# General solution methods for orthotropic solids with body forces

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**Abstract**

A three-dimensional elasticity solution with arbitrary body forces for orthotropic solids is derived from the displacement equilibrium equations for orthotropic solids. Three potential functions are used to derive the solution. One is used for a basic solution such as Boussinesq's function in isotropic solids, and the other two are used for adjoint functions. The solution is automatically in agreement with thermoelastic potentials of orthotropic solids, if three body forces are replaced with three temperature gradients.

## 1. Introduction

The latest studies on three-dimensional problems in the linear theory of elasticity have turned to those of anisotropic solids. There are two reasons for this trend. The first is that studies on isotropic and homogenious solids have been virtually accomplished in theory. The second is that the elucidation of the mechanical properties of anisotropic solids has grown in importance due to the recent increase in the use of anisotropic or composite materials. The theory of anisotropic solids was established in early times and has been stated in Love's (1944), Lekhnitskii's (1981) and Ting's (1996) books. Although there are various classes of anisotropic solids, transversely isotropic, orthotropic and cylindrically anisotropic solids seem to be important from the viewpoint of practical use of materials and the theory of elasticity.

The solutions for transversely isotropic solids were found by Elliott (1948), Lodge (1955) and Lekhnitskii (1981) in early times. However, the three-dimensional solutions for orthotropic solids are very few at the present time, and the authors only know Hata's (1956) and the Sonoda-Horikawa (1981) solutions. Hata (1956) obtained a solution expressed by one potential function by means of differential operators in that a system of the displacement equilibrium equations was reduced to a single equation. Sonoda and Horikawa (1981) extended Hata's method into a solution with body forces and obtained a solution by the superposition of three solutions for independent body forces of each other. However, their method is inapplicable to the case such that body forces are coupled with each other as thermoelastic potentials of orthotropic solids. Although orthotropic materials are

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seen in wood or cross-ply materials and are of high utility in technology, they are not so much treated in the latest studies on three-dimensional problems of elasticity. The cause is unclear, but one of that is considered to be that simply applicable solutions to boundary-value problems have not been discovered so far. Furthermore, the three-dimensional solutions for orthotropic solids taking body forces into account are applicable to thermoelastic potentials and dynamic problems of elastic wave propagation and are highly significant for various three-dimensional problems of orthotropic solids. However, they are not yet found in general form.

This paper is concerned with a three-dimensional elasticity solution with arbitrary body forces for orthotropic solids, in view of insufficient studies on three-dimensional problems of elasticity in orthotropic solids. Three potential functions are used for the derivation of the solution. One is used for a basic solution such as Boussinesq's function in isotropic solids and other two are used for adjoint functions due to a system of the displacement equilibrium equations with three unknowns. The solution presented in this paper is automatically in agreement with thermoelastic potentials of orthotropic solids, if three body forces are replaced with three temperature gradients. Furthermore, the solution yields a particular solution for body forces in isotropic solids, if the elastic constants of orthotropic solids are replaced with those of isotropic solids.

## 2. Displacement equilibrium equations

Using rectangular Cartesian coordinates $(x, y, z)$ such that the axes of $x$, $y$ and $z$ are taken parallel to the three principal axes of elasticity, the equilibrium equations are

$$\sigma_{ji,j} + b_i = 0 \quad (i, j = x, y, z), \tag{1}$$

where $\sigma_{ji}$ and $b_i$ denote stress components and body forces, respectively. Furthermore, the strain-displacement relations are

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{2}$$

where $\varepsilon_{ij}$ and $u_i$ denote strain components and displacement components, respectively. The stress-strain relations for orthotropic solids with nine elastic constants are expressed as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \\ 2\varepsilon_{xy} \end{Bmatrix}, \tag{3}$$

where $c_{ij}$ denotes elastic constants of the orthotropic solid.

