# The Mathematical Theory of Thick and Thin Elastic Rectangular Disks in the States of Plane Stress and Plane Strain

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**Abstract**

The mathematical theory of thick and thin elastic rectangular disks in the states of plane stress and plane strain is derived from the three- and two-dimensional theories of elasticity. By verifying the three-dimensional strain-displacement relations it is found that the exact state of plane stress in two-dimensional problems of elasticity for thin disks exists only when either Poisson's ratio=0 or the area dilatation=a constant, and does not exist except under these conditions. Two kinds of the exact stress-strain relations of thin disks in the state of plane stress are derived from the conditions. The exact displacement equilibrium equations of thick and thin disks in three- and two-dimensional problems are stated. The solutions to the displacement equilibrium equations and the stress-strain relations of thick and thin disks in the states of plane stress and plane strain are given.

## 1. Introduction

There are three- and two-dimensional theories of elasticity. However, the two-dimensional theory of elasticity seems to have been treated with no consideration to consistency with the three-dimensional theory of elasticity thus far. If the three-dimensional theory is considered to be the most general theory, the two-dimensional theory is only a specialization of that under some certain conditions. Therefore, the two theories should be consistent with each other under those conditions. In other words, exact solutions, even in the two-dimensional theory, must exactly satisfy all of the equilibrium equations, the stress-strain relations and the strain-displacement relations under those conditions.

There are two theories on the states of plane stress and plane strain in the two-dimensional theory. However, the theory on the state of plane stress seems to be an incorrect one constructed by disregarding the strain-displacement relations for transverse shearing strains. This causes Airy's stress function [1] of thin disks in the state of plane stress to fail to satisfy three of the six compatibility equations. If the theory on the state of plane stress cannot be exactly formulated in the category of the two-dimensional theory, it should be formulated in the category of the three-dimensional theory. Love's solution [2] for thick elastic rectangular disks by means of stress functions is considered to be one in that. However, the theory on the state of plane stress is notionally regarded as that to be applied just to thin disks. Therefore, even if a specific condition is required, it is useful to con-

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struct an exact theory in the category of the two-dimensional theory that avoids inconsistency with the three-dimensional theory of elasticity.

On the other hand, the theory on the state of plane strain exists exactly in the category of the two-dimensional theory of elasticity, unlike that on the state of plane stress. In the state of plane strain, the condition that the transverse normal strain vanishes makes the existence of the two-dimensional theory possible. The theory on the state of plane strain has been considered to be a situation of thin disks cut out from infinitely thick disks thus far. However, it seems to be unnecessary to restrict the theory to thin disks, because an examination of the strain-displacement relations in the state of plane strain suggests the existence of solutions for thick disks in the category of the three-dimensional theory of elasticity. If the solutions for thick disks are found, solutions for thin disks can be obtained simply by specializing them into two-dimensional solutions.

This paper is concerned with an examination of the stress-strain relations, the strain-displacement relations and the displacement equilibrium equations of thick and thin elastic rectangular disks in the states of plane stress and plane strain. Furthermore, the solutions for the thick and thin disks in terms of displacement potentials are given through that examination. The thick and thin disks which are called here are considered to be three- and two-dimensional problems, respectively. It is found from the verification of the strain-displacement relations that the theory on the state of plane stress exists exactly as a three-dimensional problem of elasticity, whereas that as a two-dimensional problem of elasticity exists exactly only under specific conditions. The solutions for the thin disks are obtained by specializing the solutions for the thick disks into two-dimensional solutions by making use of the specific conditions.

## 2. Basic equations in the three-dimensional theory of elasticity

We should state some basic equations in the three-dimensional theory of elasticity which will be necessary for the construction of the theory. The strain-displacement relation is

