2 in [6].

Let J be the duality mapping from E into 2^{E^*} (i. e., for each x in E, $J(x)=\{x^*\in E^* ;\ x^*(x)=\|x\|^2=\|x^*\|^2\}$.
For each x, y in E, define

$$\langle x, y\rangle_i = \inf\left\{\mathrm{Re}\big(x^*(y)\big) ;\ x^*\in J(x)\right\} .$$

Then for each $x\neq 0$ and y in E, $[x, y]=\langle x, y\rangle_i/\|x\|$ (see [11]). Thus the condition (K_2) is equivalent to the following:

$$\big\langle x-y,\ f(t, x)-f(t, y)\big\rangle_i \leqq \|x-y\|\, g(t, \|x-y\|) \tag{5. 1}$$

for all $x, y\in D$ and for $a. e.\ t\in(0, \infty.)$
We note also that Proposition 3. 1 remains valid even if F is a relatively closed subset of D. Hence, this fact and (5. 1) imply that our Theorem gives a generalization of Theorems 3 and 4 in R. M. Redheffer [15] into a general Banach space.

REMARK 2. Let β be a real-valued function defined on $(0, \infty)$ satisfying the following conditions:

(β_1) For each $t_1, t_2\in(0, \infty)$ with $t_1<t_2$, β is Lebesgue integrable on (t_1, t_2).

(β_2) For each $t>0$, $\limsup_{\varepsilon\to+0}\left[\varepsilon\exp\left(\int_\varepsilon^t\beta(\tau)\,d\tau\right)\right]<+\infty$.

The condition (β_2) was considered by C. V. Pao [13] to prove the uniqueness of solutions to $(CP;\ 0, u_0)$.
If $g(t, \tau)=\beta(t)\tau$, then the conclusion of our Theorem remains valid. In fact, it is obvious that this function $\beta(t)\tau$ satisfies the condition (i) in § 1. To prove that $\beta(t)\tau$ satisfies also the condition (ii) in § 1, let w be a solution of the equation $w'(t)=\beta(t)w(t)$ on $[0, T]$ satisfying $w(0)=(D^+w)(0)=0$. Then for each $\varepsilon>0$, we have

$$\begin{aligned} 0\leqq w(t) &= w(\varepsilon)\exp\left(\int_\varepsilon^t\beta(\tau)\,d\tau\right)\\ &=\varepsilon\exp\left(\int_\varepsilon^t\beta(\tau)\,d\tau\right)\big(w(\varepsilon)-w(0)\big)/\varepsilon\end{aligned}$$

for $t\in[\varepsilon, T]$. This implies that $w\equiv 0$ on $[0, T]$. Thus $\beta(t)\tau$ satisfies (i) and (ii) in § 1. However, the function $\beta(t)\tau$ need not be nondecreasing in τ for fixed t. The nondecreasing nature is used only in establishing

Lemma 2.3 (see [6]) which is valid for $g(t, \tau)=\beta(t)\tau$. Thus our result extends those of [10, 11, 14] when $g(t, \tau)=\beta(t)\tau$.

REMARK 3. Recently, N. Kenmochi and T. Takahashi [8] proved the following theorem which gives an improvement of [12].

THEOREM A. *Let F be a closed subset of E. Suppose that f satisfies the following conditions*:

$$f \text{ is continuous from } [0, \infty)\times F \text{ into } E. \tag{5.2}$$

$$\langle x-y, f(t, x)-f(t, y)\rangle_i \leqq \omega(t)\|x-y\|^2 \tag{5.3}$$

for all (t, x), $(t, y)\in[0, \infty)\times F$, where ω is a real-valued continuous function defined on $[0, \infty)$. Suppose furthermore that

$$\liminf_{h\to+0} d\big(x+hf(t, x), F\big)/h = 0 \tag{5.4}$$

for all $(t, x)\in[0, \infty)\times F$. Then $(CP;\ 0, u_0)$ has a unique global solution u defined on $[0, \infty)$ for each $u_0\in F$.

This result is intimately related to the notion of flow-invariant sets. If we consider this theorem from the view-point of the notion of flow-invariant sets we have the following

THEOREM B. *Let D be an open set in E and let F be a closed set in E such that $F\subset D$. Suppose that f satisfies* (5.4) *and the following conditions*:

$$f \text{ is continuous from } [0, \infty)\times D \text{ into } E. \tag{5.5}$$

$$\langle x-y, f(t, x)-f(t, y)\rangle_i \leqq \omega(t)\|x-y\|^2 \tag{5.6}$$

for all (t, x), $(t, y)\in[0, \infty)\times D$. Then the set F is flow-invariant for f.

Since (5.1) implies (5.6), our Theorem contains Theorem B.

The following examples show that the condition (K_2) is strictly more general than (5.6).

EXAMPLE 1. Let $a(t)$ be the function defined by

$$a(t)=\begin{cases} t^{3/2} & (0\leqq t\leqq \rho) \\ \rho^{3/2} & (t\geqq \rho), \end{cases}$$

where ρ is a constant such that $\rho>1$. Consider the function G defined by

$$G(t, u)=\begin{cases} \dfrac{\sqrt[3]{u}}{1+\sqrt[3]{a(t)}}+b(t)u^3 & \big(t\geqq 0,\ u\geqq a(t)\big) \\[2ex] \dfrac{\sqrt[3]{a(t)}}{1+\sqrt[3]{a(t)}}+b(t)u^3 & \big(t\geqq 0,\ u< a(t)\big), \end{cases}$$

where b is a real-valued continuous function from $[0, \infty)$ into $(-\infty, 0]$. It is easily verified that the function G satisfies the following inequality:

$$\begin{aligned} (5.7) \qquad & \left|u-v+h\left(G(t, u)-G(t, v)\right)\right| \\ & \geqq \left(1+h/3\sqrt[3]{a(t)^2}\left(1+\sqrt[3]{a(t)}\right)\right)|u-v| \end{aligned}$$

for all $h \leqq 0$, $t>0$ and $u, v \in(-\infty, \infty)$.

Let us take as E the Banach space ℓ^∞ of bounded sequences of real numbers. For eabh $x=(x_n)$ and $t \geqq 0$, define $f(t, x)=(G(t, x_n))$. Then f is continuous from $[0, \infty) \times E$ into E. For each $x=(x_n)$, $y=(y_n)$ in E, $h<0$, we have by (5.7)

$$\begin{aligned} & \sup_n \left|x_n-y_n+h\left(G(t, x_n)-G(t, y_n)\right)\right|-\sup_n |x_n-y_n| \\ & \geqq \frac{h}{3\sqrt[3]{a(t)^2}\left(1+\sqrt[3]{a(t)}\right)} \sup_n |x_n-y_n| . \end{aligned}$$

This implies that

$$\left[x-y, f(t, x)-f(t, y)\right] \leqq \|x-y\| / 3\sqrt[3]{a(t)^2}\left(1+\sqrt[3]{a(t)}\right)$$

for all x, y in E and $t>0$. Let $\beta(t)=1/3\sqrt[3]{a(t)^2}\left(1+\sqrt[3]{a(t)}\right)$. Then $\int_0^\rho \beta(\tau)\, d\tau = \int_0^\rho dt/3t(1+\sqrt{t})=+\infty$. However, it is easy to see that $\beta(t)$ satisfies the condition (β_1) in Remark 2. Moreover, by a simple calculation, we have

$$\begin{aligned} & \varepsilon \exp\left(\int_\varepsilon^t \beta(\tau)\, d\tau\right) \\ & \leqq \begin{cases} (\varepsilon^2 t)^{1/3} & (0<\varepsilon<t \leqq \rho) \\ (\varepsilon^2 t)^{1/3} \exp\left((t-\rho)/3\rho(1+\sqrt{\rho})\right) & (0<\varepsilon<\rho<t) . \end{cases} \end{aligned}$$

Thus, $\beta(t)$ satisfies also the condition (β_2).
Consequently, for each $(t_0, u_0) \in [0, \infty) \times E$, $(CP;\, t_0, u_0)$ has a unique global solution for the above defined f.

On the other hand, for each $x=(x_n)$ and $y=(y_n)$ in E such that $x_1>y_1>0$ and $x_n=y_n=0$ for $n \geqq 2$,

$$\begin{aligned} & \left[x-y, f(0, x)-f(0, y)\right] \\ & =\left\{\frac{1}{\sqrt[3]{x_1^2}-\sqrt[3]{x_1 y_1}+\sqrt[3]{y_1^2}}-b(0)\left(x_1^2-x_1 y_1+y_1^2\right)\right\}\|x-y\| . \end{aligned}$$

Hence we can not apply [8, 10, 11, 12, 13] to this example for the Cauchy problem $(CP;\, 0, u_0)$.

EXAMPLE 2. Next, let us take as E the Banach space ℓ^p $(1<p<\infty)$ of sequences of real numbers. Let $a(t)$ be as in Example 1 and let $M=(\sum_{n=1}^{\infty} 1/n^p)^{1/p}$. For each $x=(x_n)\in E$, define

$$f_n(t, x)=\begin{cases} \dfrac{\sqrt[3]{x_n}}{n(1+\sqrt[3]{a(t)})}-b(t)\, x_n & \big(t\geqq 0,\ x_n\geqq a(t)\big)\\[2mm] \dfrac{\sqrt[3]{a(t)}}{n(1+\sqrt[3]{a(t)})}-b(t)\, x_n & \big(t\geqq 0,\ x_n< a(t)\big). \end{cases}$$

Here $b(t)$ is a real-valued continuous function defined on $[0, \infty)$ satisfying $b(t)>M/\sqrt{\rho}$ for all $t\geqq 0$.
Define $f(t, x)=(f_n(t, x))$ for $(t, x)\in[0, \infty)\times E$. Then f is continuous from $[0, \infty)\times E$ into E. Let

$$F=\Big\{x\,;\ E\ni x=(x_n) \text{ such that } x_n\geqq 0 \text{ for } n\geqq 1 \text{ and } \|x\|\leqq\rho\Big\}.$$

Then F is closed in E. We shall show that the mapping f does not satisfy (5. 3) but does satisfy all the conditions of our Theorem. For this note that

$$[x, y]=\sum_{n=1}^{\infty}\operatorname{sgn}(x_n)|x_n|^{p-1}y_n/\|x\|^{p-1} \tag{5. 8}$$

for all $x\neq 0$ and y in E.

Using (5. 8) we can verify easily that

$$\Big[x-y,\, f(t, x)-f(t, y)\Big]$$
$$\leqq\Big(b(t)+1/3\,\sqrt[3]{a(t)^2}\,\big(1+\sqrt[3]{a(t)}\big)\Big)\,\|x-y\|$$

for all x, y in E and $t>0$. Let $\beta(t)=1/3\,\sqrt[3]{a(t)^2}\,\big(1+\sqrt[3]{a(t)}\big)$. Then $\int_0^\rho\beta(t)\,dt=+\infty$. But $\beta(t)$ satisfies the conditions (β_1) and (β_2) in Remark 2 by the same argument as in Example 1. Thus the above defined f satisfies (K_1) and (K_2) in § 1.

To show that f satisfies (5. 4) we note that

$$x+hf(t, x)$$
$$=\Big(\big(1-hb(t)\big)\, x_n+h\big(\sqrt[3]{x_n}\ \text{or}\ \sqrt[3]{a(t)}\big)/n\big(1+\sqrt[3]{a(t)}\big)\Big)$$

for each $x=(x_n)\in F$ and $t\geqq 0$. Thus it follows that

$$\Big\|x+hf(t, x)\Big\|\leqq\big(1-hb(t)\big)\,\|x\|+h\sqrt{\rho}\ M$$
$$\leqq\rho+\big(\sqrt{\rho}\ M-\rho b(t)\big)\, h\,.$$

By the assumption on b we have for each $x\in F$ and $t\geqq 0$

$$x+hf(t, x)\in F \qquad \text{for } 0<h<\mathrm{Min}\left\{1/b(t), \sqrt{\rho}/(\sqrt{\rho}\, b(t)-M)\right\}.$$

Consequently, the set F is flow-invariant for f by our Theorem.

On the other hand, for each $x=(x_n)$ and $y=(y_n)$ in F such that $\rho\geqq x_1>y_1>0$ and $x_n=y_n=0$ for $n\geqq 2$,

$$\left[x-y, f(0, x)-f(0, y)\right]$$
$$=\left(1/(\sqrt[3]{x_1^2}+\sqrt[3]{x_1 y_1}+\sqrt[3]{y_1^2})-b(0)\right)\|x-y\|,$$

so that we can not apply [8] to this example.

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[From PROCEEDINGS OF THE JAPAN ACADEMY, Vol. 50, No. 4 (1974)]

A Note on Nonlinear Differential Equation in a Banach Space

By

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71. *A Note on Nonlinear Differential Equation in a Banach Space*

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(Comm. by Kinjirô KUNUGI, M. J. A., April 18, 1974)

1. Let E be a Banach space with the dual space E^*. The norms in E and E^* are denoted by $\|\ \|$. We denote by $S(u, r)$ the closed sphere of center u with radius r.

It is our object in this note to give a sufficient condition for the existence of the unique solution to the Cauchy problem of the form

(1.1) $$u'(t)=f(t, u(t)), \qquad u(0)=u_0 \in E,$$

where f is a E-valued mapping defined on $[0, T]\times S(u_0, r)$.

We compare the differential equation (1.1) with the scalar equation

(1.2) $$w'(t)=g(t, w(t)),$$

where $g(t, w)$ is a function defined on $(0, a]\times[0, b]$ which is measurable in t for fixed w, and continuous monotone nondecreasing in w for fixed t. We say w is a solution of (1.2) on an interval I contained in $[0, a]$ if w is absolutely continuous on I and if $w'(t)=g(t, w(t))$ for a.e. $t \in I^\circ$, where I° is the set of all interior points of I.

We assume that g satisfies the following conditions:

(i) There exists a function m defined on $(0, a)$ such that $g(t, w) \leqq m(t)$ for $(t, w) \in (0, a)\times[0, b]$ and for which m is Lebesgue integrable on (ε, a) for every $\varepsilon>0$.

(ii) For each $t_0 \in (0, a]$, $w\equiv 0$ is the only solution of the equation (1.2) on $[0, t_0]$ satisfying the conditions that $w(0)=(D^+w)(0)=0$, where D^+w denotes the right-sided derivative of w.

2. Let g be as in Section 1. Then we have the following lemmas.

Lemma 2.1. *Let $\{w_n\}$ be a sequence of functions from $[0, a]$ to $[0, b]$ converging pointwise on $[0, a]$ to a function w_0. Let $M>0$ such that $|w_n(t)-w_n(s)|\leqq M|t-s|$ for $s, t \in [0, a]$ and $n\geqq 1$. Suppose further that for each $n\geqq 1$*

$$w_n'(t)\leqq g(t, w_n(t)) \qquad \text{for } t \in (0, a)$$

such that $w_n'(t)$ exists. Then w_0 is a solution of (1.2) on $[0, a]$.

For a proof see [4].

Lemma 2.2. *Let $M>0$ and let $\{w_n\}$ be a sequence of functions from $[0, a]$ to $[0, b]$ with the property that $|w_n(t)-w_n(s)|\leqq M|t-s|$ for all $s, t \in [0, a]$ and $n\geqq 1$. Let $w=\sup_{n\geqq 1} w_n$, and suppose that $w_n'(t) \leqq g(t, w_n(t))$ for $t \in (0, a)$ such that $w_n'(t)$ exists. Then w is a solution of (1.2) on $[0, a]$.*

For a proof see [2].

Lemma 2.3. *Let w be an absolutely continuous function from $[0, a]$ to $[0, b]$ such that $w(0)=(D^+w)(0)=0$ and $w'(t)\leqq g(t, w(t))$ for $t\in(0, a)$ such that $w'(t)$ exists. Then $w\equiv 0$ on $[0, a]$.*

The proof of this lemma is quite similar to that of Theorem 2.2 in [1] and is omitted.

3. For each u in E let $F(u)$ denote the set of all x^* in E^* such that $(u, x^*)=\|u\|^2=\|x^*\|^2$, where (u, x^*) denotes the value of x^* at u.

Theorem. *Let f be a strongly continuous mapping of $[0, T]\times S(u_0, r)$ into E such that*

$$2\,\mathrm{Re}\,(f(t, u)-f(t, v), x^*)\leqq g(t, \|u-v\|^2) \tag{3.1}$$

for (t, u), $(t, v)\in(0, T]\times S(u_0, r)$ and for some $x^*\in F(u-v)$, where g satisfies the conditions in Section 1 with $a=T$ and $b=\mathrm{Max}\,\{4r^2, 8rMT\}$. Then (1.1) has a unique strongly continuously differentiable solution u defined on some interval $[0, T_0]$.

Proof. Since f is strongly continuous on $[0, T]\times S(u_0, r)$ there exist constants $0<r_0\leqq r$, $0<T_1\leqq T$ and $M>0$ such that $\|f(t, u)\|\leqq M$ for $(t, u)\in[0, T_1]\times S(u_0, r_0)$. Let $T_0=\mathrm{Min}\,\{r_0/M, T_1\}$ and let n be a positive integer. We set $t_0^n=0$, and $u_n(t_0^n)=u_0$. Inductively, for each positive integer i, define $\delta_i^n, t_i^n, u_n(t_i^n)$ as follows:

$$\delta_i^n\geqq 0, \qquad t_{i-1}^n+\delta_i^n\leqq T_0. \tag{3.2}$$

If

$$\|v-u_n(t_{i-1}^n)\|\leqq M\delta_i^n \quad\text{and}\quad |t-t_{i-1}^n|\leqq\delta_i^n, \tag{3.3}$$

then $\|f(t, v)-f(t_{i-1}^n, u_n(t_{i-1}^n))\|\leqq 1/n$.

$$\|u_n(t_{i-1}^n)-u_0\|\leqq r_0, \tag{3.4}$$

and δ_i^n is the largest number such that (3.2) to (3.4) hold. Define $t_i^n=t_{i-1}^n+\delta_i^n$ and define for each $t\in[t_{i-1}^n, t_i^n]$

$$u_n(t)=u_n(t_{i-1}^n)+\int_{t_{i-1}^n}^t f(s, u_n(t_{i-1}^n))ds. \tag{3.5}$$

Then we have

$$\|u_n(t)-u_n(s)\|\leqq M|t-s|,\ \|u_n(t)-u_0\|\leqq r_0 \qquad \text{for } s, t\in[0, T_0], \tag{3.6}$$

and $t_N^n=T_0$ for some positive integer $N=N(n)$. For some detail see [6] and [3].