Representing the stress components by the displacement components by making use of Eqs. (2) and (3) and substituting them into Eq. (1), the displacement equilibrium equations for the orthotropic solid are obtained in the form

$$c_{11}\frac{\partial^2 u_x}{\partial x^2} + c_{66}\frac{\partial^2 u_x}{\partial y^2} + c_{55}\frac{\partial^2 u_x}{\partial z^2} + \left(c_{12} + c_{66}\right)\frac{\partial^2 u_y}{\partial x \partial y} + \left(c_{13} + c_{55}\right)\frac{\partial^2 u_z}{\partial x \partial z} + b_x = 0, \tag{4a}$$

$$\left(c_{12} + c_{66}\right)\frac{\partial^2 u_x}{\partial x \partial y} + c_{66}\frac{\partial^2 u_y}{\partial x^2} + c_{22}\frac{\partial^2 u_y}{\partial y^2} + c_{44}\frac{\partial^2 u_y}{\partial z^2} + \left(c_{23} + c_{44}\right)\frac{\partial^2 u_z}{\partial y \partial z} + b_y = 0, \tag{4b}$$

$$(c_{13}+c_{55})\frac{\partial^2 u_x}{\partial x\partial z}+(c_{23}+c_{44})\frac{\partial^2 u_y}{\partial y\partial z}+c_{55}\frac{\partial^2 u_z}{\partial x^2}+c_{44}\frac{\partial^2 u_z}{\partial y^2}+c_{33}\frac{\partial^2 u_z}{\partial z^2}+b_z=0. \tag{4c}$$

## 3. Governing equations of potential functions

We now express the displacement components by means of potential functions, i.e., $\phi$ , $\psi_1$ and $\psi_2$ as

$$u_x=\frac{\partial\phi}{\partial x},\quad u_y=j\frac{\partial\phi}{\partial y}+\frac{\partial\psi_1}{\partial y},\quad u_z=k\frac{\partial\phi}{\partial z}+\frac{\partial\psi_2}{\partial z}, \tag{5a-c}$$

where *j* and *k* denote coefficients as determined later.
Substituting the displacement components of Eqs. (5a-c) into Eqs. (4a-c), we consecutively obtain

$$\frac{\partial}{\partial x}\left\{c_{11}\frac{\partial^2\phi}{\partial x^2}+\left[c_{66}+j(c_{12}+c_{66})\right]\frac{\partial^2\phi}{\partial y^2}+\left[c_{55}+k(c_{13}+c_{55})\right]\frac{\partial^2\phi}{\partial z^2}+(c_{12}+c_{66})\frac{\partial^2\psi_1}{\partial y^2}+(c_{13}+c_{55})\frac{\partial^2\psi_2}{\partial z^2}\right\}+b_x =0, \tag{6a}$$

$$\frac{\partial}{\partial y}\left\{(c_{66}j+c_{12}+c_{66})\frac{\partial^2\phi}{\partial x^2}+c_{22}j\frac{\partial^2\phi}{\partial y^2}+\left[c_{44}j+k(c_{23}+c_{44})\right]\frac{\partial^2\phi}{\partial z^2}+(c_{23}+c_{44})\frac{\partial^2\psi_2}{\partial z^2}+c_{66}\frac{\partial^2\psi_1}{\partial x^2}+c_{22}\frac{\partial^2\psi_1}{\partial y^2} +c_{44}\frac{\partial^2\psi_1}{\partial z^2}\right\}+b_y=0, \tag{6b}$$

$$\frac{\partial}{\partial z}\left\{(c_{55}k+c_{13}+c_{55})\frac{\partial^2\phi}{\partial x^2}+\left[c_{44}k+j(c_{23}+c_{44})\right]\frac{\partial^2\phi}{\partial y^2}+c_{33}k\frac{\partial^2\phi}{\partial z^2}+(c_{23}+c_{44})\frac{\partial^2\psi_1}{\partial y^2}+c_{55}\frac{\partial^2\psi_2}{\partial x^2}+c_{44}\frac{\partial^2\psi_2}{\partial y^2} +c_{33}\frac{\partial^2\psi_2}{\partial z^2}\right\}+b_z=0. \tag{6c}$$

To exclude $\partial/\partial x$ , $\partial/\partial y$ and $\partial/\partial z$ from Eqs. (6a-c), we replace the body forces with

$$b_x=\frac{\partial b_x^*}{\partial x},\quad b_y=\frac{\partial b_y^*}{\partial y},\quad b_z=\frac{\partial b_z^*}{\partial z}. \tag{7a-c}$$