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{1}$$

where $\varepsilon_{ij}$ and $\boldsymbol{u}_i$ denote strain and displacement components, respectively. The stress-strain relation is

$$\sigma_{ij} = 2G\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}e\right), \tag{2}$$

where $\sigma_{ij}$, $G$, $\nu$, $e$ and $\delta_{ij}$ denote stress component, the shear modulus, Poisson's ratio, the cubical dilatation and the Kronecker delta, respectively. Using rectangular Cartesian coordinates ($\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$), the equilibrium equation is

$$\sigma_{ji,j} + \boldsymbol{K} = \boldsymbol{0}, \tag{3}$$

where $\boldsymbol{K}$ ($K_x$, $K_y$, $K_z$) denotes the body force vector. The Beltrami-Michell compatibility equations [3] are as follows:

$$\nabla^2\sigma_{xx} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2} = -2\frac{\partial K_x}{\partial x} - \frac{\nu}{1-\nu}\operatorname{div}\boldsymbol{K}, \tag{4a}$$

$$\nabla^2\sigma_{yy} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2} = -2\frac{\partial K_y}{\partial y} - \frac{\nu}{1-\nu}\operatorname{div}\boldsymbol{K}, \tag{4b}$$

$$\nabla^2\sigma_{zz}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2}=-2\frac{\partial K_z}{\partial z}-\frac{\nu}{1-\nu}\operatorname{div}\boldsymbol{K},\tag{4c}$$

$$\nabla^2\sigma_{yz}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z}=-\frac{\partial K_y}{\partial z}-\frac{\partial K_z}{\partial y},\tag{4d}$$

$$\nabla^2\sigma_{zx}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z\partial x}=-\frac{\partial K_z}{\partial x}-\frac{\partial K_x}{\partial z},\tag{4e}$$

$$\nabla^2\sigma_{xy}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial y}=-\frac{\partial K_x}{\partial y}-\frac{\partial K_y}{\partial x},\tag{4f}$$

where $\nabla^2$ denotes the Laplacian operator in the rectangular Cartesian coordinates, and

$$\Theta=\sigma_{xx}+\sigma_{yy}+\sigma_{zz}.\tag{5}$$

## 3. A state of plane stress

### 3.1 The stress-strain relations

The conditions for a state of plane stress are expressed as

$$\sigma_{zz}=0,\quad \sigma_{yz}=0,\quad \sigma_{zx}=0.\tag{6a-c}$$

Imposing condition (6a) on stress-strain relation (2), we obtain the following relationship:

$$\varepsilon_{zz}=-\frac{\nu}{1-\nu}e^*,\tag{7}$$

where $e^*$ denotes the area dilatation in the form

$$e^*=\varepsilon_{xx}+\varepsilon_{yy}.\tag{8}$$

**a. Thick disks**

Substituting (7) into stress-strain relation (2) and regarding conditions (6b,c), the stress-strain relations of a thick disk are obtained in the form

$$\begin{aligned}&\sigma_{xx}=2G\left(\varepsilon_{xx}+\frac{\nu}{1-\nu}e^*\right),\quad \sigma_{yy}=2G\left(\varepsilon_{yy}+\frac{\nu}{1-\nu}e^*\right),\quad \sigma_{xy}=2G\varepsilon_{xy},\\&\sigma_{zz}=0,\quad \sigma_{yz}=G\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)=0,\quad \sigma_{zx}=G\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right)=0,\end{aligned}\tag{9}$$

where the stress components are functions of $x$, $y$ and $z$. Although (9) is in agreement with the conventional stress-strain relations of a thin disk in two-dimensional problems of elasticity, it is exact only for the thick disk in three-dimensional problems of elasticity. The last two of (9) are also necessary for three-dimensional problems to determine the out-of-plane displacement.

**b. Thin disks**

If condition (6a) is satisfied by (7), conditions (6b,c) should be satisfied. Imposing conditions (6b,c) on stress-strain relation (2), we obtain

$$\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}=0,\quad \frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}=0.\tag{10}$$

If we regard the in-plane displacements $u_x$ and $u_y$ in (10) as functions of $x$ and $y$ under the condition of two-dimensional problems, we obtain

$$\frac{\partial u_z}{\partial y}=0,\quad \frac{\partial u_z}{\partial x}=0. \tag{11}$$

Solving (11) simultaneously, the out-of-plane displacement $u_z$ becomes

$$u_z=f(z). \tag{12}$$

Substituting $\varepsilon_{zz}$ obtained from (12) into the left-hand side of (7), we obtain

$$\frac{df}{dz}=-\frac{\nu}{1-\nu}e^{*}. \tag{13}$$

To satisfy (13), both sides must equal a certain constant $c$, because the left- and right-hand sides of (13) are functions only of $z$ and of $x$ and $y$, respectively.