Let $w_{mn}(t)=\|u_m(t)-u_n(t)\|^2$ for $m>n\geqq 1$ and $t\in[0, T_0]$. Obviously $w_{mn}(0)=0$, and $|w_{mn}(t)-w_{mn}(s)|\leqq 8r_0M|t-s|$ for $s, t\in[0, T_0]$. For each $t\in(0, T_0)$ there exist positive integers i and j such that $t\in(t_{j-1}^m, t_j^m)$ and $t\in(t_{i-1}^n, t_i^n)$. By Lemma 1.3 in [5] and (3.5) we have

$$\begin{aligned} w'_{mn}(t)&=2\,\mathrm{Re}\,(u'_m(t)-u'_n(t), x^*_{mn}(t))\\ &=2\,\mathrm{Re}\,(f(t, u_m(t_{j-1}^m))-f(t, u_n(t_{i-1}^n)), x^*_{mn}(t))\\ &\leqq g(t, w_{mn}(t))+2(1/m+1/n)\,\|u_m(t)-u_n(t)\|\\ &\leqq g(t, w_{mn}(t))+8r_0/n \end{aligned} \tag{3.7}$$

for a.e. $t\in(0, T_0)$ and for some $x^*_{mn}(t)\in F(u_n(t)-u_n(t))$.

Let $w_n(t)=\sup_{m>n} w_{mn}(t)$ for $t\in[0, T_0]$. Then obviously $w_n(0)=0$

for $n \geqq 1$. By Lemma 2.2 and (3.7) we have

(3.8) $$|w_n(t)-w_n(s)| \leqq 8r_0M|t-s| \qquad \text{for } s, t \in [0, T_0],$$

and

(3.9) $$w_n'(t)=g(t, w_n(t))+8r_0/n \qquad \text{for a.e. } t \in (0, T_0).$$

On the other hand, $0 \leqq w_n(t) \leqq w_n(0)+8r_0Mt \leqq 8r_0MT_0$ for $n \geqq 1$ and $t \in [0, T_0]$. Thus the sequence $\{w_n\}$ is equicontinuous and uniformly bounded, and hence it has a subsequence $\{w_{nj}\}$ converging uniformly on $[0, T_0]$ to a function w, and obviously $w(0)=0$. It follows from (3.9) and Lemma 2.1 that $w'(t)=g(t, w(t))$ for a.e. $t \in (0, T_0)$.

We shall next show that $(D^+w)(0)=0$. Since f is continuous at $(0, u_0)$, given $\varepsilon>0$ we can find $\delta>0$ such that $\|f(t, u)-f(0, u_0)\|<\varepsilon$ whenever $0 \leqq t \leqq \delta$ and $\|u-u_0\| \leqq \delta$. Let $\delta_0=\operatorname{Min}\{\delta, \delta/M\}$. Then, by (3.6), $\|u_n(t)-u_0\| \leqq \delta_0$ for all n and $t \in [0, \delta_0]$, and therefore $\|f(t, u_m(t)) - f(t, u_n(t))\|<2\varepsilon$ whenever $m>n \geqq 1$ and $t \in [0, \delta_0]$. By (3.3) and (3.7) we have

$$\begin{aligned} w_{mn}'(t) &= 2 \operatorname{Re}(f(t, u_m(t_{j-1}^m))-f(t, u_n(t_{i-1}^n)), x_{mn}^*(t)) \\ &\leqq 4r_0\|f(t, u_m(t_{j-1}^m))-f(t, u_n(t_{i-1}^n))\| \leqq 8r_0(\varepsilon+1/n) \\ &\qquad \text{for a.e. } t \in (0, \delta_0), \end{aligned}$$

and hence, by integrating the above inequality, we have

$$0 \leqq w_{mn}(t) \leqq 8r_0(\varepsilon+1/n)t,$$

whence $(D^+w)(0)=0$. From Lemma 2.3, we deduce now that $w \equiv 0$, and this implies that the sequence $\{u_n\}$ is uniformly convergent on $[0, T_0]$. The limit of this sequence satisfies

$$u(t)=u_0+\int_0^t f(s, u(s))ds \qquad \text{for } t \in [0, T_0]$$

(see [3]). Consequently u is a strongly continuously differentiable solution of (1.1) on $[0, T_0]$.

Let v be another strongly continuously differentiable solution of (1.1) on $[0, T_0]$. Let $z(t)=\|u(t)-v(t)\|^2$. Then obviously $z(0)=0$, and

$$z'(t)=2 \operatorname{Re}(f(t, u(t))-f(t, v(t)), x^*(t)) \leqq g(t, z(t))$$

for a.e. $t \in (0, T_0)$ and for some $x^*(t) \in F(u(t)-v(t))$. The fact $(D^+z)(0)=0$ follows from $0 \leqq z(t)/t = t\|(u(t)-v(t))/t\|^2 \to 0$ as $t \downarrow 0$. Therefore by Lemma 2.3 $z \equiv 0$, and the proof is complete.

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With the Author's Compliments

Some remarks on nonlinear differential equations in Banach spaces

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Reprinted from Hokkaido Mathematical Journal
Vol. IV No. 2

SAPPORO, JAPAN

May 1975

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§ 1. Introduction and results.

Let E be a Banach space with the dual space E^*. The norms in E and E^* are denoted by $\|\|$. We denote by $S(u, r)$ the closed sphere of center u with radius r.

In this paper we are concerned with nonlinear abstract Cauchy problems of the forms

$$(D_1) \qquad \frac{d}{dt} u(t) = f(t, u(t)), \qquad u(0) = u_0 \in E,$$

and

$$(D_2) \qquad \frac{d}{dt} u(t) = Au(t) + f(t, u(t)), \qquad u(0) = u_0 \in D(A).$$

Here A is a nonlinear operator with domain $D(A)$ and range $R(A)$ in E, and f is a E-valued mapping defined on $[0, T] \times S(u_0, r)$ or on $[0, \infty) \times E$.

It is well known that in the case of $E = R^n$, the n-dimensional Euclidean space, the continuity of f in a neighbourhood of $(0, u_0)$ alone implies the existence of a local solution of (D_1). This is the classical Peano's theorem. However, this theorem cannot be generalized to the infinite-dimensional case (see [3], [16]).

It is our object in this paper to give sufficient conditions for the existence of the unique solutions to the Cauchy problems of the forms (D_1) and (D_2).

Let the functionals $\langle , \rangle_1$ and $\langle , \rangle_2$ be defined as follows (cf. M. Hasegawa [6]):

$$\langle u, v \rangle_1 = \lim_{h \downarrow 0} \frac{1}{h} (\|u + hv\| - \|u\|),$$

and

$$\langle u, v \rangle_2 = \frac{1}{2} (\langle u, v \rangle_1 - \langle u, -v \rangle_1)$$

for u, v in E.

In order to prove the existence of the unique solution of the equation

(D_1) we consider the following scalar equation

(1. 1) $$w'(t) = g(t, w(t)),$$

where $g(t, \tau)$ is a scalar-valued function defined on $(0, a] \times [0, b]$ which is measurable in t for fixed τ, and continuous nondecreasing in τ for fixed t.

We say w is a solution of (1. 1) on an interval I contained in $[0, a]$ if w is absolutely continuous on I and if

$$w'(t) = g(t, w(t)) \qquad \text{for a. e. } t \in I^0,$$

where I^0 is the set of all interior points of I.

We assume furthermore that g satisfies the following conditions: (i_a) There exists a function m defined on $(0, a)$ such that

$$|g(t, \tau)| \leqq m(t) \qquad \text{for } (t, \tau) \in (0, a] \times [0, b]$$

and for which m is Lebesgue integrable on (ε, a) for every $\varepsilon > 0$. (ii_a) For each $t_0 \in (0, a]$, $w \equiv 0$ is the only solution of the equation (1 .1) on $[0, t_0]$ satisfying the conditions that $w(0) = (D^+ w)(0) = 0$, where $D^+ w$ denotes the right-sided derivative of w.

First, we can state the following result.

THEOREM 1. *Let f be a strongly continuous mapping of $[0, T] \times S(u_0, r)$ into E such that*

(1. 2) $$\langle u - v, f(t, u) - f(t, v) \rangle_2 \leqq g(t, \|u - v\|)$$

for all (t, u), $(t, v) \in (0, T] \times S(u_0, r)$, where g satisfies (i_a), (ii_a) with $a = T$ and $b = 2r$.
Then (D_1) has a unique strongly continuously differentiable solution u defined on some interval $[0, T_0]$.

We next consider a global analogue of Theorem 1, and we assume that $g(t, \tau)$ is a scalar-valued function defined on $(0, \infty) \times [0, \infty)$ which is measurable in t for fixed τ, and continuous nondecreasing in τ for fixed t. We assume furthermore that g satisfies the following conditions: (i_b) $g(t, 0) = 0$ for all $t \in (0, \infty)$, and for every bounded subset B of $(0, \infty) \times [0, \infty)$ let there exist a locally Lebesgue integrable function m_B defined on $(0, \infty)$ such that

$$|g(t, \tau)| \leqq m_B(t) \qquad \text{for } (t, \tau) \in B.$$

(ii_b) There exists a strictly increasing continuous function α defined on $[0, \infty)$ satisfying $\alpha(0) = 0$ and

$$|g(t, \tau) - g(t, \tilde{\tau})| \leqq m_B(t)\, \alpha(|\tau - \tilde{\tau}|)$$

for (t, τ), $(t, \tilde{\tau}) \in B$.

(iii$_b$) For every $\delta > 0$, $\int_0^\delta d\tau/\alpha(\tau) = \infty$.

Under these conditions we can prove the following

THEOREM 2. *Let f be a strongly continuous mapping of $[0, \infty) \times E$ into E, carring bounded sets in $[0, \infty) \times E$ into bounded sets in E. Suppose furthermore that*

$$(1.3) \qquad \langle u-v, f(t, u)-f(t, v)\rangle_2 \leqq g(t, \|u-v\|)$$

for (t, u), $(t, v) \in (0, \infty) \times E$.
Then (D_1) has a unique strongly continuously differentiable solution u defined on $[0, \infty)$.

Finally, we consider the equation (D_2) in a Banach space E whose dual space E^* is uniformly convex.

We say u is a solution of (D_2) on $[0, \infty)$ with $u(0) = u_0$ if u is strongly absolutely continuous on any finite interval of $[0, \infty)$ and if

$$u(t) \in D(A), \qquad \frac{d}{dt} u(t) = Au(t) + f(t, u(t))$$

for a. e. $t \in [0, \infty)$.

We assume that A satisfies

$$(1.4) \qquad \langle u-v, Au-Av\rangle_2 \leqq 0 \qquad \text{for } u, v \in D(A),$$

and $R(I-\lambda_0 A) = E$ for some $\lambda_0 > 0$.
If the strongly continuous mapping f of $[0, \infty) \times E$ into E has the strongly continuous derivative f_t with respect to t and if both f and f_t carry bounded sets in $[0, \infty) \times E$ into bounded sets in E, then we have

THEOREM 3. *Let A, f and f_t satisfy the assumptions mentioned above. Furthermore, if f satisfies*

$$(1.5) \qquad \langle u-v, f(t, u)-f(t, v)\rangle_1 \leqq \beta(t)\|u-v\|$$

for (t, u), $(t, v) \in (0, \infty) \times E$, where β is a locally Lebesgue integrable function defined on $(0, \infty)$.
Then (D_2) has a unique solution u on $[0, \infty)$ for each $u_0 \in D(A)$.

In the paper [1] F. E. Browder proved the global existence in a Hilbert space of the unique solution of (D_1) under the monotonicity condition.

Recently T. M. Flett ([4], [5]) has given the sufficient conditions for both local and global existence in Banach and Hilbert spaces of the unique solution of (D_1).

The contents of this paper are as follows: Some lemmas concerning the scalar differential equation (1. 1) are given in § 2. Theorems 1, 2 and 3 are proved in § 3, 4 and 5, respectively. In § 6 we shall give a simple example and some remarks about the relations between our results and those of F. E. Browder and T. M. Flett.

§ 2. Some lemmas.

In the following Lemmas 2. 1, 2. 2 and 2. 3 we assume that g satisfies the assumptions (i_a) and (ii_a) stated in § 1.

LEMMA 2. 1. *Let* $\{w_n\}$ *be a sequence of functions from* $[0, a]$ *into* $[0, b]$ *converging uniformly on* $[0, a]$ *to a function* w_0. *Let* $M>0$ *such that*

$$|w_n(t)-w_n(s)| \leqq M|t-s| \qquad \text{for } s, t\in[0, a] \text{ and } n\geqq 1.$$

Suppose furthermore that for each $n\geqq 1$

$$w_n'(t)\leqq g(t, w_n(t)) \qquad \text{for } t\in(0, a) \text{ such that } w_n'(t) \text{ exists.}$$

Then

$$w_0'(t)\leqq g(t, w_0(t)) \qquad \text{for a. e. } t\in(0, a).$$

PROOF. Since $|w_0(t)-w_0(s)|\leqq M|t-s|$ for $s, t\in[0, a]$, $w_0'(t)$ exists for a. e. $t\in[0, a]$.
Let $A_n=\{t\in[0, a]\,;\ w_n'(t)$ does not exist$\}$ and let $A=\bigcup_{n=0}^{\infty}A_n$, then mes $(A)=0$.
Set

$$B=\{t\in(0, a]\,;\ \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} g(s, w_0(s))\,ds=g(t, w_0(t))\}.$$

Then, by (i_a), we have mes $([0, a]-B)=0$.
For each $t\in\{[0, a]-A\}\cap B$, $n\geqq 1$ and for sufficiently small $h>0$

$$w_n(t+h)-w_n(t)\leqq\int_t^{t+h} g(s, w_n(s))\,ds.$$

By the Lebesgue's dominated convergence theorem, we have

$$w_0(t+h)-w_0(t)\leqq\int_t^{t+h} g(s, w_0(s))\,ds.$$

Dividing both sides by $h>0$ and letting $h\to 0$, we have $w_0'(t)\leqq g(t, w_0(t))$.
Thus we have the inequality

$$w_0'(t)\leqq g(t, w_0(t)) \qquad \text{for a. e. } t\in(0, a).$$

LEMMA 2. 2. *Let $M>0$ and Φ be a set of functions from* $[0, a]$ *into* $[0, b]$ *with the property that for all* $s, t\in[0, a]$ *and* $w\in\Phi$

$$|w(t)-w(s)|\leqq M|t-s|. \tag{2.1}$$

Let $z=\sup\{w ; w\in\Phi\}$, *and suppose that for each* $w\in\Phi$

$$w'(t)\leqq g(t, w(t)) \qquad \textit{for } t\in(0, a) \textit{ such that } w'(t) \textit{ exists.} \tag{2.2}$$

Then

$$z'(t)\leqq g(t, z(t)) \qquad \textit{for a. e. } t\in(0, a).$$

PROOF. We follow an argument essentially given in T. M. Flett [4]. By the definition of z and (2. 1), z satisfies

$$|z(t)-z(s)|\leqq M|t-s|$$

and

$$0\leqq z(t)-w(t)\leqq z(s)-w(s)+2M|t-s|$$

for all $s, t\in[0, a]$ and all $w\in\Phi$. From this it follows that for each positive integer n we can find a positive integer k, a partition of $[0, a]$ into k subintervals of equal length, and k functions $w_1, \cdots, w_k\in\Phi$ such that in the jth subinterval

$$0\leqq z(t)-w_j(t)\leqq 1/n.$$

We put $w^{(n)}=\mathrm{Max}\{w_1, \cdots, w_k\}$. Then $w^{(n)}$ satisfies (2. 1) and (2. 2). Since

$$0\leqq z(t)-w^{(n)}(t)\leqq 1/\mathrm{n}$$

for all $t\in[0, a]$, the sequence $\{w^{(n)}\}$ converges uniformly to z on $[0, a]$, and the required result follows from Lemma 2. 1.

LEMMA 2. 3. *Let w be an absolutely continuous function from* $[0, a]$ *into* $[0, b]$ *such that* $w(0)=(D^+w)(0)=0$ *and*

$$w'(t)\leqq g(t, w(t)) \qquad \textit{for a. e. } t(0, a).$$

Then $w\equiv 0$ *on* $[0, a]$.

PROOF. The method of the following proof is essentially due to that of Theorem 2. 2 in [2].

Suppose that there exists a σ, $0<\sigma\leqq$a such that $w(\sigma)>0$. Then there exists a solution z of (1. 1) with $z(\sigma)=w(\sigma)$ on some interval to the left of σ. As far to the left of σ as z exists, it satisfies the inequality $z(t)\leqq w(t)$, for if this were not the case there would exist a positive σ_1 to the left of σ where $z(\sigma_1)=w(\sigma_1)$, and $z(t)>w(t)$ for $t<\sigma_1$, and sufficiently near σ.

By the assumptions on w we have for sufficiently small $h>0$

$$w(\sigma_1)-w(\sigma_1-h)\leqq\int_{\sigma_1-h}^{\sigma_1} g(t, w(t))\,dt.$$

On the other hand, from the definition of z we have, since $z(\sigma_1)=w(\sigma_1)$,

$$w(\sigma_1)-z(\sigma_1-h)=\int_{\sigma_1-h}^{\sigma_1} g(t, z(t))\,dt,$$

where h is assumed so small that z exists on $[\sigma_1-h, \sigma_1]$.
Thus

$$z(\sigma_1-h)-w(\sigma_1-h)\leqq\int_{\sigma_1-h}^{\sigma_1}[g(t, w(t))-g(t, z(t))]\,dt.$$

Since g is nondecreasing in τ and $z(t)>w(t)$ on $[\sigma_1-h, \sigma_1)$ we have the contradiction $z(\sigma_1-h)\leqq w(\sigma_1-h)$.

