# Oriented Perfectly Matched Layer With Flexible Parameters for Waveguide Discontinuity Problems

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***Abstract*—A new arbitrarily oriented perfectly matched layer (PML) is developed for truncating finite element meshes in waveguiding problems. The PML can have any orientation with flexible parameters such as stretching variables and thicknesses. It is based on the PMLs stretched step-by-step along each normal direction of their surface. The formulation is easily adapted within the finite element method frameworks, providing good simulation results.**

***Index Terms*—Absorbing boundary, finite element method (FEM), perfectly matched layer (PML), photonic crystals (PCs).**

## I. Introduction

When NUMERICAL solvers based on domain discritization, such as the finite element method (FEM) and the finite-difference time-domain method, are applied to open boundary structures, the analysis region must be truncated with an absorbing boundary condition. The perfectly matched layer (PML) [1], [2] is an effective reflectionless boundary condition and has been extensively used.

The PML can also be used for truncating ports of FEM analysis in waveguide discontinuity problems. In these problems, PML stretching variables have to be set along the input and output waveguides. Therefore, in order to be applied to the arbitrarily oriented input and output waveguides, PMLs have to be oriented in any direction. Recently, the arbitrarily oriented PML was introduced [3]. Although this approach overcomes the limit of coordinate axes, the stretching variable is assumed to be a common constant to make matching between interfacing PMLs and so the thickness of PMLs is also limited to a common length. When we consider the photonic crystal (PC) waveguides which have periodic structures even nonbending ones, the constant feature on stretching variables is not acceptable [4], [5]. The PML for PC is not rigorous and so the absorption from PML surface has to be gradual. For example, parabolic functions can be employed as effective stretching variables.

In this letter, we propose a more flexible oriented PML that permits us to analyze waveguide discontinuity problems with the appropriate shapes and parameters, by using a step-by-step algorithm. The new procedure itself is very simple and its integration within the FEM frameworks is quite easy. When applied to oriented waveguiding problems such as waveguide bends and junctions, this method leads to reliable numerical results and reducing the analysis region by using different thicknesses of PMLs between truncating ports and absorbing radiation modes. Also, it provides effective absorptions by changing the stretching values when different media are to be used.

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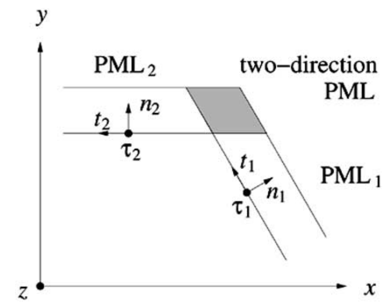


Fig. 1. Two-dimensional display of oriented PML.

## II. Theory

We consider two-dimensional oriented PMLs in the $xy$ plane, as shown in Fig. 1. Rotated rectangular coordinate systems are used for emulating various orientations of PMLs. The $n$ axes indicate the normal direction of the interface between the free-space and a one-direction-oriented PML. Also, the $t$ and $\tau$ are the first and second tangential axes, respectively. We use a similar structure as in [3]. However, in our case, the interfaces between two one-direction PMLs and a two-direction PML are given along the $t$ axes, as shown in Fig. 1.

In linear media, Maxwell's equations are written as

$$\nabla \times \boldsymbol{E} = -j\omega \boldsymbol{B} \tag{1a}$$
$$\nabla \times \boldsymbol{H} = j\omega \boldsymbol{D} \tag{1b}$$

with

$$\boldsymbol{D} = [\varepsilon(x,y,z)]\boldsymbol{E} \tag{1c}$$
$$\boldsymbol{B} = [\mu(x,y,z)]\boldsymbol{H}. \tag{1d}$$