That is

$$\frac{df}{dz}=-\frac{\nu}{1-\nu}e^{*}=c\ \text{(const.)}. \tag{14}$$

To satisfy (14), we need the following two cases:

Case 1: $c=0$ and $\nu=0$.

Case 2: $c\neq 0$, $\nu\neq 0$ and $e^{*}=c_0$ (const).

Therefore, either Case 1 or Case 2 is necessary for the thin disk in two-dimensional problems exactly to satisfy conditions (6a-c). In other words, the exact state of plane stress of the thin disk in two-dimensional problems does not exist except for Case 1 and Case 2. We can verify the two cases from another approach.

Introducing the stress function $\chi$ which is a function of $x$, $y$ and $z$, and expressing the stress components as

$$\sigma_{xx}=\frac{\partial^2\chi}{\partial y^2},\quad \sigma_{yy}=\frac{\partial^2\chi}{\partial x^2},\quad \sigma_{xy}=-\frac{\partial^2\chi}{\partial x\partial y}, \tag{15}$$

they satisfy equilibrium equations (3) under conditions (6a-c) and $K_x=K_y=K_z=0$. Therefore, it is sufficient for solutions to satisfy compatibility equations (4a-f). To satisfy compatibility equations (4c-e) simultaneously, we need the following relationship:

$$\frac{\partial\Theta}{\partial z}=\beta'\ \text{(const.)}. \tag{16}$$

Integrating (16) with respect to z, we obtain

$$\Theta=\beta' z+\Theta_0(x,y)+c_0'=\nabla_1^2\chi, \tag{17}$$

where

$$\nabla_1^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

Since $\Theta$ in (5) becomes a harmonic function for the case of zero body forces, we obtain

$$\nabla^2\Theta=0. \tag{18}$$

Applying $\nabla^2$ to (17) and using (18), we obtain

$$\nabla_1^2\Theta_0=0. \tag{19}$$

Substituting (15) and (17) into compatibility equations (4a,b,f), we successively obtain

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 \chi}{\partial z^2}+\frac{\nu}{1+\nu}\Theta_0\right)=0,\quad \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \chi}{\partial z^2}+\frac{\nu}{1+\nu}\Theta_0\right)=0,\quad -\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 \chi}{\partial z^2}+\frac{\nu}{1+\nu}\Theta_0\right)=0. \tag{20}$$

To satisfy (20) simultaneously, we need

$$\frac{\partial^2 \chi}{\partial z^2}+\frac{\nu}{1+\nu}\Theta_0 = xf_1(z)+yf_2(z)+f_3(z). \tag{21}$$

Integrating (21) twice with respect to z, we obtain

$$\chi=-\frac{\nu}{2(1+\nu)}z^2\Theta_0+z\chi_1(x,y)+\chi_0(x,y)+x\int\int f_1(z)dzdz+y\int\int f_2(z)dzdz+\int\int f_3(z)dzdz. \tag{22}$$

If $\chi$ is taken as a function of ***x*** and ***y*** under the condition of two-dimensional problems, the following relationships should hold:

$$\beta'=\chi_1(x,y)=f_1(z)=f_2(z)=f_3(z)=0,\qquad -\frac{\nu}{2(1+\nu)}\Theta_0=0. \tag{23a,b}$$

Then, (22) and (15) yield

$$\chi=\chi_0(x,y), \tag{24}$$

$$\sigma_{xx}=\frac{\partial^2 \chi_0}{\partial y^2},\quad \sigma_{yy}=\frac{\partial^2 \chi_0}{\partial x^2},\quad \sigma_{xy}=-\frac{\partial^2 \chi_0}{\partial x \partial y}. \tag{25}$$

To satisfy (23b), we need the following two cases:

Case A: $\nu = 0$ ($\Theta_0 \neq 0$).

Case B: $\Theta_0 = 0$ ($\nu \neq 0$).