We shall next show that $z(t)>0$ on $0<t\leqq\sigma$, as far as it exists. Otherwise $z(t_0)=0$ for some t_0, $0<t_0<\sigma$, and the function $\tilde{z}$ defined by

$$\tilde{z}(t)=\begin{cases}0 & (0\leqq t\leqq t_0)\\ z(t) & (t_0\leqq t\leqq\sigma)\end{cases}$$

would be a function on $[0, \sigma]$ not identically zero, which satisfies

$$\tilde{z}'(t)=g(t, \tilde{z}(t)), \qquad \tilde{z}(0)=(D^+\tilde{z})(0)=0.$$

This contradicts the assumption (ii_a). Therefore

$$0<z(t)\leqq w(t)$$

as far to the left of σ as z exists.
It therefore follows that z can be continued as a solution, call it z again, on the whole interval $0<t\leqq\sigma$. Since $\lim_{t\downarrow 0} z(t)=0$, we define $z(0)=0$. Since

$$0<z(t)/t\leqq w(t)/t \qquad \text{for } 0<t\leqq\sigma$$

and $(D^+w)(0)=0$, we have $(D^+z)(0)=0$.
From (ii_a) it follows $z\equiv 0$ on $[0, \sigma]$, but this contradicts the fact $z(\sigma)=w(\sigma)>0$.

LEMMA 2.4. *If g satisfies the assumptions* (i_b), (ii_b) *and* (iii_b) *stated in* § 1, *then for each $T>0$ and $d\geqq 0$ there exists a unique solution w of* (1.1) *on* $[0, T]$ *with the initial condition* $w(0)=d$.

PROOF. Suppose that there are two solutions w_1 and w_2 of (1.1) on $[0, T]$ satisfying $w_1(0)=w_2(0)=d$. Let z be the function defined by

$$z(t)=|w_1(t)-w_2(t)| \qquad \text{for } t\in[0, T].$$

Then there exist $\sigma\in(0, T]$ and $\sigma_0\in[0, \sigma)$ such that $z(\sigma_0)=0$ and $z(t)>0$ for $t\in(\sigma_0, \sigma]$.
Since z is absolutely continuous, $z'(t)$ exists for a. e. $t\in[\sigma_0, \sigma]$ and, by ($\mathrm{ii_b}$), we have

$$\begin{aligned} z'(t) &\leqq |w_1'(t)-w_2'(t)| = |g(t, w_1(t))-g(t, w_2(t))| \\ &\leqq m_B(t)\,\alpha(z(t)), \end{aligned}$$

where $B=\{(t, w_1(t)),\ (t, w_2(t))\,;\ t\in[\sigma_0, \sigma]\}$.
Since α is continuous and z is absolutely continuous, we have for sufficiently small $\varepsilon>0$

$$\int_{\sigma_0+\varepsilon}^{\sigma} z'(t)/\alpha\Big(z(t)\Big)dt = \int_{z(\sigma_0+\varepsilon)}^{z(\sigma)} d\tau/\alpha(\tau) \leqq \int_{\sigma_0+\varepsilon}^{\sigma} m_B(\tau)\,d\tau$$

(see [13], p. 211).

By ($\mathrm{iii_b}$) and by letting $\varepsilon\downarrow 0$, we have a contradiction.

§ 3. Proof of Theorem 1.

Let the functionals $\langle,\rangle_1$ and $\langle,\rangle_2$ be as in § 1.
We shall give the following two lemmas which are used throughout this paper.

LEMMA 3. 1. (*cf. M. Hasegawa* [6]). *For u, v and w in E,*

(1) $|\langle u, v\rangle_1| \leqq \|v\|$,

(2) $\langle u, v+w\rangle_1 \leqq \langle u, v\rangle_1 + \langle u, w\rangle_1$

(3) $\langle u, du+v\rangle_2 = d\|u\| + \langle u, v\rangle_2$ *for real number d,*

(4) $\langle u, v\rangle_2 \leqq \langle u, v\rangle_1$,

(5) $\langle u, v+w\rangle_2 \leqq \langle u, v\rangle_2 + \langle u, w\rangle_1$,

(6) $\langle u, v\rangle_2 \leqq \langle u, v-w\rangle_2 + \|w\|$.

PROOF. (1) and (2) are easy consequences of the definition. For any real number d we have

$$\begin{aligned} \langle u, du+v\rangle_2 &= \frac{1}{2}\lim_{h\downarrow 0}\frac{1}{h}\Big(\|u+h(du+v)\|-\|u-h(du+v)\|\Big) \\ &= \frac{1}{2}\Big\{\lim_{h\downarrow 0}\frac{1+dh}{h}\Big(\|u+\frac{1}{1+dh}v\|-\|u\|\Big) \\ &\qquad -\lim_{h\downarrow 0}\frac{1-dh}{h}\Big(\|u-\frac{h}{1-dh}v\|-\|u\|\Big)\Big\} + d\|u\| \\ &= d\|u\| + \frac{1}{2}\Big(\langle u,v\rangle_1 - \langle u, -v\rangle_1\Big) = d\|u\| + \langle u, v\rangle_2, \end{aligned}$$

which proves (3).
(4) follows readily from (2). By the definitions and (2) we have

$$\begin{aligned}
&\langle u, v\rangle_2+\langle u, w\rangle_1-\langle u, v+w\rangle_2 \\
&\geqq \frac{1}{2}\Big(\langle u, w\rangle_1+\langle u, -(v+w)\rangle_1-\langle u, -v\rangle_1\Big) \\
&\geqq \frac{1}{2}\Big(\langle u, -v\rangle_1-\langle u, -v\rangle_1\Big)=0,
\end{aligned}$$

which implies (5).
To prove (6) we note that

$$\begin{aligned}
\langle u, v\rangle_2 &= \frac{1}{2}\lim_{h\downarrow 0}\frac{1}{h}\Big(\|u+hv\|-\|u-hv\|\Big) \\
&= \frac{1}{2}\lim_{h\downarrow 0}\frac{1}{h}\Big(\|u+h(v-w)+hw\|-\|u-h(v-w)-hw\|\Big) \\
&\leqq \frac{1}{2}\lim_{h\downarrow 0}\frac{1}{h}\Big(\|u+h(v-w)\|-\|u-h(v-w\|+2h\|w\|\Big) \\
&= \langle u, v-w\rangle_2+\|w\|.
\end{aligned}$$

LEMMA 3.2. *Let $u(t)$ be a E-valued function defined on a real interval I such that $u'(t)$ and $\frac{d}{dt}\|u(t)\|$ exist for a. e. $t\in I$. Then*

$$\frac{d}{dt}\|u(t)\|=\langle u(t),\ u'(t)\rangle_2 \qquad \textit{for a. e. } t\in I.$$

PROOF. If we denote $D^+u(t)$ and $D^-u(t)$ respectively the right and left derivatives of $u(t)$. Then

$$\begin{aligned}
&\Big|\frac{1}{h}\Big(\|u(t+h)\|-\|u(t-h)\|\Big)-\frac{1}{h}\Big(\|u(t)+hD^+u(t)\|-\|u(t)\|\Big) \\
&\qquad\qquad +\frac{1}{h}\Big(\|u(t)-hD^-u(t)\|-\|u(t)\|\Big)\Big| \\
&=\frac{1}{h}\Big|\|u(t+h)\|-\|u(t-h)\|-\|u(t)+hD^+u(t)\|+\|u(t)-hD^-u(t)\|\Big| \\
&\leqq \Big\|\frac{1}{h}\Big(u(t+h)-u(t)\Big)-D^+u(t)\Big\|+\Big\|\frac{1}{h}\Big(u(t-h)-u(t)\Big)+D^-u(t)\Big\| \\
&\qquad\qquad \to 0 \text{ as } h\downarrow 0 \qquad \text{for a. e. } t\in I.
\end{aligned}$$

Thus we have

$$D^+\|u(t)\|+D^-\|u(t)\|=\langle u(t),\quad D^+u(t)\rangle_1-\langle u(t),\quad -D^-u(t)\rangle_1$$
$$\text{for a. e. } t\in I.$$

It follows from the assumptions that

$$\frac{d}{dt}\|u(t)\| = \langle u(t),\ u'(t)\rangle_2 \qquad \text{for a. e. } t\in I.$$

PROOF of THEOREM 1. Since f is strongly continuous on $[0, T]\times S(u_0, r)$ there exist constants $0<r_0\leqq r$, $0<T_1\leqq T$ and $M>0$ such that

$$\|f(t, u)\|\leqq M \text{ for } (t, u)\in[0, T_1]\times S(u_0, r_0).$$

Let $T_0=\mathrm{Min}\,\{r_0/M, T_1\}$ and let n be a positive integer.
We set $t_0^n=0$, and $u_n(t_0^n)=u_0$. Inductively for each positive integer i, define δ_i^n, t_i^n, $u_n(t_{i-1}^n)$ as follows (cf. G. Webb [14]):

(3. 1) $\qquad \delta_i^n\geqq 0, \qquad t_{i-1}^n+\delta_i^n\leqq T_0\,;$

(3. 2) $\qquad$ If $\|v-u_n(t_{i-1}^n)\|\leqq M\delta_i^n$ and $t_{i-1}^n\leqq t\leqq t_{i-1}^n+\delta_i^n$, then $\|f(t, v)-f(t_{i-1}^n, u_n(t_{i-1}^n))\|\leqq 1/n\,;$

(3. 3) $\qquad \|u_n(t_{i-1}^n)-u_0\|\leqq r_0,$

and δ_i^n is the largest number such that (3. 1), (3. 2) and (3. 3) hold.
Let $t_i^n=t_{i-1}^n+\delta_i^n$. We set

$$u_n(t) = u_n(t_{i-1}^n)+\int_{t_{i-1}^n}^{t} f\big(s, u_n(t_{i-1}^n)\big)\,ds \qquad \text{for each } t\in[t_{i-1}^n, t_i^n].$$

Then for each $t\in[t_{k-1}^n, t_k^n]$

$$\begin{aligned} u_n(t) &= u_n(t_{k-1}^n)+\int_{t_{k-1}^n}^{t} f\big(s, u_n(t_{k-1}^n)\big)\,ds \\ &= u_n(t_{k-1}^n)+\int_{t_{k-2}^n}^{t_{k-1}^n} f\big(s, u_n(t_{k-2}^n)\big)\,ds+\int_{t_{k-1}^n}^{t} f\big(s, u_n(t_{k-1}^n)\big)\,ds \\ &= \cdots = u_0+\sum_{j=1}^{k-1}\int_{t_{j-1}^n}^{t_j^n} f\big(s, u_n(t_{j-1}^n)\big)\,ds+\int_{t_{k-1}^n}^{t} f\big(s, u_n(t_{k-1}^n)\big)\,ds. \end{aligned}$$

For each t, s (say $t>s$) in $[0, T_0]$ there exist i, k such that $t\in[t_{i-1}^n, t_i^n]$ and $s\in[t_{k-1}^n, t_k^n]$. Then

$$\begin{aligned} \|u_n(t)-u_n(s)\| \leqq \int_{s}^{t_k^n} \big\|f\big(s, u_n(t_{k-1}^n)\big)\big\|\,ds+\sum_{j=k+1}^{i-1}\int_{t_{j-1}^n}^{t_j^n} \big\|f\big(s, u_n(t_{j-1}^n)\big)\big\|\,ds \\ +\int_{t_{i-1}^n}^{t} \big\|f\big(s, u_n(t_{i-1}^n)\big)\big\|\,ds \end{aligned}$$

$$\leqq M(t_k^n - s) + \sum_{j=k+1}^{i-1} M(t_j^n - t_{j-1}^n) + M(t - t_{i-1}^n)$$
$$= M(t-s).$$

On the other hand

$$\|u_n(t) - u_0\| \leqq \sum_{j=1}^{i-1} \int_{t_{j-1}^n}^{t_j^n} \left\|f\left(s, u_n(t_{j-1}^n)\right)\right\| ds + \int_{t_{i-1}^n}^{t} \left\|f\left(s, u_n(t_{i-1}^n)\right)\right\| ds$$
$$\leqq Mt \leqq r_0.$$

We shall show that there exists some positive integer $N = N(n)$ such that $t_N^n = T_0$. Suppose, on the contrary, that this were not true. Then, since $\{t_i^n\}$ is a nondecreasing sequence bounded from above, there is a t_0 in $(0, T_0]$ such that $\lim_{i\to\infty} t_i^n = t_0$.
Since $\|u_n(t_i^n) - u_n(t_k^n)\| \leqq M|t_i^n - t_k^n| \to 0$ as $i, k \to \infty$, $\lim_{i\to\infty} u_n(t_i^n) = v_0$ exists. Let $\sigma_1 > 0$ such that

$$\|f(t, v) - f(t_0, v_0)\| \leqq 1/2n \tag{3. 5}$$

whenever $\|v - v_0\| \leqq 2\sigma_1$ and $|t - t_0| \leqq 2\sigma_1$.

Since $\lim_{k\to\infty} f(t_k^n, u_n(t_k^n)) = f(t_0, v_0)$ there exist $\sigma_2 > 0$ and sufficiently large positive integer i such that

$$\left\|f(t_0, v_0) - f\left(t_{i-1}^n, u_n(t_{i-1}^n)\right)\right\| \leqq 1/2n \tag{3. 6}$$

whenever $t_0 - t_{i-1}^n \leqq \sigma_2$ and $\|v_0 - u_n(t_{i-1}^n))\| \leqq \sigma_2$.

Set $\sigma = \text{Min}\ \{\sigma_1, \sigma_2\}$. Then there exists a positive integer j such that

$$\delta_j^n < \text{Min}\ \{\sigma/2M, \sigma\}. \tag{3. 7}$$

Thus (3. 5), (3. 6) and (3. 7) hold for σ and $k = \text{Max}\ \{i, j\}$.
Consequently, if $\|v - u_n(t_{k-1}^n)\| \leqq M(\delta_k^n + \sigma/4M)$ and $t_{k-1}^n \leqq t \leqq t_{k-1}^n + \sigma$, then

$$\|v - v_0\| \leqq \|v - u_n(t_{k-1}^n)\| + \|u_n(t_{k-1}^n) - v_0\| \leqq 3\sigma/4 + \sigma < 2\sigma,$$

and

$$|t - t_0| \leqq |t - t_{k-1}^n| + |t_0 - t_{k-1}^n| \leqq 2\sigma.$$

It therefore follows that

$$\left\|f(t, v) - f\left(t_{k-1}^n, u_n(t_{k-1}^n)\right)\right\| \leqq \|f(t, v) - f(t_0, v_0)\|$$
$$+ \left\|f(t_0, v_0) - f\left(t_{k-1}^n, u_n(t_{k-1}^n)\right)\right\|$$
$$\leqq 1/2n + 1/2n = 1/n.$$

This is a contradiction to the choice of δ_k^n.

We next show that the sequence of continuous functions $\{u_n(t)\}$ converges uniformly to a E-valued function $u(t)$ on $[0, T_0]$.
For this we set $w_{mn}(t)=\|u_m(t)-u_n(t)\|$ for $m>n\geqq 1$ and $t\in[0, T_0]$, and remark first that, since

(3.8) $$|w_{mn}(t)-w_{mn}(s)|\leqq 2M|t-s| \qquad \text{for } s, t\in[0, T_0],$$

$w'_{mn}(t)$ exists for a. e. $t\in[0, T_0]$.

For each $t\in(0, T_0)$ such that $w'_{mn}(t)$ exists there exist positive integers i and j such that $t\in(t^n_{i-1}, t^n_i)$ and $t\in(t^m_{j-1}, t^m_j)$.
By Lemma 3.1 (1), (6) and Lemma 3.2 we have

(3.9) $$\begin{aligned} w'_{mn}(t) &= \langle u_m(t)-u_n(t), f(t, u_m(t^m_{j-1}))-f(t, u_n(t^n_{i-1}))\rangle_2 \\ &\leqq g(t, w_{mn}(t))+\|f(t, u_m(t))-f(t, u_m(t^m_{j-1}))\| \\ &\qquad +\|f(t, u_n(t))-f(t, u_n(t^n_{i-1}))\|. \end{aligned}$$

On the other hand

$$\|u_m(t)-u_m(t^m_{j-1})\|\leqq M|t-t^m_{j-1}|\leqq M\delta^m_j \text{ and } \|u_n(t)-u_n(t^n_{i-1})\|\leqq M\delta^n_i.$$

Thus we have by (3.2)

(3.10) $$w'_{mn}(t)\leqq g\Big(t, w_{mn}(t)\Big)+1/m+1/n\leqq g\Big(t, w_{mn}(t)\Big)+2/n$$

for a. e. $t\in(0, T_0)$. Let $w_n(t)=\sup_{m>n}\{w_{mn}(t)\}$ for $t\in[0, T_0]$.
Then $w_n(0)=(0)$ for all n. It thus follows from (3.8), (3.10) and Lemma 2.2 that

(3.11) $$|w_n(t)-w_n(s)|\leqq 2M|t-s| \qquad \text{for } s, t\in[0, T_0],$$

and

(3.12) $$w'_n(t)\leqq g(t, w_n(t))+2/n \qquad \text{for a. e. } t\in(0, T_0).$$

Since

$$0\leqq w_n(t)\leqq w_n(0)+2Mt\leqq 2MT_0 \qquad \text{for } n\geqq 1 \text{ and } t\in[0, T_0]$$

the sequence $\{w_n\}$ is equicontinuous and uniformly bounded, and hence it has a subsequence converging uniformly on $[0, T_0]$ to a function w, and obviously $w(0)=0$. From (3.12) and the proof of Lemma 2.1 we have

$$w'(t)\leqq g\Big(t, w(t)\Big) \qquad \text{for a. e. } t\in(0, T_0).$$

We show next that $(D^+w)(0)=0$. Since f is continuous at $(0, u_0)$, given $\varepsilon>0$ we can find $\delta>0$ such that $\|f(t, u)-f(t, u_0)\|<\varepsilon$ whenever $0\leqq t\leqq\delta$ and $\|u-u_0\|\leqq\delta$. Let $\delta_0=\mathrm{Min}\,\{\delta, \delta/M\}$. Since $\|u_n(t)-u_0\|\leqq Mt\leqq\delta$, $\|f(t, u_m(t))-f(t, u_n(t))\|<2\varepsilon$ whenever $m>n\geqq 1$ and $t\in[0, \delta_0]$. By Lemma 3.1 (1) and (3.9) we have

$$\begin{aligned} w'_{mn}(t) &= \langle u_m(t)-u_n(t),\ f(t, u_m(t^m_{j-1}))-f(t, u_n(t^n_{i-1}))\rangle_2 \\ &\leqq \|f(t, u_m(t^m_{j-1}))-f(t, u_m(t^n_{i-1}))\| \\ &\leqq \|f(t, u_m(t))-f(t, u_n(t))\|+2/n \leqq 2(\varepsilon+1/n) \end{aligned}$$

for a. e $t\in(0, \delta_0)$, and hence, by integrating the above inequality,

$$0\leqq w_{mn}(t)\leqq 2(\varepsilon+1/n)\,t,$$

whence $(D^+w)\ (0)=0$.