In order to apply an analytic continuation only for the $n$ axis, we rotate the coordinate from the $x$-$y$-$z$ system to the $n$-$t$-$\tau$ system. The constitutive parameters along the rotated coordinate axes $n$-$t$-$\tau$ can be represented by components in the $x$-$y$-$z$ coordinate system as

$$[\varepsilon(n,t,\tau)] = [R]^T[\varepsilon(x,y,z)][R] \tag{2a}$$
$$[\mu(n,t,\tau)] = [R]^T[\mu(x,y,z)][R] \tag{2b}$$

with

$$[R] = \begin{bmatrix} \cos\theta_{n,x} & \cos\theta_{n,y} & \cos\theta_{n,z} \\ \cos\theta_{t,x} & \cos\theta_{t,y} & \cos\theta_{t,z} \\ \cos\theta_{\tau,x} & \cos\theta_{\tau,y} & \cos\theta_{\tau,z} \end{bmatrix} \tag{2c}$$

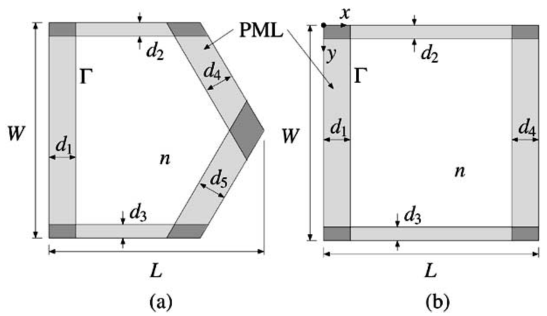


Fig. 2. Layouts for Gaussian beam propagation simulation. (a) Oriented PML structure. (b) Conventional PML structure.

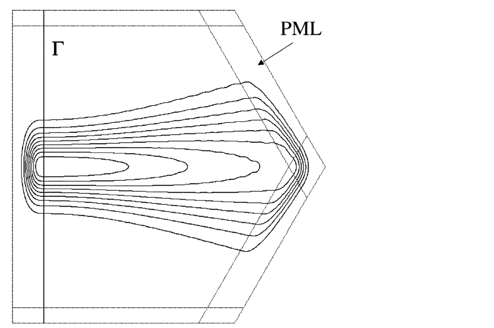


Fig. 3. Electric field distribution obtained with oriented PML structure.

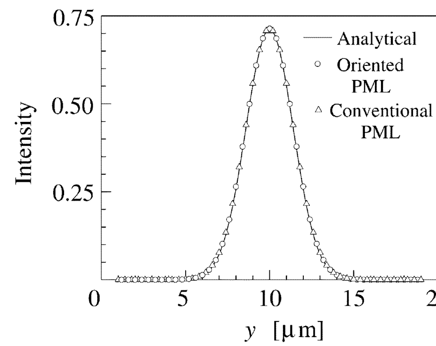


Fig. 4. Intensity profiles along $y$ axis obtained after a propagation distance of 10 $\mu$m.

where $\theta_{i,j}(i = n, t, \tau, j = x, y, z)$ is the angle between the relevant coordinate axes.

After the coordinate rotation, using anisotropic PML [2] along the $n$ direction, we obtain the following constitutive parameters for a PML:

$$[\varepsilon(n,t,\tau)]_{\mathrm{PML}} = |S|^{-1}[S]([R][\varepsilon(x,y,z)][R]^T)[S] \quad (3a)$$

$$[\mu(n,t,\tau)]_{\mathrm{PML}} = |S|^{-1}[S]([R][\mu(x,y,z)][R]^T)[S] \quad (3b)$$

with

$$[S] = \begin{bmatrix} 1/s_n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3c)$$

where $s_n$ is the stretching variable for the $n$ axis. The value of $s_n$ can be taken arbitrarily.