Substituting Eqs. (7a-c) into Eqs. (6a-c), Eqs. (6a-c) are rewritten into

$$c_{11}\frac{\partial^2\phi}{\partial x^2}+\left[c_{66}+j(c_{12}+c_{66})\right]\frac{\partial^2\phi}{\partial y^2}+\left[c_{55}+k(c_{13}+c_{55})\right]\frac{\partial^2\phi}{\partial z^2}+(c_{12}+c_{66})\frac{\partial^2\psi_1}{\partial y^2}+(c_{13}+c_{55})\frac{\partial^2\psi_2}{\partial z^2}+b_x^*=0, \tag{8a}$$

$$(c_{66}j+c_{12}+c_{66})\frac{\partial^2\phi}{\partial x^2}+c_{22}j\frac{\partial^2\phi}{\partial y^2}+\left[c_{44}j+k(c_{23}+c_{44})\right]\frac{\partial^2\phi}{\partial z^2}+c_{66}\frac{\partial^2\psi_1}{\partial x^2}+c_{22}\frac{\partial^2\psi_1}{\partial y^2}+c_{44}\frac{\partial^2\psi_1}{\partial z^2}+(c_{23}+c_{44}) \times\frac{\partial^2\psi_2}{\partial z^2}+b_y^*=0, \tag{8b}$$

$$(c_{55}k+c_{13}+c_{55})\frac{\partial^2\phi}{\partial x^2}+\left[c_{44}k+j(c_{23}+c_{44})\right]\frac{\partial^2\phi}{\partial y^2}+c_{33}k\frac{\partial^2\phi}{\partial z^2}+c_{55}\frac{\partial^2\psi_2}{\partial x^2}+c_{44}\frac{\partial^2\psi_2}{\partial y^2}+c_{33}\frac{\partial^2\psi_2}{\partial z^2}+(c_{23}+c_{44}) \times\frac{\partial^2\psi_1}{\partial y^2}+b_z^*=0. \tag{8c}$$

Now, we set the coefficients *j* and *k* in Eqs. (8a-c) as

$$j=\frac{c_{11}\mu-c_{66}}{c_{12}+c_{66}},\quad k=\frac{c_{11}\nu-c_{55}}{c_{13}+c_{55}}, \tag{9a,b}$$

where $\mu$ denotes the root of a quadratic equation as determined later, and $\nu$ is a constant to be determined from $\mu$ .
Substituting Eqs. (9a,b) into Eq. (8a), we obtain

$$\frac{\partial^2\phi}{\partial x^2}+\mu\frac{\partial^2\phi}{\partial y^2}+\nu\frac{\partial^2\phi}{\partial z^2}+\frac{c_{12}+c_{66}}{c_{11}}\frac{\partial^2\psi_1}{\partial y^2}+\frac{c_{13}+c_{55}}{c_{11}}\frac{\partial^2\psi_2}{\partial z^2}+\frac{b_x^*}{c_{11}}=0. \tag{10}$$

Equation (8b) is rewritten into

$$\frac{\partial^2\phi}{\partial x^2}+\frac{c_{22}j}{c_{66}j+c_{12}+c_{66}}\frac{\partial^2\phi}{\partial y^2}+\frac{c_{44}j+k(c_{23}+c_{44})}{c_{66}j+c_{12}+c_{66}}\frac{\partial^2\phi}{\partial z^2}+\frac{1}{c_{66}j+c_{12}+c_{66}}\left[c_{66}\frac{\partial^2\psi_1}{\partial x^2}+c_{22}\frac{\partial^2\psi_1}{\partial y^2}+c_{44}\frac{\partial^2\psi_1}{\partial z^2}\right.$$
$$\left.+(c_{23}+c_{44})\frac{\partial^2\psi_2}{\partial z^2}+b_y^*\right]=0. \tag{11}$$

Comparing Eq. (10) with Eq. (11), it is convenient to set $\mu$ and $\nu$ as

$$\frac{c_{22}j}{c_{66}j+c_{12}+c_{66}}=\mu,\quad \frac{c_{44}j+k(c_{23}+c_{44})}{c_{66}j+c_{12}+c_{66}}=\nu. \tag{12a,b}$$

Substituting Eq. (9a) into Eq. (12a), we obtain such a quadratic equation as

$$c_{11}c_{66}\mu^2+\left[c_{12}(c_{12}+2c_{66})-c_{11}c_{22}\right]\mu+c_{22}c_{66}=0. \tag{13}$$

Denoting two roots in Eq. (13) by $\mu_1$ and $\mu_2$ and using the root-coefficient relations, the following expressions hold:

$$\mu_1\mu_2=\frac{c_{22}}{c_{11}},\quad \mu_1+\mu_2=-\frac{1}{c_{11}c_{66}}\left[c_{12}(c_{12}+2c_{66})-c_{11}c_{22}\right]. \tag{14a,b}$$