From Lemma 2. 3 we deduce now that $w\equiv 0$, and this implies that the sequence $\{u_n\}$ is uniformly convergent on $[0, T_0]$. The limit u of this sequence satisfies

$$u(t) = u_0+\int_0^t f\big(s, u(s)\big)ds \qquad \text{for } t\in[0, T_0].$$

To show this, note that

$$\int_0^t f\big(s, u(s)\big)ds = \sum_{j=1}^{k-1}\int_{t^n_{j-1}}^{t^n_j} f\big(s, u(s)\big)ds + \int_{t^n_{k-1}}^{t} f\big(s, u(s)\big)ds$$

for $t\in[t^n_{k-1}, t^n_k]$. Then we have by (3. 4)

$$\begin{aligned} &\Big\|u_n(t)-\Big(u_0+\int_0^t f\big(s, u(s)\big)ds\Big)\Big\| \\ &\leqq \sum_{j=1}^{k-1}\int_{t^n_{j-1}}^{t^n_j} \Big\|f\big(s, u_n(t^n_{j-1})\big)-f\big(s, u(s)\big)\Big\|ds \\ &\qquad\qquad + \int_{t^n_{k-1}}^{t} \Big\|f\big(s, u_n(t^n_{k-1})\big)-f\big(s, u(s)\big)\Big\|ds \\ &\leqq \Big[1/n+\operatorname*{Max}_{0\leqq s\leqq T_0}\Big\|f\big(s, u_n(s)\big)-f\big(s, u(s)\big)\Big\|\Big]T. \end{aligned}$$

Because of the uniform convergence of $\{u_n\}$ to u on $[0, T_0]$, $C=\{u_n(t), u(t)\,;\ 0\leqq t\leqq T_0,\ n=1, 2, \cdots\}$ is a compact set in E. Since $f(t, u)$ is uniformly continuous on $[0, T_0]\times C$ we have

$$\operatorname*{Max}_{0\leqq s\leqq T_0}\Big\|f\big(s, u_n(s)\big)-f\big(s, u(s)\big)\Big\|\to 0 \text{ as } n\to\infty,$$

and hence the required result follows.

Thus u is a strongly continuously differentiable solution of (D_1) on $[0, T_0]$.

Let v be another strongly continuously differentiable solution of (D_1) on $[0, T_0]$ and let $z(t)=\|u(t)-v(t)\|$. Then $z(0)=0$, and

$$z'(t) = \langle u(t)-v(t),\ f(t, u(t))-f(t, v(t))\rangle_2 \leqq g(t, z(t))$$

for a. e. $t\in(0, T_0)$. The fact $(D^+z)(0)=0$ follows from

$$0\leqq z(t)/t = \left\|\left(u(t)-v(t)\right)/t\right\|\to 0 \text{ as } t\to 0.$$

It therefore follows from Lemma 2. 3 that $z\equiv 0$. The proof is complete.

§ 4. Proof of Theorem 2.

PROOF of THEOREM 2. It follows from Lemma 2. 4 and Theorem 1 that there exists a unique local solution u of (D_1) on some interval $[0, T_0^*)$. We assume that $[0, T_0^*)$ is a maximal interval of existence of u. We have only to show that $T_0^*<\infty$ leads to a contradiction.

Let $w(t)=\|u(t)-u_0\|$ for $t\in[0, T_0^*)$. Then, by Lemma 3. 1 (6), we have

$$\begin{aligned} w'(t) &= \langle u(t)-u_0,\ f(t, u(t))\rangle_2 \\ &\leqq \langle u(t)-u_0,\ f(t, u(t))-f(t, u_0)\rangle_2+\|f(t, u_0)\| \\ &\leqq g(t, w(t))+L \end{aligned} \tag{4. 1}$$

for a. e. $t\in(0, T_0^*)$, where $L=\operatorname*{Max}_{0\leqq t\leqq T_0^*}\|f(t, u_0)\|$.

In virtue of $(\mathrm{i_b})$, $(\mathrm{ii_b})$ and $(\mathrm{iii_b})$ the differential equation

$$z'(t) = g(t, z(t))+L \tag{4. 2}$$

has a unique solution z on $[0, T_0^*]$ with the initial condition $z(0)=0$. It therefore follows from (4. 1) that

$$w(t)\leqq z(t) \qquad \text{for all } t\in[0, T_0^*). \tag{4. 3}$$

In fact, if we assume that there exists a $\sigma\in(0, T_0^*)$ such that $w(\sigma)>z(\sigma)$. Then there exists a $\sigma_0\in[0, \sigma)$ such that $w(\sigma_0)=z(\sigma_0)$ and $w(t)>z(t)$ for $t\in(\sigma_0, \sigma]$.
Let $\theta(t)=w(t)-z(t)$. Then, by (4. 1), (4. 2) and $(\mathrm{ii_b})$, we have

$$\theta'(t)=w'(t)-z'(t)\leqq g(t, w(t))-g(t, z(t))\leqq m_B(t)\alpha(\theta(t))$$

for a. e. $t\in[\sigma_0, \sigma]$, where $B=\{(t, w(t)),\ (t, z(t))\,;\ \sigma_0\leqq t\leqq\sigma\}$.
Since α is continuous and θ is absolutely continuous, we have for sufficiently small $\varepsilon>0$

$$\int_{\sigma_0+\varepsilon}^{\sigma}\theta'(t)/\alpha\left(\theta(t)\right)dt = \int_{\theta(\sigma_0+\varepsilon)}^{\theta(\sigma)}d\tau/\alpha(\tau)\leqq\int_{\sigma_0+\varepsilon}^{\sigma}m_B(t)\,dt.$$

By $(\mathrm{iii_b})$ and by letting $\varepsilon\downarrow 0$, we have a contradiction.

(4. 3) implies that

$$\|u(t)\|\leqq\|u_0\|+\operatorname*{Max}_{0\leqq t\leqq T_0^*}\{z(t)\} \text{ for } t\in[0, T_0^*).$$

Since $\{f(t, u(t))\,;\ t\in[0, T_0^*)\}$ is a bounded set in E, we have

$$\|u(t)-u(s)\| \leqq \left|\int_s^t \|f\big(\tau, u(\tau)\big)\| d\tau\right| \to 0 \text{ as } s, t \uparrow T_0^*.$$

Let $v_0 = \lim_{t \uparrow T_{0*}} u(t)$, then we can apply Theorem 1 once more with the initial condition $u(T_0^*) = v_0$, and obtain a unique continuation of the solution u beyond T_0^*, which contradicts the assumption on T_0^*.

§ 5. Proof of Theorem 3.

Throughout this section we assume that the dual space E^* is uniformly convex.

We say that F is a duality mapping of E into E^* if to each u in E it assigns (in general a set) $F(u)$ in E^* determined by

$$F(u) = \left\{x^* ;\ x^* \in E^* \text{ such that } (u, x^*) = \|u\|^2 = \|x^*\|^2\right\},$$

where (u, x^*) denotes the value of x^* at u.

Since E^* is uniformly convex F is single-valued and uniformly continuous on any bounded subset of E (see [9]).

LEMMA 5. 1. *For each $u \neq 0$ and v in E*

$$\langle u, v\rangle_2 = Re(v, F(u))/\|u\|.$$

PROOF. Since $\langle u, v\rangle_1 = \mathrm{Re}(v, F(u))/\|u\|$ for each $u \neq 0$ and v in E (see the proof of Proposition 2. 5 in [11]),

$$\langle u, v\rangle_2 = \frac{1}{2}\mathrm{Re}\Big(v, F(u)\Big) - \mathrm{Re}\Big(-v, F(u)\Big) = \mathrm{Re}\Big(v, F(u)\Big).$$

We recall that A satisfies

(5. 1) $$\langle u-v,\ Au-Av\rangle_2 \leqq 0 \qquad \text{for } u, v \in D(A),$$

and $\mathrm{R}(I-\lambda_0 A) = E$ for some $\lambda_0 > 0$.
For such an operator A we have

LEMMA 5. 2. *$(I-\lambda A)^{-1}$ exists for any $\lambda > 0$.*
Set $J_n = (I - \frac{1}{n} A)^{-1}$ *and* $A_n = A\ J_n = n(J_n - I)$ *for* $n = 1, 2, \cdots,$.
Then

(1) $$\|J_n u - J_n v\| \leqq \|u-v\| \qquad \textit{for } u, v \in E,$$

(2) $$\|A_n u\| \leqq Au\| \qquad \textit{for } u \in D(A),$$

(3) $$\langle u-v,\ A_n u - A_n v\rangle_2 \leqq 0 \qquad \textit{for } u, v \in E,$$

and

(4) *A is demiclosed, that is, if $u_n \in D A)$, $n=1, 2, \cdots$, $u_n \to u$ (strongly in E) and $Au_n \to v$ (weakly in E), then $u \in D(A)$ and $v=Au$.*

PROOF. In virtue of Lemma 5.1, $-A$ is m-monotonic in the sense of T. Kato [9], and hence, the existence of $(I-\lambda A)^{-1}$ and (1), (2) and (4) follows from Lemma 2.5 in [9]. To prove (3) note that

$$\begin{aligned}&\langle u-v,\ A_n u-A_n v\rangle_2 = n\langle u-v,\ J_n u-J_n v-(u-v)\rangle_2\\&= n(\langle u-v,\ J_n u-J_n v\rangle_2-\|u-v\|)\\&\leqq n(\|J_n u-J_n v\|-\|u-v\|)\leqq 0,\end{aligned}$$

where we used (1) and Lemma 3.1 (1), (4).

In Theorem 2, if $g(t, \tau)=\beta(t)\tau$, where β is a locally Lebesgue integrable function defined on $(0, \infty)$, then the conclusion of Theorem 2 remains valid. In fact, it is obvious that this function $\beta(t)\tau$ satisfies the conditions ($\mathrm{i_b}$), ($\mathrm{ii_b}$) and ($\mathrm{iii_b}$) except that $\beta(t)\tau$ need not be nondecreasing in τ for fixed t. However, the nondecreasing nature of g in τ was used in establishing Lemma 2.3 which is valid for this $\beta(t)\tau$.

LEMMA 5.3. *Under the hypothesis of Theorem 3 the differential equation*

$$\frac{d}{dt}u_n(t) = A_n u_n(t)+f\Big(t,\ u_n(t)\Big), \qquad u_n(0) = u_0 \in E,$$

has a unique strongly continuously differentiable solution u_n defined on $[0, \infty)$.

PROOF. Since $\|A_n u-A_n v\| \leqq 2n\|u-v\|$ for u, v in E, $A_n u+f(t, u)$ carries bounded sets in $[0, \infty)\times E$ into bounded sets in E. By Lemma 3.1 (5) and Lemma 5.2 (3) we have

$$\begin{aligned}&\langle u-v,\ A_n u+f(t, u)-(A_n v+f(t, v))\rangle_2\\&\leqq \langle u-v,\ A_n u-A_n v\rangle_2+\langle u-v,\ f(t, u)-f(t, v)\rangle_1\\&\leqq \beta(t)\|u-v\|\end{aligned}$$

for (t, u), $(t, v) \in [0, \infty)\times E$.

Hence the assertion follows directly from Theorem 2 and the above mentioned remark.

We shall now deduce some estimates for $u_n(t)$.

LEMMA 5.4. *Let $u_0 \in D(A)$. Then $\{u_n(t)\}$ and $\{u_n' t)\}$ are bounded on any finite interval of $[0, \infty)$.*

PROOF. By Lemma 3.1 (3) and Lemma 5.2 (2), (3)

$$\frac{d}{dt}\|u_n(t)-u_0\| = \langle u_n(t)-u_0,\ A_n u_n(t)+f(t, u_n(t))\rangle_2$$
$$\leqq \langle u_n(t)-u_0,\ A_n u_n(t)\rangle_2+\langle u_n(t)-u_0,\ f(t, u_n(t))\rangle_1$$
$$\leqq \langle u_n(t)-u_0,\ f(t, u_n(t))-f(t, u_0)\rangle_1+\|f(t, u_0)\|+\|A_n u_0\|$$
$$\leqq \beta(t)\|u_n(t)-u_0\|+\|f(t, u_0)\|+\|A\ u_0\|.$$

Thus we have

$$\|u_n(t)-u_0\| \leqq \int_0^t \exp\left(\int_s^t \beta(\tau)\,d\tau\right)\left(\|f(s, u_0)\|+\|A\ u_0\|\right)ds$$

for $n=1, 2, \cdots$. This implies

$$\|u_n(t)\| \leqq \|u_0\|+\int_0^t \exp\left(\int_s^t \beta(\tau)\,d\tau\right)\left(\|f(s, u_0)\|+\|A\ u_0\|\right)ds \tag{5. 2}$$

for $t\in[0, \infty)$ and $n=1, 2, \cdots$.

For each fixed $h>0$ we have, by Lemma 3. 1 (5) and Lemma 5. 2 (3),

$$\frac{d}{dt}\|u_n(t+h)-u_n(t)\| = \langle u_n(t+h)-u_n(t),\ A_n u_n(t+h)-A_n u_n(t)$$
$$+f(t+h,\ u_n(t+h))-f(t, u_n(t))\rangle_2$$
$$\leqq \langle u_n(t+h)-u_n(t),\ f(t+h, u_n(t+h))-f(t, u_n(t))\rangle_1$$
$$\leqq \langle u_n(t+h)-u_n(t),\ f(t+h, u_n(t+h))-f(t, u_n(t))\rangle_1$$
$$+\|f(t+h, u_n(t))-f(t, u_n(t))\|$$
$$\leqq \beta(t+h)\|u_n(t+h)-u_n(t)\|+\|f(t+h, u_n(t))-f(t, u_n(t))\|.$$

It follows that

$$\|u_n(t+h)-u_n(t)\| \leqq \|u_n(h)-u_n(0)\|$$
$$+\int_0^t \exp\left(\int_s^t \beta(\tau+h)\,d\tau\right)\left\|f\left(s+h, u_n(s)\right)-f\left(s, u_n(s)\right)\right\|ds$$

By dividing the above inquality by h and letting $h\downarrow 0$, we have

$$\|u_n'(t)\| \leqq \|u_n'(0)\|+\int_0^t \exp\left(\int_s^t \beta(\tau)\,d\tau\right)\left\|f_s\left(s, u_n(s)\right)\right\|ds \tag{5. 3}$$

for $n=1, 2, \cdots$. This completes the proof.

We shall now give the proof of Theorem 3.

Proof of Theorem 3. By (5. 2) and (5. 3) there exists constant $M_T>0$ for each $T>0$ such that

$$\|u_n'(t)\|+\|f(t, u_n(t))\| \leqq M_T \qquad \text{for } t\in[0, T] \text{ and } n\geqq 1. \tag{5. 4}$$

By Lemma 3. 1 (5) and Lemma 5. 1, for each $t\in[0, T]$ such that

$$\frac{d}{dt}\|u_n(t)-u_m(t)\| \text{ exists and } u_n(t)-u_m(t)\neq 0,$$

$$\frac{d}{dt}\|u_n(t)-u_m(t)\| = \langle u_n(t)-u_m(t),\ A_nu_n(t)-A_mu_m(t) + f(t, u_n(t))-f(t, u_m(t))\rangle_2$$

$$\leqq \beta(t)\|u_n(t)-u_m(t)\| + 2M_T\|F(u_n(t)-u_m(t))-F(J_nu_n(t)-J_mu_m(t))\|/\|u_n(t)-u_m t)\|.$$

It follows that

$$\frac{d}{dt}\|u_n(t)-u_m(t)\|^2 \leqq 2\beta(t)\|u_n(t)-u_m(t)\|^2 + 4M_T\|F(u_n(t)-u_m(t))-F(J_nu_n(t)-J_mu_m(t))\|.$$

On the other hand, for each $t\in[0, T]$ such that $\frac{d}{dt}\|u_n(t)-u_m(t)\|$ exists and $u_n(t)-u_m(t)=0$,

$$\frac{d}{dt}\|u_n(t)-u_m(t)\| = \langle 0,\ A_nu_n(t)-A_mu_m(t)\rangle_2 = 0.$$

Thus we have

$$\frac{d}{dt}\|u_n(t)-u_m(t)\|^2 \leqq 2\beta(t)\|u_n(t)-u_m(t)\|^2 + 4M_T\|F(u_n(t)-u_m(t))-F(J_nu_n(t)-J_mu_m(t))\|$$

for a. e. $t\in[0, T]$ and $n, m\geqq 1$.
Consequently

$$\|u_n(t)-u_m(t)\|^2 \leqq 4M_T\int_0^t \exp\left(\int_s^t 2\beta(\tau)\,d\tau\right)\|F(u_n(s)-u_m(s))-F(J_nu_n(s) - J_mu_m(s))\|\,ds$$

for $t\in[0, T]$ and $n, m\geqq 1$.
In virtue of (5. 4) and the definition of A_n

$$\|u_n(s)-u_m(s)-(J_nu_n(s)-J_mu_m(s))\| \leqq \frac{1}{n}\|A_nu_n(s)\| + \frac{1}{m}\|A_mu_m(s)\|$$

$$\leqq M_T(1/n+1/m)\to 0 \text{ as } n, m\to\infty.$$

Since $F(u)$ is uniformly continuous on any bounded set in E, $\{u_n(t)\}$ converges uniformly to a continuous function $u(t)$ on $[0, T]$ for each $T>0$. The absolute continuity of $u(t)$ on $[0, T]$ follows from the inequality

$$\|u_n(t)-u_n(s)\| \leqq \left|\int_s^t \|u_n'(\tau)\|\,d\tau\right| \leqq M_T|t-s| \qquad \text{for } t, s\in[0, T].$$

We show next that $u(t)$ is a solution of (D_1).
By (5.4) we have

$$\|A_n u_n(t)\| \leqq \|u_n'(t)\| + \|f(t, u_n(t))\| \leqq M_T \tag{5.5}$$

for $t \in [0, T]$ and $n \geqq 1$.