We finally reverse the rotated coordinate axes to the $x$-$y$-$z$ coordinate axes, then we obtain the following constitutive parameters:

$$[\varepsilon(x,y,z)]_{\mathrm{PML}} = [R]^T\{|S|^{-1}[S]([R][\varepsilon(x,y,z)][R]^T)[S]\}[R] \quad (4a)$$

$$[\mu(x,y,z)]_{\mathrm{PML}} = [R]^T\{|S|^{-1}[S]([R][\mu(x,y,z)][R]^T)[S]\}[R]. \quad (4b)$$

On corner regions, which are shaded in Fig. 1, (4a) and (4b) are iterated for each interfacing PML. This simple procedure provides the matching between the relevant PMLs. Moreover, comparing the present method with the previous oriented PML proposed in [3], our approach can set PML parameters, such as stretching variables $s_n$ and thicknesses, independently of other PMLs.

## III. Numerical Examples

### A. Gaussian Beam

To show the reliability of the present method, we calculated Gaussian beam propagation in the two-dimensional free space surrounded by the oriented PMLs, as shown in Fig. 2(a), where $L = W = 20$ $\mu$m, $d_1 = d_4 = d_5 = 2.0$ $\mu$m, $d_2 = d_3 = 1.0$ $\mu$m, and refractive index $n = 1.0$. The transverse electric Gaussian beam with spot size $W_0 = 2.0$ $\mu$m and wavelength $\lambda = 1.55$ $\mu$m is input on the incidence plane $\Gamma$. Fig. 3 shows the electric field distribution obtained by FEM analysis [5] using the present PML. The input Gaussian beam is absorbed into PML without any reflection.

Intensity profiles obtained after a propagation distance of 8 $\mu$m, with the oriented PML structure and the compatible conventional PML structure, which is shown in Fig. 2(b), are then compared with a known analytical solution for Helmholtz propagation. In our simulations, stretching variables $s_n$ are taken as

$$s_n = 1 - j\left(\frac{\rho}{d}\right)^2 \tan\delta \quad (5)$$

where $\rho$ is the distance from the beginning of PML (PML surface) and $\delta$ is the loss angle at the end of PML ($\rho = d$), chosen as $\tan\delta = 0.8$. The oriented PML structure and the conventional PML structure are discretized, respectively, in 84 232 and 80 000 quadratic triangular elements, as the number of division is the same along the propagation direction on the center of lateral position.

In Fig. 4, we show the resulting intensity profiles computed with the three methods just described. As is clearly seen, FEM results with both PML structures are in a good agreement with the analytical results. We define the error to be

$$\mathrm{ERR} = \frac{\int |\phi_f - \phi_a|^2 dy}{\int |\phi_a|^2 dy} \quad (6)$$

where $\phi_f$ and $\phi_a$ are the complex fields obtained with the FEM and the analytical solution, respectively. The error involved in the present method is 0.024%, while that of the conventional PML is 0.031%. These values are negligible.

Besides, the size of the computational overhead associated with the use of the proposed oriented PML is negligible, com-

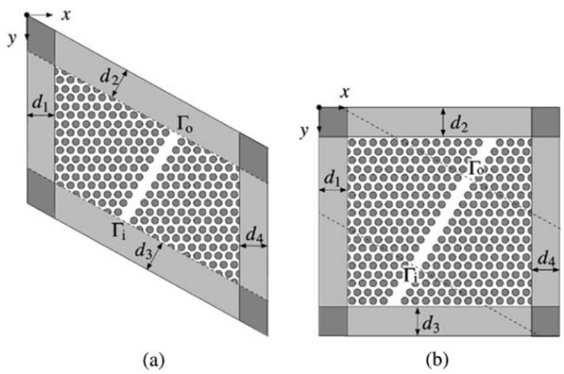


Fig. 5. Schematics for 60° oriented waveguides. (a) Oriented PML structure. (b) Conventional PML structure.