By making use of the two roots, Eq. (9a) is expressed as

$$j_1=\frac{c_{11}\mu_1-c_{66}}{c_{12}+c_{66}},\quad j_2=\frac{c_{11}\mu_2-c_{66}}{c_{12}+c_{66}}. \tag{15a,b}$$

Substituting Eq. (9b) into Eq. (12b), we obtain

$$\nu=\frac{c_{55}(c_{23}+c_{44})-c_{44}(c_{13}+c_{55})j}{c_{11}(c_{23}+c_{44})-(c_{13}+c_{55})(c_{66}j+c_{12}+c_{66})}. \tag{16}$$

Since $j$ takes either value of $j_1$ or $j_2$, $\nu$ in Eq. (16) takes also either value of $\nu_1$ or $\nu_2$ as

$$\nu_1=\frac{c_{55}(c_{23}+c_{44})-c_{44}(c_{13}+c_{55})j_1}{c_{11}(c_{23}+c_{44})-(c_{13}+c_{55})(c_{66}j_1+c_{12}+c_{66})}, \tag{17a}$$

$$\nu_2=\frac{c_{55}(c_{23}+c_{44})-c_{44}(c_{13}+c_{55})j_2}{c_{11}(c_{23}+c_{44})-(c_{13}+c_{55})(c_{66}j_2+c_{12}+c_{66})}. \tag{17b}$$

Since $j_1$ and $j_2$ are equal to one in the case of isotropic solids, $\nu_1$ and $\nu_2$ in Eqs. (17a,b) yield the following indeterminate form:

$$\nu_1=\nu_2=\frac{0}{0}. \tag{18}$$

However, if we take the limit value of Eq. (18), we obtain $\nu_1=\nu_2=1$ for isotropic solids. By making use of Eqs. (17a,b), Eq. (9b) is expressed as

$$k_1=\frac{c_{11}\nu_1-c_{55}}{c_{13}+c_{55}},\quad k_2=\frac{c_{11}\nu_2-c_{55}}{c_{13}+c_{55}}. \tag{19a,b}$$

Since $\nu_1=\nu_2=1$ for isotropic solids, $k_1$ and $k_2$ in Eqs. (19a,b) are equal to one for isotropic solids. Substituting Eqs. (12a,b) into Eq. (11), we obtain

$$\frac{\partial^2\phi}{\partial x^2}+\mu\frac{\partial^2\phi}{\partial y^2}+\nu\frac{\partial^2\phi}{\partial z^2}+\frac{1}{c_{66}j+c_{12}+c_{66}}\left[c_{66}\frac{\partial^2\psi_1}{\partial x^2}+c_{22}\frac{\partial^2\psi_1}{\partial y^2}+c_{44}\frac{\partial^2\psi_1}{\partial z^2}+(c_{23}+c_{44})\frac{\partial^2\psi_2}{\partial z^2}+b_y^*\right]=0. \tag{20}$$

Substituting Eq. (10) into Eq. (20) and eliminating $\phi$, we obtain

$$c_{66}\left(\frac{\partial^2\psi_1}{\partial x^2}+\frac{c_{22}}{c_{11}\mu}\frac{\partial^2\psi_1}{\partial y^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2\psi_1}{\partial z^2}\right)+\left[c_{23}+c_{44}-\frac{c_{22}j}{c_{11}\mu}(c_{13}+c_{55})\right]\frac{\partial^2\psi_2}{\partial z^2}+b_y^*-\frac{c_{22}j}{c_{11}\mu}b_x^*=0. \tag{21}$$

From Eq. (21), $\psi_2$ is expressed by $\psi_1$ as

$$\frac{\partial^2\psi_2}{\partial z^2}=-\frac{c_{11}\kappa}{c_{13}+c_{55}}\left[\frac{\partial^2\psi_1}{\partial x^2}+\frac{c_{22}}{c_{11}\mu}\frac{\partial^2\psi_1}{\partial y^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2\psi_1}{\partial z^2}+\frac{1}{c_{66}}\left(b_y^*-\frac{c_{22}j}{c_{11}\mu}b_x^*\right)\right], \tag{22}$$

where

$$\kappa=\frac{c_{13}+c_{55}}{c_{11}}\frac{c_{11}c_{66}\mu}{c_{11}\mu(c_{23}+c_{44})-c_{22}j(c_{13}+c_{55})}. \tag{23}$$