This implies that $\{A_n u_n(t)\}$ is a bounded set in $L_E^2\,[0, T]$ for each $T>0$, where $L_E^2\,[0, T]$ denotes the set of all square integrable E-valued strongly measurable functions on $[0, T]$.

Thus some subsequence of $\{A_n u_n(t)\}$ converges to an element z weakly in $L_E^2\,[0, T]$. For notational convenience we assume that $\{A_n u_n(t)\}$ itself converges to z weakly in $L_E^2\,[0, T]$.

Let $C[t]$ be the set of all weak limit in E of a subsequence of $\{A_n u_n(t)\}$ for each fixed $t \in [0, T]$.

We will show that $u(t) \in D(A)$ for all $t \in [0, T]$ and $z(t) = A\ u(t)$ for a. e. $t \in [0, T]$ (cf. T. Kato [10]).
To show this we note that for each $v \in C[t]$ there exists a subsequence $\{A_{nm} u_{nm}(t)\}$ such that $w\text{-}\lim\limits_{m \downarrow \infty} A_{nm} u_{nm}(t) = v$, where w-lim denotes weak limit in E. Since $J_{nm} u_{nm}(t) \to u(t)$, $J_{nm} u_{nm}(t) \in D(A)$ and $A_{nm} u_{nm}(t) = A\ J_{nm} u_{nm}(t)$, it follows from the demiclosedness of A that

$$u(t) \in D(A) \text{ and } v = A\ u(t).$$

Hence $C[t]$ consists of only one element for each $t \in [0, T]$. Since any subsequence of $\{A_n u_n(t)\}$ has a subsequence converging weakly to the same element $v = v(t)$, $\{A_n u_n(t)\}$ itself converges weakly to $v(t)$ for each $t \in [0, T]$. Since $\{A_n u_n(t)\}$ converges to z weakly in $L_E^2\,[0, T]$, z is the strong limit of the type $\sum\limits_i a_i A_{n+i} u_{n+i}$. Here $\{a_i\}$ is a finite set of nonnegative numbers such that $\sum\limits_i a_i = 1$.
Thus we can find a subsequence of the above sequence converging to $z(t)$ strongly in E for a. e. $t \in [0, T]$.

Let U be any open convex neighbourhood of 0 in the weak topology of E. Then there exists an open convex neighbourhood V of 0 in the same topology of E such that $V + V \subset U$.
Since $v(t) + V$ is open convex in the weak topolopy of E, there is a n_0 such that

$$A_n u_u(t) \in v(t) + V \qquad \text{for } n \geqq n_0.$$

Thus the convex combination of the type $\sum\limits_i a_i A_{n+i} u_{n+i}(t)$ belongs to $v(t) + V$ for $n \geqq n_0$. Hence $z(t) \in (v(t) + V)^{-w}$, where $(v(t) + V)^{-w}$ denotes the closure of $v(t) + V$ with respect to the weak topology of E. Since

$$(v(t)+V)^{-w}\subset(v(t)+V)+V\subset v(t)+U,$$

it follows that $z(t)-v(t)\in U$. This implies that

$$z(t)=v(t) \qquad \text{for a. e. } t\in[0, T].$$

Since $\|A_n u_n(t)\|\leqq M_T$ the norm of a convex combination of $A_n u_n(t)$'s is also $\leqq M_T$. It follows that $\|z(t)\|\leqq M_T$ for a. e. $t\in[0, T]$ and that $z(t)$ is Bochner integrable on $[0, T]$. Since $L_E^2[0, T]^*=L_{E^*}^2[0, T]$ and since

$$(u_n(t), x^*)=(u_0, x^*)+\int_0^t\Big(A_n u_n(s)+f\big(s, u_n(s)\big),\ x^*\Big)ds$$

for each $x^*\in E^*$ and $t\in[0, T]$, we have by going to $n\to\infty$

$$(u(t), x^*)=(u_0, x^*)+\int_0^t\Big(z(s)+f\big(s, u(s)\big), x^*\Big)ds.$$

Thus we obtain that $\frac{d}{dt}u(t)$ exists for a. e. $t\in[0, T]$ and

$$\frac{d}{dt}u(t)=z(t)+f(t, u(t)=A\ \ u(t)+f(t, u(t)) \qquad \text{for a. e. } t\in[0, T].$$

Since T is arbitrary, the proof is complete.

§ 6. Remarks and an example.

In this section we give some remarks about the relations between our results and those of F. E. Browder and T. M. Flett. We give also a simple example to which our Theorem 2 applies.

REMARK 1. In the papers [4] and [5] T. M. Flett has given sufficient conditions for the existence in Banach and Hilbert spaces of the unique local solution of (D_1) on some interval $[0, T_0]$ under the following conditions:
(A) E is a Banach space and f is a continuous mapping of $[0, T]\times S(u_0, r)$ into E such that for all (t, u), $(t, v)\in(0, T]\times S(u_0, r)$

$$\|f(t, u)-f(t, v)\|\leqq g(t, \|u-v\|); \tag{6.1}$$

(B) E is a Hilbert space with inner product $(,)$ and f is a continuous mapping of $[0, T]\times S(u_0, r)$ into E such that for all (t, u), $(t, v)\in(0, T]\times S(u_0, r)$

$$\mathrm{Re}\,(f(t, u)-f(t, v),\ u-v)\leqq\|u-v\|g(t, \|u-v\|), \tag{6.2}$$

where g is a continuous function defined on $(0, T]\times[0, 2r]$ satisfying the condition (ii_a) in § 1 in this paper.

In Theorem 1 if we assume that $g(t, \tau)$ is continuous on $(0, T]\times[0, 2r]$, then we can drop the assumption that $g(t, \tau)$ is nondecreasing in τ for fixed t (cf. [2]).

In virtue of this fact and Lemma 3. 1 (1), (4) our result is an extension of (A). If E is a Hilbert space with inner product $(,)$, then we can easily see that

$$\langle u, v\rangle_2 = \mathrm{Re}(v, u)/\|u\| \qquad \text{for } u \neq 0 \text{ and } v \text{ in } E,$$

and hence, our condition of Theorem 1 becomes

$$\mathrm{Re}(f(t, u)-f(t, v),\ u-v) \leqq \|u-v\| g(t, \|u-v\|)$$

for all (t, u) $(t, v) \in (0, T] \times S(u_0, r)$. Thus our result is also an extension of (B).

Let $F(u)$ be the duality mapping of E into E^* defined in § 5. Then for each $u \neq 0$ and v in E

$$\langle u, v\rangle_2 \leqq \mathrm{Re}(v, x^*)/\|u\| \qquad \text{for some } x^* \in F(u)$$

(see the proof of Proposition 2. 5 in [11]).

Thus we can replace the condition of Theorem 1 by the following one.

$$\mathrm{Re}(f(t, u)-f(t, v), x^*) \leqq \|u-v\| g(t, \|u-v\|)$$

for (t, u), $(t, v) \in (0, T] \times S(u_0, r)$ and for all $x^* \in F(u-v)$.

Hence our result is a generalization of (B) into a general Banach space.

Remark 2. In [1] F. E. Browder proved the existence and uniqueness of a strongly continuously differentiable solution of (D_1) on $[0, \infty)$ under the following conditions:

(I) E is a Hilbert space with inner product $(,)$ and f is a continuous mapping of $[0, \infty) \times E$ into E, carring bounded sets in $[0, \infty) \times E$ into bounded sets in E.

(II) There exists a real-valued continuous function $c(t)$ defined on $[0, \infty)$ such that

$$\mathrm{Re}\,(f(t, u)-f(t, v),\ u-v) \leqq c(t)\|u-v\|^2 \tag{6. 3}$$

for all (t, u), $(t, v) \in [0, \infty) \times E$.

By the same argument as in Remark 1 we see that Theorem 2 is a generalization into a general Banach space of the above result of F. E. Browder.

The following example shows that the conditions of Theorem 2 are more general than those of F. E. Browder.

EXAMPLE. Let $E=R^1$ and let $a(t)$ be the function defined by

$$a(t) = \begin{cases} t\,(0 \leqq t \leqq \varepsilon) \\ \varepsilon\,(t > \varepsilon) \end{cases}$$

where ε is a positive constant. We consider the differential equation

$$\frac{d}{dt}u = f(t, u) = \begin{cases} 1+\dfrac{1}{1+\sqrt{u}} & (t \geqq 0,\ u > a(t)) \\ 1+\dfrac{1}{1+\sqrt{a(t)}} & (t \geqq 0,\ u \leqq a(t)). \end{cases}$$

Obviously, the function $f(t, u)$ is continuous from $[0, \infty) \times R^1$ into R. However the function $f(t, u)$ does not satisfy the monotonicity condition (6. 3) but does satisfy all our conditions of Theorem 2.

In fact, for $u \neq v$ and $t > 0$

$$\begin{aligned} &\langle u-v, f(t, u)-f(t, v)\rangle_2 = (f(t, u)-f(t, v))\ (u-v)/|u-v| \\ &= \pm(f(t, u)-f(t, v)) \\ &\leqq \begin{cases} (1/2\sqrt{a(t)}\,)|u-v| & (u, v > a(t), t > 0) \\ (1/2\sqrt{a(t)}\,)|u-v| & (u > a(t),\ 0 \leqq v \leqq a(t),\ t > 0) \\ (1/2\sqrt{a(t)}\,)|u-v| & (u > a(t),\ v < 0, t > 0) \\ 0 & (u, v \leqq a(t,)\ t > 0). \end{cases} \end{aligned}$$

Thus we have

$$\langle u-v,\ f(t, u)-f(t, v)\rangle_2 \leqq \ (1/2\sqrt{a(t)}\,)|u-v|$$

for all (t, u), $(t, v) \in (0, \infty) \times R^1$.

Set $g(t, \tau) = (1/2\sqrt{a(t)}\,)\,\tau$ and $\alpha(t)=t$, then it follows easily that g and α satisfy all our conditions of Theorem 2.

On the other hand we have

$$(f(t, u)-f(t, v),\ u-v) \leqq (1/2\sqrt{a(t)}\,)|u-v|^2$$

for all (t, u), $(t, v) \in (0, \infty) \times R^1$.

Since $1/2\sqrt{a(t)}$ is discontinuous at 0, the condition (6. 3) does not hold.

REMARK 3. In Theorem 3 if A is linear and $D(A)$ is dense in E, then A is the infinitesimal generator of a strongly continuous contraction semi-group $\{T(t)\,;\ t \geqq 0\}$ (see M. Hasegawa [6]).
In this case the integral equation

$$v(t) = u_0 + \int_0^t T(t-s)\, f\Big(s, v(s)\Big) ds$$

has a unique solution for each $u_0 \in D(A)$ by the same argument as G. Webb [15]. We don't know whether the solution of the above integral equation is a solution of (D_2).

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(Received February 7, 1974)
(Revised June 15, 1974)

Funkcialaj Ekvacioj

(Serio Internacia)

VOLUMO 19, NUMERO 3

DECEMBRO, 1976

On Existence and Uniqueness Conditions for Nonlinear Ordinary Differential Equations in Banach Spaces

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Coden FESIAT	Pages 203–300
Volume 19	December 1976
Number 3	Whole No. 47

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JAPANA MATEMATIKA SOCIETO

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Funkcialaj Ekvacioj, **19** (1976) 239-245

On Existence and Uniqueness Conditions for Nonlinear Ordinary Differential Equations in Banach Spaces

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1. Introduction and results.

Let E be a Banach space with the dual space E^*. The norms in E and E^* are denoted by $\| \quad \|$. We denote by $S(u, r)$ the closed sphere of center u with radius r.

We consider the Cauchy problem

$$\text{(CP)} \qquad u'(t)=f(t, u(t)), \qquad u(0)=u_0 \in E,$$

where f is a E-valued mapping defined on $[0, T]\times S(u_0, r)$ or on $[0, \infty)\times E$.

Many authors have studied this problem and some of their articles are listed in our references.

It is our object in this paper to give sufficient conditions for both local and global existence of strongly continuous, once weakly continuously differentiable solutions to (CP).

Throughout this paper, whenever we speak of a solution to (CP), we will mean a strongly continuous, once weakly continuously differentiable function u on some interval $[0, a]$ (or $[0, a)$).

The techniques employed in this paper are similar to those of the present author [8] and J. M. Bownds and J. B. Diaz [1].

Let E_ω denote the space E with its weak topology and let f be a mapping from $[0, T]\times S(u_0, r)$ (or $[0, \infty)\times E$) into E. We say f is weakly continuous if it is continuous from $[0, T]\times S(u_0, r)$ (or $[0, \infty)\times E_\omega$) endowed with relative topology of $[0, \infty)\times E_\omega$ into E_ω.

We now state the following result.

Theorem 1. *Let f be a weakly continuous mapping from $[0, T]\times S(u_0, r)$ into E. Suppose further that the range $f([0, T]\times S(u_0, r))$ is relatively compact in E_ω. Then* (CP) *has at least one solution u defined on some interval* $[0, a]$.

Remark 1. The solution u to (CP) mentioned in Theorem 1 is strongly differentiable for a.e.t $\in [0, a]$ and satisfies $\frac{d}{dt}u(t)=f(t, u(t))$ for a.e.t $\in [0, a]$, where $\frac{d}{dt}u$ denotes the strong derivative of u (see [5]).

If E is a reflexive Banach space then we have the following result similar to that of Theorem 7 in F. E. Browder [2].

Theorem 2. *Let E be a reflexive Banach space and let f be a weakly continuous mapping from $[0, \infty)\times E$ into E.*

Then for each $r>0$ there exists $a(r)>0$ such that, for each u_0 in E with $\|u_0\| \leqq r$, (CP) *has at least one solution u defined on $[0, a(r)]$.*

Remark 2. In Theorem 2 if E is a Hilbert space, then F. E. Browder proved that (CP) has a strongly C^1 solution defined on $[0, a(r)]$ (see [2]).

We next consider the global existence and uniqueness of solutions to (CP). We define $\langle\ ,\ \rangle: E\times E\rightarrow R$ by

$$\langle v, w\rangle=\frac{1}{2}\lim_{h\rightarrow+0}\frac{1}{h}(\|v+hw\|-\|v-hw\|).$$

For the properties of the functional $\langle\ ,\ \rangle$ see, for example, [8], where $\langle\ ,\ \rangle$ was denoted by $\langle\ ,\ \rangle_2$.

Theorem 3. *Let E be a reflexive Banach space and let f be a weakly continuous mapping from $[0, \infty)\times E$ into E. Suppose further that*

$$\langle v-w, f(t, v)-f(t, w)\rangle\leqq\beta(t)\|v-w\| \tag{1.1}$$

for all v, w in E and a.e.t $\in (0, \infty)$, where $\beta\in L^1_{\mathrm{loc}}(0, \infty)$. Then for each u_0 in E (CP) *has a unique solution u defined on $[0, \infty)$.*

Remark 3. In Theorem 3 if E is a Hilbert space with inner product denoted by $(\ ,\ )$. Then it is easy to see that

$$\langle v, w\rangle=\mathrm{Re}\,(v, w)/\|v\| \qquad \text{for } v\neq 0 \text{ and } w \text{ in } E,$$

and hence, the condition (1.1) becomes

$$\mathrm{Re}\,(f(t, v)-f(t, w), v-w)\leqq\beta(t)\|v-w\|^2$$

for all v, w in E and a.e.t>0. This implies that $f(t, \cdot)-\beta(t)I$ is dissipative for a.e.t>0, where I denotes the identity mapping.

2. Proof of Theorem 1.

Since $f([0, T]\times S(u_0, r))$ is relatively compact in E_ω there exists a constant $M>0$ such that

$$\|f(t, v)\|\leqq M \qquad \text{for } (t, v)\in[0, T]\times S(u_0, r).$$

Let $a=\min\{T, r/M\}$. For each positive integer n, let

$$\Delta_n: 0=t_0^n<t_1^n<\cdots<t_{N(n)}^n=a$$

be a partition of the interval $[0, a]$ such that

$$(2.1) \qquad |\Delta_n|=\max\{t_k^n-t_{k-1}^n;\ 1\leqq k\leqq N(n)\}\to 0 \quad \text{as} \quad n\to\infty.$$

For each Δ_n the approximate solution u_n to (CP) on $[0, a]$ is defined inductively as follows:

$$(2.2) \qquad u_n(t_0^n)=u_0, \qquad u_n(t)=u_n(t_i^n)+(t-t_i^n)f(t_i^n, u_n(t_i^n))$$

for $t\in[t_i^n, t_{i+1}^n]$ and $i=1, 2, \cdots, N(n)-1$. Then it is easy to see that u_n is well defined on $[0, a]$ and satisfies

$$(2.3) \qquad \|u_n(t)-u_n(s)\|\leqq M|t-s|$$

for $s, t\in[0, a]$ and $n\geqq 1$. Thus $u_n(t)\in S(u_0, r)$ for $t\in[0, a]$ and $n\geqq 1$. Let $t\in(0, a]$ be such that $t\in[t_i^n, t_{i+1}^n]$ for some i. Then

$$(u_n(t)-u_0)/t=\sum_{k=1}^{i}\frac{t_k^n-t_{k-1}^n}{t}f(t_{k-1}^n, u_n(t_{k-1}^n))+\frac{t-t_i^n}{t}f(t_i^n, u_n(t_i^n)).$$

It follows that $(u_n(t)-u_0)/t\in co(f([0, T]\times S(u_0, r)))$, where $co(f([0, T]\times S(u_0, r)))$ denotes the convex hull of $f([0, T]\times S(u_0, r))$. Since $f([0, T]\times S(u_0, r))$ is relatively compact in E_ω, it follows that $co(f([0, T]\times S(u_0, r)))$ is also relatively compact in E_ω (see [5]). In E_ω the relative compactness is equivalent to the relative sequential compactness (see [5]). Consequently, the sequence $\{(u_n(t)-u_0)/t\}_{n=1}^\infty$ contains a subsequence which converges in E_ω, and this implies that the sequence $\{u_n(t)\}$ also contains a subsequence which converges in E_ω. Thus we have proved that the sequence $\{u_n(t)\}_{n=1}^\infty$ has a subsequence which converges in E_ω for each $t\in[0, a]$. In virtue of (2.3) it follows, by the diagonal method, that $\{u_n\}$ has a subsequence $\{u_{n'}\}$ such that ω-lim $u_{n'}(t)=u(t)$ exists for all $t\in[0, a]$, simultaneously, where ω-lim means limit in E_ω. For notational convenience we assume that $\{u_n(t)\}$ itself converges to $u(t)$ in E_ω. The limit function u satisfies

$$\|u(t)-u(s)\|\leqq M|t-s| \qquad \text{for } s, t\in[0, a].$$

In fact, for each $\varepsilon>0$ and for each $x^*\in E^*$ such that

$$(u(t)-u(s), x^*)=\|x^*\|^2=\|u(t)-u(s)\|^2,$$

there exists an n_0 such that

$$|(u(t)-u(s), x^*)|<|(u_n(t)-u_n(s), x^*)|+\varepsilon$$

for $n\geqq n_0$, where (v, x^*) denotes the value of x^* at v. Thus

$$\begin{aligned}\|u(t)-u(s)\|^2&<|(u_n(t)-u_n(s), x^*)|+\varepsilon\\&\leqq M\,|t-s|\,\|x^*\|+\varepsilon\\&=M\,\|u(t)-u(s)\|\,|t-s|+\varepsilon\end{aligned}$$

and hence $\|u(t)-u(s)\|\leqq M\,|t-s|$.