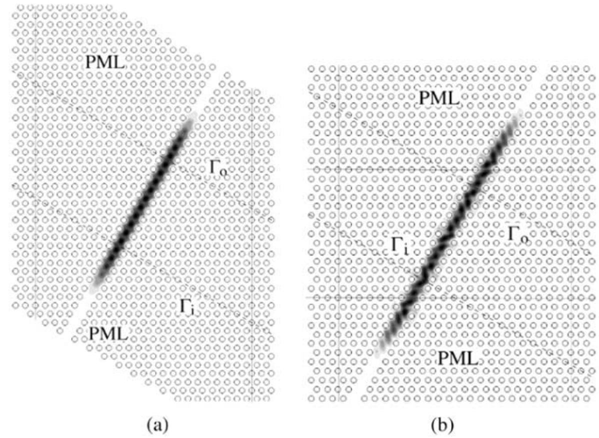


Fig. 6. Magnetic field distributions for 60° oriented PC waveguides. (a) Oriented PML structure. (b) Conventional PML structure.

paring to the total computational load. To confirm this point, we consider the strucure shown in Fig. 2(b) again. The calculations with the oriented PML algorithm and the conventional PML algorithm alike, take ~31.5 s for computational time and ~548 MB for memory on a PC with Athlon XP 1533 MHz.

### *B. Oriented PC Waveguide*

In this example, we demonstrate the accuracy and utility of the proposed method for truncating finite element meshes for wave propagation problems following the comparison of oriented PC waveguides, surrounded by two different types of PML, as shown in Fig. 5. The waveguides are created by removing a row of air holes from a PC composed of air holes in a semiconducting material on triangular array with lattice constant $a$, where the radius of air holes is $r = 0.29a$ and the refractive index of the semiconductor is $n = 3.4$. With these values, the crystal exhibits a photonic bandgap (PBG) for transverse magnetic (TM) modes which extends from $\omega a/(2\pi c) = 0.21$ to 0.27, where $\omega$ is the angular frequency and $c$ is the light velocity. The PML, in which the original PC structure remains as is [4], has thicknesses $d_1 = d_4 = 2a$ and $d_2 = d_3 = 10a$, and the stretching variables $s_n$ are the same as in previous example, shown in (5). The comparison is performed by propagating the fundamental TM mode input on the incidence plane $\Gamma_i$ with $\omega a/(2\pi c) = 0.24$. Each analysis region is discretized in about 36 000 quadratic triangular elements.

Fig. 6(a) shows the magnetic field distribution obtained by FEM analysis with the oriented PML structure, in which the input wave is absorbed into PML without any reflection. In contrast, the field distribution calculated with the conventional PML structure, shown in Fig. 6(b), is distorted by the spurious reflection from the PMLs because the stretching parameter is not set along the waveguide.

Intensity profiles obtained after a propagation distance of $10a$ with two different PML structures are then compared with an eigenmode of the PC waveguide computed by the FEM, with the periodic boundary condition in the propagation direction. The error that is defined as in (6) is 0.020% for the newly proposed PML and that for the conventional PML is 4.5%. This shows that the oriented PML contributes a better alternative for truncating ports of oriented waveguides than the conventional PML. In addition, it is remarkable that the obtained accuracy depends on the stretching variables given by the parabolic function. To prove this point, we set the stretching value as $s_n = 1 - j0.8$. In this case, the error was calculated to be 6.9%. Comparing the two results, namely 0.020% and 6.9%, it is obvious that the error using the parabolic function is drastically reduced compared to that of the constant value.

## IV. Conclusion

We have developed a new arbitrarily oriented PML that can be used with any stretching value and thickness. It is simply based on the PMLs stretched step-by-step along each normal direction of the surface and its formulation includes the conventional PML that is arranged along the $x$-$y$-$z$ coordinate system. The newly proposed PML structure is appropriate to be used for FEM analysis of waveguide discontinuity problems with arbitrary orientation and allows us to get accurate results even in case of periodically varying input and output waveguides by using parabolic functions as stretching variables. Finally, even though the present technique has been used in the frequency domain examples, its extension to the time domain cases is straightforward. In a time-domain simulation of PC waveguides, the PML can emulate the radiation condition outside PBG by surrounding the analysis region.

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