Using (17) and (19), Case A yields

$$\nabla_1^2\chi=\nabla_1^2\chi_0=\Theta_0+c_0',\quad \nabla_1^2\nabla_1^2\chi_0=0 \tag{26}$$

and $\chi_0$ is in agreement with Airy's stress function. This indicates that Airy's stress function of the thin disk in the state of plane stress is exact only for the case of $\nu = 0$, because the existence of the state of plane stress in the thin disk requires the condition of $\nu = 0$. On the other hand, Case B yields

$$\nabla_1^2\chi=\nabla_1^2\chi_0=c_0'. \tag{27}$$

Then, from (17) and (27), the area dilatation $e^*$ becomes the following constant:

$$e^*=\frac{1-\nu}{E}c_0', \tag{28}$$

where $E$ denotes Young's modulus. Therefore, the condition of Case A or Case B is necessary for the exact state of plane stress in two-dimensional problems and is in agreement with the condition of Case 1 or Case 2 as stated above.

From the examinations described above, we obtain the stress-strain relations of the thin disk in the following forms.

Case 1: $\nu = 0$.

$$\sigma_{xx}=E\varepsilon_{xx},\quad \sigma_{yy}=E\varepsilon_{yy},\quad \sigma_{xy}=E\varepsilon_{xy},\quad \sigma_{zz}=\sigma_{yz}=\sigma_{zx}=0. \tag{29}$$

Case 2: $e^* = c_0\ (\nu \neq 0)$.

$$\sigma_{xx} = 2G\left(\varepsilon_{xx} + \frac{\nu}{1-\nu}c_0\right), \quad \sigma_{yy} = 2G\left(\varepsilon_{yy} + \frac{\nu}{1-\nu}c_0\right), \quad \sigma_{xy} = 2G\varepsilon_{xy}, \tag{30}$$
$$\sigma_{zz} = \sigma_{yz} = \sigma_{zx} = 0.$$

The stress components in (29) and (30) are functions of $x$ and $y$.

### 3.2 Displacement equilibrium equations

**a. Thick disks**

Substituting stress-strain relations (9) into equilibrium equations (3) and using strain-displacement relation (1), the displacement equilibrium equations of the thick disk are obtained in the form

$$\nabla_1^2 u_x + \frac{1+\nu}{1-\nu}\frac{\partial e^*}{\partial x} + \frac{K_x}{G} = 0, \tag{31a}$$

$$\nabla_1^2 u_y + \frac{1+\nu}{1-\nu}\frac{\partial e^*}{\partial y} + \frac{K_y}{G} = 0, \tag{31b}$$

$$\frac{\partial u_z}{\partial z} = -\frac{\nu}{1-\nu}e^*, \tag{31c}$$

where $u_x$, $u_y$ and $u_z$ are functions of $x$, $y$ and $z$. Equation (31c) is obtained from (7). Furthermore, the out-of-plane displacement $u_z$ must satisfy (10). The solutions to (31a-c) and (10) for the case of zero body forces are given in Appendix 6.1.

**b. Thin disks**

The displacement equilibrium equations of the thin disk are divided into two cases below.

Case 1: $\nu = 0$.

Substituting stress-strain relations (29) into equilibrium equations (3) and using (1), (12) and (14), the displacement equilibrium equations of the thin disk are obtained in the form

$$\nabla_1^2 u_x + \frac{\partial e^*}{\partial x} + \frac{2K_x}{E} = 0, \quad \nabla_1^2 u_y + \frac{\partial e^*}{\partial y} + \frac{2K_y}{E} = 0, \quad u_z = a_0, \tag{32a-c}$$

where $u_x$ and $u_y$ are functions of $x$ and $y$. The solutions to (32a-c) for the case of zero body forces are given in Appendix 6.1.

Case 2: $e^* = c_0\ (\nu \neq 0)$.

Substituting stress-strain relations (30) into equilibrium equations (3) and using (1), (12) and (14), the displacement equilibrium equations of the thin disk are obtained in the form

$$\nabla_1^2 u_x + \frac{K_x}{G} = 0, \quad \nabla_1^2 u_y + \frac{K_y}{G} = 0, \quad u_z = -\frac{\nu}{1-\nu}c_0 z + b_0, \tag{33a-c}$$

where $u_x$ and $u_y$ are functions of $x$ and $y$ and are plane harmonic functions for the case of zero body forces. The solutions to (33a-c) for the case of zero body forces are given in Appendix 6.1.