We next show that u is a solution to (CP) on $[0, a]$. Rewriting (2.2) we have

$$u_n(t)=u_0+\int_0^t f(\tau, u_n(\tau))d\tau+\int_0^t (f_{\Delta_n}(\tau)-f(\tau, u_n(\tau)))d\tau,$$

where

$$f_{\Delta_n}(\tau)=f(t_i^n, u_n(t_i^n)) \qquad \text{for } \tau\in[t_i^n, t_{i+1}^n] \text{ and } 0\leqq i\leqq N(n)-1.$$

Since $C=\{u_n(t)\,;\, t\in[0, a], n=1, 2, \cdots\}\cup\{u(t)\,;\, t\in[0, a]\}$ is compact in E_ω, $[0, a]\times C$ is also compact in $[0, \infty)\times E_\omega$. Thus, for each $x^*\in E^*$, $(f(t, v), x^*)$ is uniformly continuous on $[0, a]\times C$, that is, for each $\varepsilon>0$ there exist a neighbourhood U of 0 in E_ω and a $\delta=\delta(\varepsilon, x^*)>0$ such that

$$|(f(t, v)-f(t, w), x^*)|<\varepsilon$$

whenever $|t-s|<\delta$ $(s, t\in[0, a])$ and $v-w\in U$ $(v, w\in C)$. Here we may choose U such that for some $y_j^*\in E^*$ $(j=1, 2, \cdots, p)$

$$U=\{v\,;\, |(v, y_j^*)|<\delta,\ j=1, 2, \cdots, p\}.$$

By (2.1) we can choose an n_0 such that

$$|\Delta_n|<\min\left\{\delta, \delta/M \max_{1\leqq j\leqq p}\|y_j^*\|\right\} \qquad \text{for } n\geqq n_0.$$

For each $\tau\in[0, a]$ we can choose i with $0\leqq i\leqq N(n)-1$ such that $\tau\in[t_i^n, t_{i+1}^n]$. Thus we have for $n\geqq n_0$

$$|\tau-t_i^n|\leqq t_{i+1}^n-t_i^n\leqq|\Delta_n|<\delta$$

and

$$|(u_n(t_i^n)-u_n(\tau), y_j^*)|\leqq M\,\|y_j^*\|\,|\tau-t_i^n|<\delta$$

for $j=1, 2, \cdots, p$, which imply that

$$|(f_{\Delta_n}(\tau)-f(\tau, u_n(\tau)), x^*)|<\varepsilon.$$

Consequently, for $n\geqq n_0$

$$\begin{aligned}&\left|\left(u_n(t)-u_0-\int_0^t f(\tau, u_n(\tau))d\tau, x^*\right)\right|\\ &\leqq\int_0^t |(f_{\Delta_n}(\tau)-f(\tau, u_n(\tau)), x^*)|\, d\tau\\ &\leqq\varepsilon a.\end{aligned}$$

Since $x^*\in E^*$ was arbitrary, it thus follows that

$$u(t)=u_0+\int_0^t f(\tau, u(\tau))d\tau \qquad \text{for } t\in[0, a].$$

Since f is weakly continuous, u is strongly continuous, once weakly continuously differentiable on $[0, a]$ and satisfies

$$u'(t)=f(t, u(t)) \qquad \text{for } t\in[0, a],$$

where u' denotes the weak derivative of u. The proof is complete.

3. Proof of Theorem 2.

Since E is a reflexive Banach space and f is weakly continuous, f maps bounded sets of $[0, \infty)\times E$ into bounded sets of E. Thus for each $r>0$ there exists $M(r)>0$ such that

$$\|f(t, v)\|\leqq M(r)$$

for each $t\in[0, 1]$, $v\in E$ with $\|v\|\leqq r$. Let $a(r)=\min\{1, r/M(2r)\}$ and let $\{u_n\}$ be the sequence of the approximate solutions to (CP) on $[0, a(r)]$ as in the proof of Theorem 1. Then for each $s, t\in[0, a(r)]$ and $n\geqq 1$

$$\|u_n(t)-u_n(s)\|\leqq M(2r)\,|t-s|$$

and

$$\|u_n(t)-u_0\|\leqq M(2r)t\leqq r.$$

Thus it follows that $\|u_n(t)\|\leqq 2r$ for $t\in[0, a(r)]$ and $n\geqq 1$. Since $co(f([0, a(r)]\times S(u_0, r)))$ is bounded in norm, it is relatively compact in E_ω. By the same argument as in the proof of Theorem 1, the sequence $\{u_n\}$ has a subsequence which converges in E_ω for all $t\in[0, a(r)]$.

The rest of the proof is the same as the corresponding part of that of Theorem 1.

4. Proof of Theorem 3.

It follows from Theorem 2 that there exists a local solution u to (CP) on some interval $[0, b)$. We assume that $[0, b)$ is a maximal interval of existence of u. We have only to show that $b<\infty$ leads to a contradiction. By Lemma 3.1, 3.2 in [8] and (1.1) we have for a.e.t $\in (0, b)$

$$\begin{aligned}\frac{d}{dt}\|u(t)-u_0\| &= \left\langle u(t)-u_0, \frac{d}{dt}u(t)\right\rangle = \langle u(t)-u_0, f(t, u(t))\rangle \\ &\leqq \langle u(t)-u_0, f(t, u(t))-f(t, u_0)\rangle + \|f(t, u_0)\| \\ &\leqq \beta(t)\, \|u(t)-u_0\| + \|f(t, u_0)\|.\end{aligned}$$

It follows that

$$\|u(t)-u_0\| \leqq \int_0^t \exp\left(\int_s^t \beta(\tau)d\tau\right) \|f(s, u_0)\|\, ds.$$

Since $f(\cdot, u_0)$ is continuous from $[0, b]$ into E_ω, $\{f(s, u_0)\,;\, s\in[0, b]\}$ is bounded in norm by the Banach-Steinhaus theorem. Thus we have

$$\|u(t)\| \leqq \|u_0\| + \int_0^b \exp\left(\int_s^b \beta(\tau)d\tau\right) \|f(s, u_0)\|\, ds$$

for $t \in [0, b)$. Since f maps bounded sets of $[0, \infty)\times E$ into bounded sets of E, it follows that

$$\|u(t)-u(s)\| \leqq \left|\int_s^t \|f(\tau, u(\tau))\|\, d\tau\right| \to 0 \quad \text{as} \quad s, t\uparrow b,$$

which implies $\lim_{t\uparrow b} u(t)=v_0$ exists. We can now apply Theorem 2 with the initial condition $u(b)=v_0$ and obtain a continuation of the solution u beyond b, which contradicts the assumption on b.

Let w be a solution to (CP) on $[0, \infty)$ with $w(0)=w_0$. Then for a.e.t $\in (0, \infty)$

$$\begin{aligned}\frac{d}{dt}\|u(t)-w(t)\| &= \langle u(t)-w(t), f(t, u(t))-f(t, w(t))\rangle \\ &\leqq \beta(t)\, \|u(t)-w(t)\|,\end{aligned}$$

and hence

$$\|u(t)-w(t)\| \leqq \|u_0-w_0\| \exp\left(\int_0^t \beta(\tau)d\tau\right).$$

This inequality means that a solution to (CP) is unique.

Acknowledgement. The author would like to express his hearty thanks to the referees for valuable advices.

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(Ricevita la 15-an de julio, 1975)
(Riviziita la 23-an de marto, 1976)

Funkcialaj Ekvacioj

(Serio Internacia)

VOLUMO 19, NUMERO 3

DECEMBRO, 1976

On Local and Global Existence Theorems for a Nonautonomous Differential Equation in a Banach Space

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Shigeo KATO

Coden FESIAT	Pages 203–300
Volume 19	December 1976
Number 3	Whole No. 47

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JAPANA MATEMATIKA SOCIETO

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Funkcialaj Ekvacioj, **19** (1976) 279-286

On Local and Global Existence Theorems for a Nonautonomous Differential Equation in a Banach Space

By

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1. Introduction and results.

Let E be a Banach space with norm $\| \quad \|$. We denote by $S_r(u_0)$ (resp. $U_r(u_0)$) the closed (resp. open) sphere of center u with radius r.

In this paper we consider the Cauchy problem

$$\text{(CP)} \qquad x' = f(t, x), \qquad x(0) = u_0 \in E,$$

where f is a E-valued mapping defined on $[0, T] \times S_r(u_0)$ or on $[0, \infty) \times E$. By a solution to (CP) or to (CP; u_0), we mean a strongly continuously differentiable function defined on some interval $[0, a]$ (or $[0, a)$) such that $u(0) = u_0$ and $u'(t) = f(t, u(t))$ for $t \in (0, a]$ (or $(0, a)$).

This kind of problem has been treated by many authors and some of their articles are listed in our references.

It is our object in this paper to establish both local and global existence theorems for (CP) under some conditions which are similar to those treated in H. Murakami [9] and P. Ricciardi-L. Tubaro [10]. Our theorems give some generalizations of those of [7, 8, 9, 10].

Let $V(t, x, y)$ be a functional from $[0, T] \times S_r(u_0) \times S_r(u_0)$ into R satisfying the following properties:

(P_1) $V(t, x, y) > 0$ if $x \neq y$; $= 0$ if $x = y$.

(P_2) $V(t, x, y)$ is uniformly Lipschitz continuous on $[0, T] \times S_r(u_0) \times S_r(u_0)$ with Lipschitz constant L.

(P_3) $\lim_{n \to \infty} V(t, x_n, y_n) = 0$ implies $\lim_{n \to \infty} \|x_n - y_n\| = 0$ for each t.

In order to prove the existence of the unique solution to (CP) we consider the following scalar equation

$$(1.1) \qquad w' = g(t, w),$$

where $g(t, \tau)$ is a real-valued continuous function defined on $(0, a] \times [0, b]$. We assume furthermore that g satisfies the followins conditions: (1_a) $g(t, 0)=0$ for $t \in (0, a]$. (2_a) For each $t_0 \in (0, a]$, $w \equiv 0$ is the only solutions to (1.1) on $(0, t_0]$ satisfying the condition that $w(+0)=0$.

Let f be a strongly continuous mapping from $[0, T] \times S_r(u_0)$ into E. Then there exist some constants $0<r_0 \leqq r$, $0<T_1 \leqq T$ and $M>0$ such that $\|f(t, x)\| \leqq M$ for all $(t, x) \in [0, T_1] \times S_{r_0}(u_0)$.

We now state the following result.

Theorem 1. *Let f and V satisfying the assumptions mentioned above. Furthermore, if f satisfies*

$$(1.2) \qquad \varliminf_{h \to -0} \frac{1}{h}[V(t+h, x+hf(t, x), y+hf(t, y)) - V(t, x, y)] \leqq g(t, V(t, x, y))$$

for all (t, x), $(t, y) \in (0, T) \times U_r(u_0)$, where g satisfies (1_a) and (2_a) with $a=T$ and $b=(2M+1)LT$. Then (CP; u_0) *has a unique solution u defied on some interval* $[0, T_0]$.

We next consider a global analogue of Theorem 1, and we assume that $V(t, x, y)$ defined on $[0, \infty) \times E \times E$ satisfies (P_1), (P_3) and

(P_2') $V(t, x, y)$ is locally Lipschitz continuous on $[0, \infty) \times E \times E$.

Let g be a real-valued continuous function defined on $[0, \infty) \times [0, \infty)$ satisfying the following conditions: (1_b) $g(t, 0)=0$ for all $t \in [0, \infty)$. (2_b) For each $t_0 \in [0, \infty)$, $w \equiv 0$ is the only solution to (1.1) on $[0, t_0]$ satisfying the condition that $w(0)=0$.

Under these conditions we can prove the following

Theorem 2. *Let f be a strongly continuous mapping from $[0, \infty) \times E$ into E. Suppose furthermore that*

$$(1.3) \qquad \varliminf_{h \to -0} \frac{1}{h}[V(t+h, x+hf(t, x), y+hf(t, y)) - V(t, x, y)] \leqq g(t, V(t, x, y))$$

for all (t, x), $(t, y) \in [0, \infty) \times E$. Then (CP; u_0) *has a unique solution u defined on $[0, \infty)$ for each $u_0 \in E$.*

2. Some lemmas.

In the following Lemmas 1, 2 and 3 we assume that V and g satisfy the assumptions $(P_1)-(P_3)$ and $(1_a)-(2_a)$ respectively stated in Section 1. In virtue of (P_2), it is easy to prove the following

Lemma 1. *Let u and v be continuous functions from $[0, T]$ into $U_r(u_0)$ which have left derivatives $u'_-(t)$ and $v'_-(t)$ respectively for some t in $(0, T]$. Then*

$$D_{-}V(t, u(t), v(t)) = \lim_{h \to -0} \frac{1}{h}[V(t+h,\ u(t)+hu'_{-}(t),\ v(t)+hv'_{-}(t)) - V(t, u(t), v(t))], \tag{2.1}$$

where $D_{-}V(t, u(t), v(t))$ denotes the lower left-hand Dini derivative of $V(t, u(t), v(t))$.

Lemma 2. *Let $M>0$ and Φ be a set of functions from $[0, a]$ into $[0, b]$ with the property that for all $s, t \in [0, a]$ and $w \in \Phi$*

$$|w(t)-w(s)| \leqq M\,|t-s|. \tag{2.2}$$

Let $z(t) = \sup\{w(t)\,;\ w \in \Phi\}$ for $t \in [0, T]$, and suppose that for each $w \in \Phi$

$$D_{-}w(t) \leqq g(t, w(t)) \qquad \text{for } t \in [0, a]-S, \tag{2.3}$$

where S is a countable subset of $[0, a]$. Then for all $s, t \in [0, a]$

$$|z(t)-z(s)| \leqq M\,|t-s| \tag{2.4}$$

and for all $t \in (0, a)$

$$D_{-}z(t) \leqq g(t, z(t)). \tag{2.5}$$

For a proof see [3, Lemma 3].

Lemma 3. *Let w be a continuous function from $[0, a]$ into $[0, b]$ such that $w(0)=0$ and*

$$D_{-}w(t) \leqq g(t, w(t)) \qquad \text{for } t \in [0, a]-S,$$

where S is a countable subset of $[0, a]$. Then $w \equiv 0$ on $[0, a]$.

The proof of this lemma is similar to that of [4, Lemma 2.3] and is omitted.

3. Proof of Theorem 1.

The following proof is essentially based on the methods in [4, 11]. Let $0 < T_0 < \mathrm{Min}\,\{T_1, r_0/M\}$, in which r_0, T_1 and M are the same as in the remark preceding Theorem 1, and let n be a positive integer. We set $t_0^n = 0$, and $u_n(t_0^n) = u_0$. Inductively for each positive integer i, define δ_i^n, t_i^n, $u_n(t_{i-1}^n)$ as follows:

$$\delta_i^n \geqq 0, \qquad t_{i-1}^n + \delta_i^n \leqq T_0\,; \tag{3.1}$$

(3.2) If $\|x-u_n(t_{i-1}^n)\| \leqq M\,\delta_i^n$ and $t_{i-1}^n \leqq t \leqq t_{i-1}^n + \delta_i^n$, then
$\|f(t, x) - f(t_{i-1}^n, u_n(t_{i-1}^n))\| \leqq 1/n\,;$

$$\|u_n(t_{i-1}^n) - u_0\| \leqq r_0\,; \tag{3.3}$$

and δ_i^n is the largest number such that (3.1), (3.2) and (3.3) hold. Let $t_i^n = t_{i-1}^n + \delta_i^n$. We set

$$u_n(t) = u_n(t_{i-1}^n) + \int_{t_{i-1}^n}^t f(s, u_n(t_{i-1}^n))ds \qquad \text{for each } t \in [t_{i-1}^n, t_i^n].$$

Then for each $t \in [t_{k-1}^n, t_k^n]$

$$u_n(t) = u_0 + \sum_{j=1}^{k-1} \int_{t_{j-1}^n}^{t_j^n} f(s, u_n(t_{j-1}^n))ds + \int_{t_{k-1}^n}^t f(s, u_n(t_{k-1}^n))ds \tag{3.4}$$

and

$$\|u_n(t) - u_n(s)\| \leqq M\,|t-s| \qquad \text{for } s, t \in [0, T_0]. \tag{3.5}$$

Moreover we see that there exists a positive integer $N = N(n)$ such that $t_N^n = T_0$. (For detail see [4, 11]). Thus we have $u_n(t) \in U_{r_0}(u_0)$ for all $t \in [0, T_0]$.

We next show that the sequence of continuous functions $\{u_n\}$ converges uniformly to a E-valued function u on $[0, T_0]$. For this we set $w_{mn}(t) = V(t, u_m(t), u_n(t))$ for $m > n \geqq 1$ and $t \in [0, T_0]$, and remark first that by using (P_2)

$$|w_{mn}(t) - w_{mn}(s)| \leqq (2M+1)L\,|t-s| \qquad \text{for } s, t \in [0, T_0]. \tag{3.6}$$

By the construction of $\{u_n\}$ and Lemma 1 we see that $D_- w_{mn}(t)$ exists for $t \in (0, T_0]$ and $m > n \geqq 1$. For each $t \in (0, T_0]$ there exist positive integers i and j such that $t \in (t_{j-1}^m, t_j^m] \cap (t_{i-1}^n, t_i^n]$. By Lemma 1 and (P_2) we have

$$\begin{aligned} D_- w_{mn}(t) &= \varliminf_{h \to -0} \frac{1}{h}[V(t+h,\ u_m(t) + h(u_m)'_-(t),\ u_n(t) + h(u_n)'_-(t)) \\ &\qquad\qquad - V(t, u_m(t), u_n(t))] \\ &= \varliminf_{h \to -0} \frac{1}{h}[V(t+h,\ u_m(t) + hf(t_{j-1}^m, u_m(t_{j-1}^m)),\ u_n(t) + hf(t_{i-1}^n, u_n(t_{i-1}^n))) \\ &\qquad\qquad - V(t, u_m(t), u_n(t))] \\ &\leqq g(t, V(t, u_m(t), u_n(t))) + L(\|f(t, u_m(t)) - f(t_{j-1}^m, u_m(t_{j-1}^m))\| \\ &\qquad\qquad + \|f(t, u_n(t)) - f(t_{i-1}^n, u_n(t_{i-1}^n))\|). \end{aligned}$$

On the other hand

$$\|u_m(t) - u_m(t_{j-1}^m)\| \leqq M\,\delta_j^m \quad \text{and} \quad \|u_n(t) - u_n(t_{i-1}^n)\| \leqq M\,\delta_i^n.$$

Thus we have by (3.2)

$$\begin{aligned} D_- w_{mn}(t) &\leqq g(t, w_{mn}(t)) + L(1/n + 1/m) \\ &\leqq g(t, w_{mn}(t)) + 2L/n \end{aligned} \tag{3.7}$$

for all $t \in (0, T_0]$ and $m > n \geqq 1$. Let $w_n(t) = \sup_{m>n} \{w_{mn}(t)\}$ for $t \in [0, T_0]$. Then $w_n(0) = 0$ for all n. It thus follows from (3.6), (3.7) and Lemma 2 that

$$|w_n(t) - w_n(s)| \leqq (2M+1)L\,|t-s| \qquad \text{for } s, t \in [0, T_0], \tag{3.8}$$

and

$$D_- w_n(t) \leqq g(t, w_n(t)) + 2L/n \qquad \text{for } t \in (0, T_0). \tag{3.9}$$

Since $0 \leqq w_n(t) \leqq w_n(0) + (2M+1)Lt \leqq (2M+1)LT_0$ for $t \in [0, T_0]$ and $n \geqq 1$ the sequence $\{w_n\}$ is equicontinuous and uniformly bounded, and hence it has a subsequence converging uniformly on $[0, T_0]$ to a function w. Obviously $w(0) = 0$ and from (3.9) and Lemma 2 we have

$$D_- w(t) \leqq g(t, w(t)) \qquad \text{for } t \in (0, T_0).$$

From Lemma 3 we deduce now that $w \equiv 0$, and this implies that

$$\lim_{m,n\to\infty} V(t, u_m(t), u_n(t)) = 0 \text{ uniformly on } [0, T_0].$$

In virtue of (P_3) the sequence $\{u_n\}$ is uniformly Cauchy on $[0, T_0]$ and the limit u of this sequence satisfies

$$u(t) = u_0 + \int_0^t f(s, u(s))ds \qquad \text{for } t \in [0, T_0].$$

In fact, for each $t \in [0, T_0]$ and $n \geqq 1$ we have

$$\left\| u_n(t) - \left(u_0 + \int_0^t f(s, u(s))ds\right) \right\|$$
$$\leqq \left[1/n + \operatorname*{Max}_{0 \leqq s \leqq T_0} \|f(s, u_n(s)) - f(s, u(s))\| \right] T_0.$$

Thus u is a solution to (CP ; u_0) on $[0, T_0]$. If v is a solution to (CP ; u_0) on $[0, T_0]$ and $z(t) = V(t, u(t), v(t))$, then $z(0) = 0$ and

$$D_- z(t) \leqq g(t, z(t)) \qquad \text{for } t \in (0, T_0].$$

It therefore follows from Lemma 3 that $z \equiv 0$, which completes the proof.

4. Proof of Theorem 2.

Before proceeding to the proof of Theorem 2, we prepare the following

Lemma 4. *Let $u_0 \in E$ and let T be a positive number such that* (CP ; u_0) *has a solution u on* $[0, T]$. *Then there exists a positive number r_0 such that for each $v_0 \in S_{r_0}^{\mathfrak{F}}(u_0)$,* (CP, v_0) *has a solution v on* $[0, T]$.

Proof. We define a continuous extension $\tilde{g}$ of g into $R\times R$ as follows:

$$\tilde{g}(t,\tau)=\begin{cases} g(t,\tau) & (t\geqq 0,\ \tau\geqq 0)\\ g(-t,\tau) & (t<0,\ \tau\geqq 0)\\ 0 & (-\infty<t<\infty,\ \tau<0). \end{cases}$$

By (2_b) $w\equiv 0$ is a maximal solution to (1.1) on $[0,T]$ with $w(0)=0$. It follows from Theorem 1.4 in [2] that there exists a $\delta>0$ such that the equation $w'=\tilde{g}(t,w)$ has a maximal solution $m(\ ,\sigma)$ for each σ, $0\leqq\sigma<\delta$ on $[0,T]$ with $m(0,\sigma)=\sigma$. Moreover, $m(\ ,\sigma)\to 0$ as $\sigma\to +0$, uniformly on $[0,T]$.

Since the set $\{(t,u(t))\,;\,t\in[0,T]\}$ is compact in $[0,T]\times E$, there exist constants $R>0$ and $M>0$ such that

$$\|f(t,x)\|\leqq M \qquad \text{for all } t\in[0,T] \text{ and all } x\in S_R(u(t)).$$

Since $V(0,v_0,u_0)\to 0$ as $v_0\to u_0$ by (P_2'), for each $\varepsilon>0$ there exists a $r>0$ such that for each $v_0\in S_r(u_0)$

$$|m(t,V(0,v_0,u_0))|<\varepsilon \qquad \text{for all } t\in[0,T].$$

Let v be a solution to $(CP\,;\,v_0)$ on $[0,T_{v_0})$. Then

$$D_-V(t,v(t),u(t))\leqq\tilde{g}(t,V(t,v(t),u(t)))$$

for $t\in(0,T_{v_0})\cap(0,T]$. Thus we have

$$V(t,v(t),u(t))\leqq m(t,V(0,v_0,u_0)) \qquad \text{for } t\in[0,T_{v_0})\cap[0,T]$$

(see [6, Theorem 1.4.1]).

Thus, by (P_3), there exists a sufficiently small $r_0>0$ such that for each $v_0\in S_{r_0}(u_0)$

$$\|v(t)-u(t)\|<R \qquad \text{for all } t\in[0,T_{v_0})\cap[0,T].$$

It follows that $\|f(t,v(t))\|\leqq M$ for all $t\in[0,T_{v_0})\cap[0,T]$, and this implies that the existence domain $[0,T_{v_0})$ of v contains $[0,T]$.

Proof of Theorem 2. Let $u_0\in E$. Then it follows from Theorem 1 that there exists a unique solution to $(CP\,;\,u_0)$ on some interval $[0,T_0]$. Let C be a connected component in E containing u_0 and let

$$D=\{x\in C\,;\,(CP\,;\,x) \text{ has a solution on } [0,T_0]\}.$$

Then $D\neq\phi$, since $u_0\in D$. By Lemma 4, D is relatively open in C. We show that D is also relatively closed in C. For this, let $\{x_n\}$ be any sequence in D which converges $x\in C$ in C and let v_n be a solution to $(CP\,;\,x_n)$ on $[0,T_0]$. Then

$$D_-V(t,v_m(t),v_n(t))\leqq\tilde{g}(t,V(t,v_m(t),v_n(t)))$$

for $t \in (0, T_0]$. Thus we have

$$V(t, v_m(t), v_n(t)) \leqq m(t, V(0, x_m, x_n)) \qquad \text{for all } t \in [0, T_0].$$

Since $\lim_{m,n \to \infty} m(t, V(0, x_m, x_n)) = 0$ uniformly on $[0, T_0]$, the sequence $\{v_n\}$ converges uniformly on $[0, T_0]$ to a function v, and clearly v is a solution to (CP ; x) on $[0, T_0]$. Hence $x \in D$. This implies that $D = C$. Since C is a connected component in E containing u_0, it follows that (CP ; u_0) has a solution on $[0, kT_0]$ for any integer $k \geqq 1$ and hence it is proved that (CP ; u_0) has a solution on $[0, \infty)$.

Remark. The idea for the proof of global existence is essentially due to N. Kenmochi and T. Takahashi [5].

5. Remarks.

Remark 1. Recently P. Ricciardi and L. Tubaro [10] proved the existence and uniqueness of the local solution to (CP ; u_0), assuming the existence of a functional $V(t, x, y)$, which, in addition to (P_1) and (P_3) stated in Section I, has the following properties:

(i) $V(t, x, y)$ is Lipschitz continuous on $S_r(u_0) \times S_r(u_0)$ uniformly in t with Lipschitz constant L;

(ii) For any $t \in [0, T]$ the mapping

$$(u, v) \to D^-_{(x,y)} V(t, x, y)(u, v)$$

from $E \times E$ into R is subadditive, where

$$D^-_{(x,y)} V(t, x, y)(u, v) = \lim_{h \to -0} \frac{1}{h}[V(t, x+hu,\ y+hv) - V(t, x, y)];$$

(iii) $V(t, x, y)$ is partially differentiable in t and $V'_t(t, x, y)$ is continuous in (t, x, y);

(iv) $V'_t(t, x, y) + D^-_{(x,y)} V(t, x, y)(f(t, x), f(t, y)) \leqq 0$ for all (t, x), $(t, y) \in [0, T] \times S_r(u_0)$.

This result is a generalization of that of H. Murakami [9]. It is easy to see that our Theorem 1 generalizes P. Ricciardi and L. Tubaro's theorem.

Remark 2. In Theorem 2, if $V(t, x, y) = \|x - y\|$ and $g(t, \tau) = \alpha(\tau)$, where α is a real-valued continuous function defined on $[0, \infty)$ such that $\alpha(0) = 0$ and $w \equiv 0$ is the only solution to $w' = \alpha(w)$ on $[0, \infty)$ with $w(0) = 0$. Then the condition (1.3) becomes

$$\lim_{h \to -0} \frac{1}{h}(\|x - y + h(f(t, x) - f(t, y))\| - \|x - y\|) \leqq \alpha(\|x - y\|)$$

for all (t, x), $(t, y) \in [0, \infty) \times E$.

Remark 3. Let β be a real-valued continuous function defined on $[0, \infty)$. If $V(t, x, y) = \|x-y\| \left(\text{or } \exp\left(-\int_0^t \beta(s)ds\right)\|x-y\|\right)$ and $g(t, \tau) = \beta(t)\tau$ (or $g \equiv 0$), then the condition (1.3) becomes

$$\lim_{h \to -0} \frac{1}{h}(\|x-y+h(f(t, x)-f(t, y))\|-\|x-y\|) \leqq \beta(t)\, \|x-y\|$$

for all (t, x), $(t, y) \in [0, \infty) \times E$.

Thus the results of R. Martin [8] and R. Martin - D.L.Lovelady [7] are the special cases of our Theorem 2.

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(Ricevita la 27-an de oktobro, 1975)

Funkcialaj Ekvacioj

(Serio Internacia)

VOLUMO 21, NUMERO 1

DECEMBRO, 1978

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Coden FESIAT	Pages 1–80
Volume 21	April 1978
Number 1	Whole No. 51

FAKO DE L'FUNKCIALAJ EKVACIOJ

JAPANA MATEMATIKA SOCIETO

Funkcialaj Ekvacioj, **21** (1978) 43-52

On the Convergence of the Successive Approximations for Nonlinear Ordinary Differential Equations in a Banach Space

By

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§ 1. Introduction and results.

Let E be a (real or complex) Banach space with norm denoted by $\| \quad \|$ and let $S(x, r)$ be a closed ball of center x with radius r. In this paper we consider the Cauchy problem

(CP) $$x' = f(t, x), \qquad x(0) = x_0 \in E,$$

where f is a E-valued mapping defined on $[0, a] \times S(x_0, r)$ or on $[0, \infty) \times E$.

Recently, G. Vidossich [4] proved the convergence of the successive approximations for (CP) under some Kamke-type condition, namely,

(1.1) $$\|f(t, x) - f(t, y)\| \leqq g(t, \|x - y\|),$$

where g is a real-valued function satisfying some uniqueness condition. However, the results obtained in [4] crucially depend on the integrability or the boundedness of g.

It is our object in this paper to establish both local and global convergence theorems for the successive approximations for (CP) under conditions which are weaker than those of [4].

Let f be a mapping from $[0, a] \times S(x_0, r)$ into E satisfying the following conditions:

(f_1) $f(\cdot, x)$ is strongly measurable in t for each fixed $x \in S(x_0, r)$, and $f(t, \cdot)$ is continuous in x for $a.e. t \in [0, a]$.

(f_2) There exists a function $\beta \in L^1(0, a)$ such that

$$\|f(t, x)\| \leqq \beta(t) \qquad \text{for } a.e. t \in [0, a] \text{ and all } x \in S(x_0, r).$$

Definition 1. Suppose that (f_1) and (f_2) are satisfied. Then a function u is said to be a strong solution to (CP) on $[0, t_0]$ if u is an absolutely continuous function defined on $[0, t_0]$ satisfying $u(0) = x_0$ and $u'(t) = f(t, u(t))$ for a.e. $t \in [0, t_0]$.

We define the successive approximations for (CP) as follows:

$$u_n(t)=x_0+\int_0^t f(s, u_{n-1}(s))ds \qquad (n\geqq 1), \tag{1.2}$$

where u_0 is an arbitrary continuous function from $[0, a]$ into $S(x_0, r)$.

In order to prove the convergence of $\{u_n\}$ we consider a Kamke-type uniqueness function g satisfying the following conditions:

(g_1) $g=g(t, \tau)$ is a nonnegative real-valued function defined on $(0, a]\times[0, 2r]$ which is Lebesgue measurable in t for each fixed τ, and continuous nondecreasing in τ for each fixed t.

(g_2) For each $\delta\in(0, a)$, $w\equiv 0$ is the only absolutely continuous function defined on $[0, \delta]$ which satisfies $w(0)=0$ and $w'(t)=g(t, w(t))$ for a.e.$t\in(0, \delta)$.

(g_3) There exists a function α defined on $(0, a]$ such that

$$g(t, \tau)\leqq\alpha(t) \qquad \text{for } (t, \tau)\in(0, a]\times[0, 2r]$$

and $\alpha\in L^1(\gamma, a)$ for every $\gamma\in(0, a)$.

In the following, for simplicity, we say that g satisfies (g_1)–(g_3) on $(t_1, t_2]\times[0, 2r]$ if g is defined on $(t_1, t_2]\times[0, 2r]$ and satisfies (g_1)-(g_3) with 0 and a replaced by t_1 and t_2 respectively.

Now, we can state the following result.

Theorem 1. *Suppose that* (f_1)–(f_2) and (g_1)–(g_3) *are satisfied. Suppose furthermore that*

$$\|f(t, x)-f(t, y)\|\leqq g(t, \|x-y\|) \tag{1.3}$$

for a.e.$t\in(0, a]$ and all $x, y\in S(x_0, r)$. Then the successive approximations $\{u_n\}$ defined by (1.2) *converges uniformly on some interval $[0, t_0]$ to a unique strong solution to* (CP).

We next consider the global convergence of $\{u_n\}$. Let f be a mapping from $[0, \infty)\times E$ into E satisfying (f_1)–(f_2) with $[0, a]$, $S(x_0, r)$ and $\beta\in L^1(0, a)$ replaced by $[0, \infty)$, E and $\beta\in L^1_{\mathrm{loc}}(0, \infty)$ respectively.

Theorem 2. *Suppose that f satisfies the above mentioned condition. Suppose furthermore that for each $(t_0, z_0)\in[0, \infty)\times E$, there exist positive constants a, r and a function g satisfying the conditions (g_1)–(g_3) on $(t_0, t_0+a]\times[0, 2r]$ such that*

$$\|f(t, x)-f(t, y)\|\leqq g(t, \|x-y\|) \tag{1.4}$$

for a.e.$t\in(t_0, t_0+a]$ and all $x, y\in S(z_0, r)$. Then the successive approximations $\{u_n\}$ converges uniformly on any compact interval of $[0, \infty)$ to a unique strong solution to (CP).

In case f is a continuous mapping we give the following

Definition 2. Let f be a continuous mapping from $[0, \infty)\times E$ into E. Then a function u is said to be a C^1 solution to (CP) on $[0, t_0]$ if u is a strongly continuously differentiable function defined on $[0, t_0]$ satisfying $u(0)=x_0$ *and* $u'(t)=f(t, u(t))$ for all $t \in [0, t_0]$.

In the following Theorem 3 the condition (g_2) can be replaced by the following $(g_2)'$ which is slightly weaker than (g_2).

$(g_2)'$ For each $\delta \in (0, a)$, $w\equiv 0$ is the only absolutely continuous function defined on $[0, \delta]$ which satisfies $w'(t)=g(t, w(t))$ for *a.e.* $t \in (0, \delta)$, and $w(0)=(D^+w)(0) =\lim_{t\to+0} w(t)/t=0$.

Theorem 3. *Let f be a continuous mapping from $[0, \infty)\times E$ into E such that*

$$\|f(t, x)\|\leqq\beta(t) \qquad \text{for a.e. } t \in [0, \infty) \text{ and all } x \in E,$$

where $\beta \in L^1_{\mathrm{loc}}(0, \infty)$. Suppose furthermore that for each $(t_0, z_0) \in [0, \infty)\times E$, there exist positive constants a, r and a function g satisfying the conditions $(g_1), (g_2)'$ and (g_3) on $(t_0, t_0+a]\times[0, 2r]$ such that

$$\|f(t, x)-f(t, y)\|\leqq g(t, \|x-y\|) \tag{1.5}$$

for all $(t, x), (t, y) \in (t_0, t_0+a]\times S(z_0, r)$. Then the successive approximations $\{u_n\}$ converges uniformly on any compact interval $[0, \infty)$ to a unique C^1 solution to (CP).

§2. Proof of Theorem 1.

Before proving Theorem 1 we prepare the following two lemmas.