## 4. A state of plane strain

### 4.1 The stress-strain relations

The conditions for a state of plane strain are expressed as

$$\varepsilon_{zz} = 0, \quad \varepsilon_{yz} = 0, \quad \varepsilon_{zx} = 0. \tag{34a-c}$$

Imposing condition (34a) on strain-displacement relation (1), we obtain

$$\frac{\partial u_z}{\partial z}=0. \tag{35}$$

Integrating (35) with respect to *z*, we obtain

$$u_z=g(x,y). \tag{36}$$

Imposing conditions (34b,c) on strain-displacement relation (1), we obtain (10).

**a. Thick disks**

Differentiating (10) with respect to *z* and regarding (35), we obtain

$$\frac{\partial^2 u_y}{\partial z^2}+\frac{\partial}{\partial y}\left(\frac{\partial u_z}{\partial z}\right)=\frac{\partial^2 u_y}{\partial z^2}=0,\quad \frac{\partial}{\partial x}\left(\frac{\partial u_z}{\partial z}\right)+\frac{\partial^2 u_x}{\partial z^2}=\frac{\partial^2 u_x}{\partial z^2}=0. \tag{37}$$

Integrating (37) twice with respect to *z*, we obtain

$$u_y=zg_1(x,y)+g_2(x,y),\quad u_x=zh_1(x,y)+h_2(x,y). \tag{38}$$

Equations (36) and (38) suggest the existence of solutions for the thick disk, in that the out-of-plane displacement $u_z$ is a function of $x$ and $y$ and the in-plane displacements $u_x$ and $u_y$ are functions of $x$ and $y$ and of $z$ of the first degree. Imposing conditions (34a-c) on stress-strain relation (2), the stress-strain relations of the thick disk are obtained in the form

$$\begin{aligned}&\sigma_{xx}=2G\left(\varepsilon_{xx}+\frac{\nu}{1-2\nu}e^*\right),\quad \sigma_{yy}=2G\left(\varepsilon_{yy}+\frac{\nu}{1-2\nu}e^*\right),\quad \sigma_{zz}=\frac{2G\nu}{1-2\nu}e^*=\nu(\sigma_{xx}+\sigma_{yy}),\\&\sigma_{xy}=2G\varepsilon_{xy},\quad \sigma_{yz}=G\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)=0,\quad \sigma_{zx}=G\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right)=0,\end{aligned} \tag{39}$$

where the stress components except for $\sigma_{zz}$ are functions of *x*, *y* and *z*.

**b. Thin disks**

Imposing the condition of two-dimensional problems that the in-plane displacements $u_x$ and $u_y$ are functions of *x* and *y* on (10) and regarding (36), we obtain

$$\frac{\partial u_z}{\partial y}=\frac{\partial g}{\partial y}=0,\quad \frac{\partial u_z}{\partial x}=\frac{\partial g}{\partial x}=0. \tag{40}$$

To satisfy (40) simultaneously, we need

$$g(x,y)=d_0, \tag{41}$$

namely,

$$u_z=d_0. \tag{42}$$

Therefore, conditions (34a-c) for the strain components in the thin disk are automatically satisfied from (42). Imposing conditions (34a-c) on stress-strain relation (2), the stress-strain relations of the thin disk are obtained in the form

$$\begin{aligned}&\sigma_{xx}=2G\left(\varepsilon_{xx}+\frac{\nu}{1-2\nu}e^*\right),\quad \sigma_{yy}=2G\left(\varepsilon_{yy}+\frac{\nu}{1-2\nu}e^*\right),\quad \sigma_{zz}=\frac{2G\nu}{1-2\nu}e^*=\nu(\sigma_{xx}+\sigma_{yy}),\\&\sigma_{xy}=2G\varepsilon_{xy},\quad \sigma_{yz}=\sigma_{zx}=0,\end{aligned} \tag{43}$$

where the stress components are functions of *x* and *y*.

### 4.2 Displacement equilibrium equations

**a. Thick disks**

Substituting stress-strain relations (39) into equilibrium equations (3) and using strain-displacement relation (1), the displacement equilibrium equations of the thick disk are obtained in the form

$$\nabla_1^2 u_x + \frac{1}{1-2\nu}\frac{\partial e^*}{\partial x} + \frac{K_x}{G} = 0, \tag{44a}$$

$$\nabla_1^2 u_y + \frac{1}{1-2\nu}\frac{\partial e^*}{\partial y} + \frac{K_y}{G} = 0, \tag{44b}$$

$$\frac{2\nu}{1-2\nu}\frac{\partial e^*}{\partial z} + \frac{K_z}{G} = 0, \tag{44c}$$

where $u_x$ and $u_y$ are functions of $x$, $y$ and $z$. Furthermore, from the last two of (39), the out-of-plane displacement $u_z$ must satisfy (10). The solutions to (44a-c) and (10) for the case of zero body forces are given in Appendix 6.2.

**b. Thin disks**

Substituting stress-strain relations (43) into equilibrium equations (3) and using strain-displacement relation (1), the displacement equilibrium equations of the thin disk are obtained in the form

$$\nabla_1^2 u_x + \frac{1}{1-2\nu}\frac{\partial e^*}{\partial x} + \frac{K_x}{G} = 0, \quad \nabla_1^2 u_y + \frac{1}{1-2\nu}\frac{\partial e^*}{\partial y} + \frac{K_y}{G} = 0, \quad u_z = d_0, \tag{45a-c}$$

where $u_x$ and $u_y$ are functions of $x$ and $y$. The solutions to (45a-c) for the case of zero body forces are given in Appendix 6.2.

## 5. Conclusion

From the examinations of the three-dimensional strain-displacement relations and the compatibility equations of thin disks in a state of plane stress, it was found that the exact state of plane stress in two-dimensional problems of elasticity exists only when either Poisson's ratio=0 or the area dilatation=a constant. In other words, it was clarified that the state of plane stress in two-dimensional problems of elasticity does not exist with no specific conditions. On the basis of the examinations, the exact stress-strain relations and the exact displacement equilibrium equations of the thick and thin elastic rectangular disks in the states of plane stress and plane strain were derived. The solutions to them will be stated in Appendix 6. The out-of-plane displacements of the thick and thin disks in the states of plane stress and plane strain are exactly determined from the displacement equilibrium equations and the strain-displacement relations. It was found that the state of plane strain exists also exactly in three-dimensional problems of elasticity. The mathematical theory of the thick and thin elastic rectangular disks in this paper is exactly consistent with the three-dimensional theory of elasticity and should be useful for the definition of the states of plane stress and plane strain in thick and thin disks.

## 6. Appendix

### 6.1 Solutions for the displacement and stress components in a state of plane stress

**a. Thick disks**

The solutions to (31a-c) and (10) for the case of zero body forces are obtained as

$$2Gu_x = -(1+\nu)\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)+2\nabla_1^2 F_1+\frac{\nu}{2}z^2\frac{\partial}{\partial x}\nabla_1^2\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)+\beta xz - z\frac{\partial \phi_1}{\partial x}-\frac{\partial \phi_0}{\partial x},$$
$$2Gu_y = -(1+\nu)\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)+2\nabla_1^2 F_2+\frac{\nu}{2}z^2\frac{\partial}{\partial y}\nabla_1^2\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)+\beta yz - z\frac{\partial \phi_1}{\partial y}-\frac{\partial \phi_0}{\partial y}, \quad (46)$$
$$2Gu_z = -\frac{\beta}{2}\left(x^2+y^2+\nu z^2\right)+\phi_1-\nu z\nabla_1^2\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)+\nu\beta^* z,$$

where

$$\nabla_1^2\nabla_1^2 F_1 = 0, \quad \nabla_1^2\nabla_1^2 F_2 = 0, \quad \nabla_1^2\phi_1 = \beta(1+\nu), \quad \nabla_1^2\phi_0 = \beta^*(1-\nu). \quad (47)$$

Furthermore, the stress components of the thick disk are expressed as

$$\sigma_{xx} = -(1+\nu)\frac{\partial^2}{\partial x^2}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)+\nabla_1^2\left[(2+\nu)\frac{\partial F_1}{\partial x}+\nu\frac{\partial F_2}{\partial y}\right]+\frac{\nu}{2}z^2\frac{\partial^2}{\partial x^2}\nabla_1^2\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)$$
$$+z\frac{\partial^2\phi_1}{\partial y^2}+\frac{\partial^2\phi_0}{\partial y^2}-\beta^*,$$

$$\sigma_{yy} = -(1+\nu)\frac{\partial^2}{\partial y^2}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)+\nabla_1^2\left[(2+\nu)\frac{\partial F_2}{\partial y}+\nu\frac{\partial F_1}{\partial x}\right]+\frac{\nu}{2}z^2\frac{\partial^2}{\partial y^2}\nabla_1^2\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)$$
$$+z\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_0}{\partial x^2}-\beta^*, \quad (48)$$