Lemma 2.1. *Let g satisfy the conditions $(g_1), (g_2)$ (or $(g_2)'$) and (g_3) on $(t_0, t_0+a]\times[0, 2r]$, and let w be an absolutely continuous function from $[t_0, t_0+a]$ into $[0, 2r]$. Suppose furthermore that $w(t_0)=0$ (or $w(t_0)=(D^+w)(t_0)=0$) and*

$$w'(t)\leqq g(t, w(t)) \qquad \text{for a.e. } t \in (t_0, t_0+a].$$

Then $w\equiv 0$ on $[t_0, t_0+a]$.

For a proof see Lemma 2.3 in [3] or [1, p. 56].

Lemma 2.2. *Suppose that f satisfies the conditions (f_1) and (f_2). Then for each strongly measurable function z from $[0, a]$ into $S(x_0, r)$, $f(t, z(t))$ is strongly measurable and Bochner integrable on $[0, a]$.*

Proof. Let $\{z_n\}$ be a sequence of finitely-valued functions on $[0, a]$ such that $\lim_{n\to\infty} z_n(t)=z(t)$ for *a.e.* $t \in [0, a]$. Then, by (f_1), $f(t, z_n(t))$ is strongly measurable on $[0, a]$ for each $n \geqq 1$ and

$$\lim_{n\to\infty} f(t, z_n(t)) = f(t, z(t)) \qquad \text{for } a.e.\, t \in [0, a].$$

It therefore follows that $f(t, z(t))$ is strongly measurable on $[0, a]$. Moreover, (f_2) implies that $f(t, z(t))$ is Bochner integrable on $[0, a]$ (see [5]).

Proof of Theorem 1. Let $t_0 \in (0, a]$ be such that $\int_0^{t_0} \beta(t)dt \leqq r$ and set $I = [0, t_0]$. Then it follows from (f_2) that

$$\|u_n(t) - u_0\| \leqq \int_0^t \|f(s, u_{n-1}(s))\| ds \leqq \int_0^t \beta(\tau) d\tau \leqq r$$

for each $t \in I$ and $n \geqq 1$. This implies that $u_n(t) \in S(x_0, r)$ for each $t \in$ I and $n \geqq 1$.

On the other hand, we have

$$\|u_n(t) - u_n(s)\| \leqq \left| \int_s^t \|f(\tau, u_{n-1}(\tau))\| d\tau \right| \leqq \left| \int_s^t \beta(\tau) d\tau \right| = |M(t) - M(s)| \tag{2.1}$$

for each $s, t \in I$ and $n \geqq 1$, where $M(t) = \int_0^t \beta(\tau) d\tau$ for $t \in I$.

Letting $s = 0$ in (2.1) we have

$$\|u_n(t)\| \leqq \|u_n(0)\| + M(t) \leqq \|x_0\| + M(t_0) \leqq \|x_0\| + r$$

and hence $\{u_n\}$ is equicontinuous and uniformly bounded on I. In order to prove that the sequence $\{u_n\}$ converges uniformly on I to a E-valued function we define the functions w_{mn} and w_n by

$$w_{mn}(t) = \|u_m(t) - u_n(t)\| \qquad \text{for } t \in I \text{ and } m \geqq n \geqq 1$$

and

$$w_n(t) = \sup_{m \geqq n} w_{mn}(t) \qquad \text{for } t \in I \text{ and } n \geqq 1.$$

Then, by (2.1), we have

$$\begin{aligned} |w_{mn}(t) - w_{mn}(s)| &= |\, \|u_m(t) - u_n(t)\| - \|u_m(s) - u_n(s)\| \,| \\ &\leqq \|u_m(t) - u_m(s) - (u_n(t) - u_n(s))\| \leqq 2\, |M(t) - M(s)| \end{aligned}$$

and hence

$$|w_n(t) - w_n(s)| \leqq \sup_{m \geqq n} |w_{mn}(t) - w_{mn}(s)| \leqq 2\, |M(t) - M(s)| \tag{2.2}$$

for all $s, t \in I$ and $n \geqq 1$.

Since $w_n(t) \leqq w_n(0) + 2M(t) \leqq 2M(t_0) \leqq 2r$ for $t \in I$ and $n \geqq 1$, the sequence $\{w_n\}$ is equicontinuous and uniformly bounded on I, and hence it has a subsequence $\{w_{n(k)}\}$ converging uniformly on I to a function $w = w(t)$, and obviously $w(0) = 0$.

Let t and $\Delta t > 0$ be such that $t, t + \Delta t \in I$. Then we have

$$\text{(2.3)} \qquad \begin{aligned} &|w_{n(k+1)}(t + \Delta t) - w_{n(k+1)}(t)| \\ &\leqq \sup_{m \geqq n(k+1)} |w_{mn(k+1)}(t + \Delta t) - w_{mn(k+1)}(t)|. \end{aligned}$$

On the other hand, for each $m \geqq n(k+1)$ we have

$$\begin{aligned} &|w_{mn(k+1)}(t + \Delta t) - w_{mn(k+1)}(t)| \\ &\leqq \| u_m(t + \Delta t) - u_m(t) - (u_{n(k+1)}(t + \Delta t) - u_{n(k+1)}(t)) \| \\ &\leqq \int_t^{t+\Delta t} \| f(s, u_{m-1}(s)) - f(s, u_{n(k+1)-1}(s)) \| \, ds \\ &\leqq \int_t^{t+\Delta t} g(s, \| u_{m-1}(s) - u_{n(k+1)-1}(s) \|) ds \\ &= \int_t^{t+\Delta t} g(s, w_{m-1n(k+1)-1}(s)) \, ds \\ &\leqq \int_t^{t+\Delta t} g(s, w_{n(k+1)-1}(s)) \, ds. \end{aligned}$$

Here we used the fact that $g = g(t, \tau)$ is nondecreasing in τ and $w_{n(k+1)-1}(s) \geqq w_{m-1n(k+1)-1}(s)$ for all $s \in [t, t + \Delta t]$ and $m \geqq n(k+1)$.

Since $n(k) \leqq n(k+1) - 1$ in general, $w_{n(k)}(t) \geqq w_{n(k+1)-1}(t)$ for all $t \in I$ and this implies that

$$g(s, w_{n(k)}(s)) \geqq g(s, w_{n(k+1)-1}(s)) \qquad \text{for all } s \in [t, t + \Delta t].$$

Consequently, we have by (2.3)

$$\text{(2.4)} \qquad |w_{n(k+1)}(t + \Delta t) - w_{n(k+1)}(t)| \leqq \int_t^{t+\Delta t} g(s, w_{n(k)}(s)) ds.$$

We show next that for each $\varepsilon > 0$ there exists an integer N_ε such that

$$\text{(2.5)} \qquad w_{n(k)}(s) \leqq w(s) + \varepsilon \qquad \text{for all } s \in [t, t + \Delta t] \text{ and } k \geqq N_\varepsilon.$$

In fact, since w is uniformly continuous and $\{w_{n(k)}\}$ is equicontinuous on I, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$|w(s) - w(\hat{s})| < \varepsilon/3, \qquad |w_{n(k)}(s) - w_{n(k)}(\hat{s})| < \varepsilon/3$$

whenever $|s - \hat{s}| < \delta$ $(s, \hat{s} \in [t, t + \Delta t])$.

Let $t = s_0 < s_1 < \cdots < s_p = t + \Delta t$ be a partition of $[t, t + \Delta t]$ such that $\mathrm{Max}_{1 \leqq i \leqq p}$

$(s_i - s_{i-1}) < \delta$. Then for each i ($i=0, 1, \cdots, p$) there exists an integer $N_\varepsilon(i)$ such that

$$w_{n(k)}(s_i) \leqq w(s_i) + \varepsilon/3 \qquad \text{for } k \geqq N_\varepsilon(i).$$

Let $N_\varepsilon = \mathrm{Max}_{0 \leqq i \leqq p} N_\varepsilon(i)$. Then we have

$$w_{n(k)}(s_i) \leqq w(s_i) + \varepsilon/3 \qquad \text{for all } k \geqq N_\varepsilon \text{ and } i \ (0 \leqq i \leqq p).$$

For each $s \in [t, t+\Delta t]$ there exists an s_i such that $s \in [s_i, s_{i+1}]$, and hence

$$\begin{aligned} w_{n(k)}(s) &\leqq |w_{n(k)}(s) - w_{n(k)}(s_i)| + w_{n(k)}(s_i) \\ &< w(s_i) + 2\varepsilon/3 < w(s) + \varepsilon \qquad \text{for all } k \geqq N_\varepsilon. \end{aligned}$$

This proves (2.5). Sicne g is nondecreasing in τ, it follows from (2.4) and (2.5) that

$$|w_{n(k+1)}(t+\Delta t) - w_{n(k+1)}(t)| \leqq \int_t^{t+\Delta t} g(s, w(s)+\varepsilon)ds \tag{2.6}$$

for $k \geqq N_\varepsilon$. Since $\lim_{k\to\infty} w_{n(k)}(t) = w(t)$ uniformly on I, it is easy to see that

$$|w(t+\Delta t) - w(t)| \leqq \lim_{k\to\infty} |w_{n(k)}(t+\Delta t) - w_{n(k)}(t)|$$

and this with (2.6) shows that

$$|w(t+\Delta t) - w(t)| \leqq \int_t^{t+\Delta t} g(s, w(s)+\varepsilon)ds.$$

By the continuity of g in τ and the dominated convergence theorem of Lebesgue, we have by letting $\varepsilon \to +0$

$$|w(t+\Delta t) - w(t)| \leqq \int_t^{t+\Delta t} g(s, w(s))ds. \tag{2.7}$$

From (2.2) and the fact that $\lim_{k\to\infty} w_{n(k)}(t) = w(t)$ uniformly on I, it follows that w is absolutely continuous on I. Consequently, $w'(t)$ exists for $a.e. t \in I$ and (2.7) implies

$$|w'(t)| \leqq g(t, w(t)) \qquad \text{for } a.e. t \in I.$$

Since $w(0)=0$, we deduce now that $w \equiv 0$ on I by Lemma 2.1, and this implies that the sequence $\{u_n\}$ is uniformly convergent on I. Let $u(t) = \lim_{n\to\infty} u_n(t)$ for $t \in I$. Then the conditions (f_1) and (f_2) imply that

$$\|f(t, u_n(t))\| \leqq \beta(t) \qquad \text{for } a.e. t \in I$$

and

$$\lim_{n\to\infty} f(t, u_n(t)) = f(t, u(t)) \qquad \text{for } a.e.t \in I.$$

It thus follows from the dominated convergence theorem for vector-valued functions that

$$\lim_{n\to\infty} \int_0^t f(s, u_n(s))ds = \int_0^t f(s, u(s))ds \qquad \text{for each } t \in I$$

and this with (1.2) shows that

$$u(t) = x_0 + \int_0^t f(s, u(s))ds \qquad \text{for } t \in I.$$

Consequently, u is a strong solution to (CP) by Lemma 2.2.

Let v be another strong solution to (CP) on I and let $w(t) = \|u(t) - v(t))\|$ for $t \in I$. Then w is absolutely continuous on I and

$$\begin{aligned} w'(t) &\leqq \|u'(t) - v'(t)\| = \|f(t, u(t)) - f(t, v(t))\| \\ &\leqq g(t, w(t)) \qquad \text{for } a.e.t \in I. \end{aligned}$$

Since $w(0) = 0$, it follows from Lemma 2.1 that $w \equiv 0$ on I. This completes the proof of Theorem 1.

Remark 2.1. In Theorem 1 if we assume f to be continuous from $[0, a] \times S(x_0, r)$ into E in place of (f_1) and (f_2), then there exist constants $0 < t_1 \leqq a$, $0 < r_1 \leqq r$ and $M > 0$ such that

$$\|f(t, x)\| \leqq M \qquad \text{for all } (t, x) \in [0, t_1] \times S(x_0, r_1).$$

Let $t_0 = \mathrm{Min}\,\{t_1, r_1/M\}$. Then the successive approximations $\{u_n\}$ converges uniformly on $[0, t_0]$ to a unique C^1 solution to (CP).

§ 3. Proof of Theorem 2.

Let $\Gamma = \{t \geqq 0;\ \{u_n\}$ converges uniformly on $[0, t]\}$ and let $t_0 = \sup \Gamma$. Then, by Theorem 1, $t_0 > 0$.

We have only to show that $t_0 < +\infty$ leads to a contradiction. Since $\{u_n\}$ is equicontinuous on $[0, T]$ for each $T \geqq t_0$, given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\|u_n(t) - u_n(t_0)\| < \varepsilon/3 \qquad \text{whenever } |t - t_0| \leqq \delta \text{ and } n \geqq 1.$$

Since $\lim_{n\to\infty} u_n(t_0 - \delta)$ exists by the definition of t_0, there exists an n_0 such that

$$\|u_n(t_0 - \delta) - u_m(t_0 - \delta)\| < \varepsilon/3 \qquad \text{for all } n, m \geqq n_0$$

and so that

$$\|u_n(t_0)-u_m(t_0)\| \leqq \|u_n(t_0)-u_n(t_0-\delta)\| + \|u_n(t_0-\delta)-u_m(t_0-\delta)\| + \|u_m(t_0-\delta)-u_m(t_0)\| < \varepsilon$$

for all $n, m \geqq n_0$. Consequently, $\lim_{n\to\infty} u_n(t_0)=z_0$ exists. Corresponding to (t_0, z_0) there exist positive constants a, r and a function g satisfying the conditions $(g_1)-(g_3)$ on $(t_0, t_0+a] \times [0, 2r]$ such that

$$\|f(t, x)-f(t, y)\| \leqq g(t, \|x-y\|)$$

for $a.e. t \in (t_0, t_0+a]$ and all $x, y \in S(z_0, r)$. By the equicontinuity of $\{u_n\}$ we can find a η such that $0<\eta \leqq \mathrm{Min}\,\{a, r\}$, $\int_{t_0}^{t_0+\eta} \beta(t)dt \leqq r$ and $\|u_n(t)-u_n(t_0)\| \leqq r/2$ for all $t \in [t_0, t_0+\eta]$ and $n \geqq 1$. Since $\lim_{n\to\infty} u_n(t_0)=z_0$, there exists an n_0 such that

$$\|u_n(t_0)-z_0\| \leqq r/2 \qquad \text{for all } n \geqq n_0.$$

It therefore follows that

$$\|u_n(t)-z_0\| \leqq \|u_n(t)-u_n(t_0)\| + \|u_n(t_0)-z_0\| \leqq r$$

for all $t \in [t_0, t_0+\eta]$ and $n \geqq n_0$, and this implies that

$$u_n([t_0, t_0+\eta]) \subset S(z_0, r) \qquad \text{for all } n \geqq n_0.$$

Now, let us define

$$w_{mn}(t)=\|u_m(t)-u_n(t)\| \qquad (t \in [t_0, t_0+], m \geqq n \geqq 1)$$

and

$$w_n(t)=\sup_{m\geqq n} w_{mn}(t) \qquad (t \in [t_0, t_0+], n \geqq 1).$$

Then just in the same way as the proof of Theorem 1, there exists a subsequence $\{w_{n(k)}\}$ of $\{w_n\}$ converging uniformly on $[t_0, t_0+\eta]$. The limit function w of $\{w_{n(k)}\}$ satisfies $w(t_0)=0$ and

$$|w'(t)| \leqq g(t, w(t)) \qquad \text{for } a.e. t \in (t_0, t_0+\eta].$$

It thus follows from Lemma 2.1 that $w \equiv 0$ on $[t_0, t_0+\eta]$ and this implies that $\{u_n\}$ converges uniformly on $[t_0, t_0+\eta]$ which contradicts to the definition of t_0. Q.E.D.

Remark 3.1. In [4] G. Vidossich proved the following

Theorem. *Let the hypothesis of Theorem* 2 *of the present paper be satisfied except that the condition* (g_3) *is replaced by*

$(g_3)'$ *There exists a function* $\alpha \in L^1(0, a)$ *such that*

$$g(t, \tau) \leqq \alpha(t) \qquad \text{for } (t, \tau) \in (0, a] \times [0, 2r].$$

Then the successive approximations $\{u_n\}$ *converges uniformly on any compact interval of* $[0, \infty)$.

Our Theroem 2 contains ovbiously the above Theorem whose proof crucially depends on the integrability of α.

§ 4. Proof of Theorem 3.

Let $\{w_{n(k)}\}$ and w be the same as in the proof of Theorem 1. Then we shall show that $(D^+w)(0)=0$. By the continuity of f, for each $\varepsilon>0$ there exists an $\eta>0$ such that

$$\|f(s, x) - f(0, 0)\| < \varepsilon/2 \qquad \text{for all } (s, x) \in [0, \eta] \times S(0, \eta).$$

Since $M(t)=\int_0^t \beta(\tau)d\tau$ is continuous and $M(0)=0$, there exists a $\delta, 0<\delta \leqq \eta$, such that $M(\delta) \leqq \eta$. Since $\|u_m(s)\| \leqq M(s) \leqq M(\delta) \leqq \eta$ for each $s \in [0, \delta]$ and $m \geqq 1$, it follows that for each $t \in (0, \delta]$

$$\|f(s, u_{m-1}(s)) - f(s, u_{n(k)-1}(s))\| < \varepsilon$$

whenever $s \in [0, t]$ and $m \geqq n(k)$. By the definition of $w_{n(k)}$ we have

$$\begin{aligned} w_{n(k)}(t) &\leqq \sup_{m \geqq n(k)} \|u_m(t) - u_{n(k)}(t)\| \\ &= \sup_{m \geqq n(k)} \left\| \int_0^t (f(s, u_{m-1}(s)) - f(s, u_{n(k)-1}(s)))ds \right\| \leqq \varepsilon t \end{aligned}$$

for all $t \in (0, \delta]$ and $k \geqq 1$, and hence

$$w(t) \leqq \varepsilon t \qquad \text{for all } t \in (0, \delta].$$

This implies that $(D^+w)(0)=0$. Therefore $w \equiv 0$ on I by Lemma 2.1 and this shows that the sequence $\{u_n\}$ is uniformly convergent on I. The rest of the proof is much the same as the corresponding part of that of Theorem 2. Q.E.D.

Remark 4.1. Theorem 3 of the present paper is an extension of Theorem 3.1 in [1, p. 54] into a general Banach space.

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(Ricevita la 4-an aprilo, 1977)