$$\sigma_{xy} = -(1+\nu)\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)+\nabla_1^2\left(\frac{\partial F_1}{\partial y}+\frac{\partial F_2}{\partial x}\right)+\frac{\nu}{2}z^2\frac{\partial^2}{\partial x\partial y}\nabla_1^2\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)$$
$$-z\frac{\partial^2\phi_1}{\partial x\partial y}-\frac{\partial^2\phi_0}{\partial x\partial y},$$

$$\sigma_{zz} = \sigma_{yz} = \sigma_{zx} = 0.$$

**b. Thin disks**

The solutions for the thin disk can be obtained by specializing solutions (46), (47) and (48) into two-dimensional solutions by making use of the specific conditions.

Case 1: $\nu = 0$.

In solutions (46) and (47), if we set

$$\nu = 0, \quad \beta = \phi_1 = 0, \quad \beta^* = \phi_0 = 0, \quad (49)$$

we obtain the solutions to (32a-c) for the case of zero body forces in the form

$$Eu_x = -\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)+2\nabla_1^2 F_1,$$
$$Eu_y = -\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)+2\nabla_1^2 F_2, \quad Eu_z = 0, \quad (50)$$

where

$$\nabla_1^2\nabla_1^2 F_1 = 0, \quad \nabla_1^2\nabla_1^2 F_2 = 0. \quad (51)$$

Substituting (49) into the solution (48) for the thick disk, the stress components of the thin disk for Case 1 are expressed as

$$\sigma_{xx} = -\frac{\partial^2}{\partial x^2}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)+2\frac{\partial}{\partial x}\nabla_1^2 F_1, \quad \sigma_{yy} = -\frac{\partial^2}{\partial y^2}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)+2\frac{\partial}{\partial y}\nabla_1^2 F_2,$$
$$\sigma_{xy} = -\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)+\nabla_1^2\left(\frac{\partial F_1}{\partial y}+\frac{\partial F_2}{\partial x}\right), \quad \sigma_{zz} = \sigma_{yz} = \sigma_{zx} = 0. \quad (52)$$

Case 2: $e^* = c_0\ (\nu \neq 0)$.

In solutions (46) and (47), if we set

$$F_1 = F_2 = 0, \quad \beta = \phi_1 = 0, \tag{53}$$

we obtain the solutions to (33a-c) for the case of zero body forces in the form

$$2Gu_x = -\frac{\partial \phi_0}{\partial x}, \quad 2Gu_y = -\frac{\partial \phi_0}{\partial y}, \quad 2Gu_z = \nu\beta^* z, \tag{54}$$

where

$$\nabla_1^2 \phi_0 = \beta^*(1-\nu), \quad \beta^* = -\frac{2G}{1-\nu}c_0. \tag{55}$$

Substituting (53) into the solution (48) for the thick disk, the stress components of the thin disk for Case 2 are expressed as

$$\sigma_{xx} = \frac{\partial^2 \phi_0}{\partial y^2} - \beta^*, \quad \sigma_{yy} = \frac{\partial^2 \phi_0}{\partial x^2} - \beta^*, \quad \sigma_{xy} = -\frac{\partial^2 \phi_0}{\partial x \partial y}, \quad \sigma_{zz} = \sigma_{yz} = \sigma_{zx} = 0. \tag{56}$$

**6.2 Solutions for the displacement and stress components in a state of plane strain**

**a. Thick disks**

The solutions to (44a-c) and (10) for the case of zero body forces are obtained as

$$\begin{aligned} 2Gu_x &= -\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) + 2(1-\nu)\nabla_1^2 F_1 - z\frac{\partial \phi_1}{\partial x} + \beta xz - \frac{\partial \phi_0}{\partial x}, \\ 2Gu_y &= -\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) + 2(1-\nu)\nabla_1^2 F_2 - z\frac{\partial \phi_1}{\partial y} + \beta yz - \frac{\partial \phi_0}{\partial y}, \\ 2Gu_z &= \phi_1 - \frac{\beta}{2}\left(x^2 + y^2\right), \end{aligned} \tag{57}$$

where

$$\nabla_1^2\nabla_1^2 F_1 = 0, \quad \nabla_1^2\nabla_1^2 F_2 = 0, \quad \nabla_1^2 \phi_1 = 2\beta, \quad \nabla_1^2 \phi_0 = \frac{1-2\nu}{1-\nu}\beta^*. \tag{58}$$

The solutions (57) and (58) satisfy (36) and (38) suggested in the examinations of strain-displacement relation (1). Furthermore, the stress components of the thick disk are expressed as

$$\begin{aligned} \sigma_{xx} &= -\frac{\partial^2}{\partial x^2}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) + \nabla_1^2\left[(2-\nu)\frac{\partial F_1}{\partial x} + \nu\frac{\partial F_2}{\partial y}\right] + z\frac{\partial^2 \phi_1}{\partial y^2} + \frac{\partial^2 \phi_0}{\partial y^2} - \beta z - \beta^*, \\ \sigma_{yy} &= -\frac{\partial^2}{\partial y^2}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) + \nabla_1^2\left[(2-\nu)\frac{\partial F_2}{\partial y} + \nu\frac{\partial F_1}{\partial x}\right] + z\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial x^2} - \beta z - \beta^*, \\ \sigma_{zz} &= \nu\nabla_1^2\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) - \frac{\nu}{1-\nu}\beta^*, \\ \sigma_{xy} &= -\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) + (1-\nu)\nabla_1^2\left(\frac{\partial F_1}{\partial y} + \frac{\partial F_2}{\partial x}\right) - z\frac{\partial^2 \phi_1}{\partial x \partial y} - \frac{\partial^2 \phi_0}{\partial x \partial y}, \\ \sigma_{yz} &= \sigma_{zx} = 0. \end{aligned} \tag{59}$$

**b. Thin disks**

The solutions for the thin disk can be obtained by specializing solutions (57), (58) and (59) into two-dimensional solutions. In solutions (57) and (58), if we set

$$\beta = \phi_1 = 0, \tag{60}$$

we obtain the solutions to (45a-c) for the case of zero body forces in the form

$$2Gu_x = -\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) + 2(1-\nu)\nabla_1^2 F_1 - \frac{\partial \phi_0}{\partial x},$$
$$2Gu_y = -\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) + 2(1-\nu)\nabla_1^2 F_2 - \frac{\partial \phi_0}{\partial y}, \quad 2Gu_z = 0, \tag{61}$$

where

$$\nabla_1^2 \nabla_1^2 F_1 = 0, \quad \nabla_1^2 \nabla_1^2 F_2 = 0, \quad \nabla_1^2 \phi_0 = \frac{1-2\nu}{1-\nu}\beta^*. \tag{62}$$

Substituting (60) into the solution (59) for the thick disk, the stress components of the thin disk are expressed as

$$\sigma_{xx} = -\frac{\partial^2}{\partial x^2}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) + \nabla_1^2\left[(2-\nu)\frac{\partial F_1}{\partial x} + \nu\frac{\partial F_2}{\partial y}\right] + \frac{\partial^2 \phi_0}{\partial y^2} - \beta^*,$$
$$\sigma_{yy} = -\frac{\partial^2}{\partial y^2}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) + \nabla_1^2\left[(2-\nu)\frac{\partial F_2}{\partial y} + \nu\frac{\partial F_1}{\partial x}\right] + \frac{\partial^2 \phi_0}{\partial x^2} - \beta^*,$$
$$\sigma_{zz} = \nu\nabla_1^2\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) - \frac{\nu}{1-\nu}\beta^*, \tag{63}$$
$$\sigma_{xy} = -\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) + (1-\nu)\nabla_1^2\left(\frac{\partial F_1}{\partial y} + \frac{\partial F_2}{\partial x}\right) - \frac{\partial^2 \phi_0}{\partial x \partial y},$$
$$\sigma_{yz} = \sigma_{zx} = 0.$$

## References

1 Timoshenko, S.P.; Goodier, J.N.: Theory of elasticity. 3rd ed., McGraw-Hill Kogakusha, Tokyo 1970.

2 Love, A.E.H.: A treatise on the mathematical theory of elasticity. 4th ed., Dover Pub., New York 1944.

3 Sokolnikoff, I.S.: Mathematical theory of elasticity. 2nd ed., McGraw-Hill, New York 1956.