Substituting Eq. (22) into Eq. (10) and eliminating $\psi_2$, we obtain

$$\frac{\partial^2\phi}{\partial x^2}+\mu\frac{\partial^2\phi}{\partial y^2}+\nu\frac{\partial^2\phi}{\partial z^2}-\kappa\left[\frac{\partial^2\psi_1}{\partial x^2}+\frac{c_{22}}{c_{11}\mu}\frac{\partial^2\psi_1}{\partial y^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2\psi_1}{\partial z^2}+\frac{1}{c_{66}}\left(b_y^*-\frac{c_{22}j}{c_{11}\mu}b_x^*\right)\right]+\frac{c_{12}+c_{66}}{c_{11}}\frac{\partial^2\psi_1}{\partial y^2}+\frac{b_x^*}{c_{11}}=0. \tag{24}$$

Differentiating Eq. (8c) by $z$ twice and substituting Eq. (22) into the result, we can eliminate $\psi_2$ from Eq. (8c) as

$$\frac{\partial^2}{\partial z^2}\left\{(c_{55}k+c_{13}+c_{55})\frac{\partial^2\phi}{\partial x^2}+\left[c_{44}k+j(c_{23}+c_{44})\right]\frac{\partial^2\phi}{\partial y^2}+c_{33}k\frac{\partial^2\phi}{\partial z^2}+(c_{23}+c_{44})\frac{\partial^2\psi_1}{\partial y^2}\right\}-\frac{c_{11}\kappa}{c_{13}+c_{55}}\left(c_{55}\frac{\partial^2}{\partial x^2}+c_{44}\frac{\partial^2}{\partial y^2}+c_{33}\frac{\partial^2}{\partial z^2}\right)\left[\frac{\partial^2\psi_1}{\partial x^2}+\frac{c_{22}}{c_{11}\mu}\frac{\partial^2\psi_1}{\partial y^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2\psi_1}{\partial z^2}+\frac{1}{c_{66}}\left(b_y^*-\frac{c_{22}j}{c_{11}\mu}b_x^*\right)\right]+\frac{\partial^2 b_z^*}{\partial z^2}=0. \tag{25}$$

From Eq. (24), we have

$$\frac{\partial^2\psi_1}{\partial y^2}=-\frac{c_{11}}{c_{12}+c_{66}}\left\{\frac{\partial^2\phi}{\partial x^2}+\mu\frac{\partial^2\phi}{\partial y^2}+\nu\frac{\partial^2\phi}{\partial z^2}-\kappa\left[\frac{\partial^2\psi_1}{\partial x^2}+\frac{c_{22}}{c_{11}\mu}\frac{\partial^2\psi_1}{\partial y^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2\psi_1}{\partial z^2}+\frac{1}{c_{66}}\left(b_y^*-\frac{c_{22}j}{c_{11}\mu}b_x^*\right)\right]+\frac{b_x^*}{c_{11}}\right\}. \tag{26}$$

Substituting Eq. (26) into Eq. (25) and eliminating $\partial^4\psi_1/\partial z^2\partial y^2$, we obtain

$$\frac{\partial^2}{\partial z^2}\left(a_1\frac{\partial^2\phi}{\partial x^2}+a_2\frac{\partial^2\phi}{\partial y^2}+a_3\frac{\partial^2\phi}{\partial z^2}\right)+\kappa\left(\frac{\partial^2}{\partial x^2}+\frac{c_{44}}{c_{55}}\frac{\partial^2}{\partial y^2}+a_4\frac{\partial^2}{\partial z^2}\right)\left[\frac{\partial^2\psi_1}{\partial x^2}+\frac{c_{22}}{c_{11}\mu}\frac{\partial^2\psi_1}{\partial y^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2\psi_1}{\partial z^2}+\frac{1}{c_{66}}\times\left(b_y^*-\frac{c_{22}j}{c_{11}\mu}b_x^*\right)\right]+\frac{c_{13}+c_{55}}{c_{11}c_{55}}\frac{\partial^2}{\partial z^2}\left(\frac{c_{23}+c_{44}}{c_{12}+c_{66}}b_x^*-b_z^*\right)=0, \tag{27}$$

where

$$a_1=\frac{c_{13}+c_{55}}{c_{55}}\frac{c_{23}+c_{44}}{c_{12}+c_{66}}-\frac{c_{13}(c_{13}+2c_{55})}{c_{11}c_{55}}-\nu,\quad a_2=\frac{c_{66}(c_{13}+c_{55})}{c_{11}c_{55}}\frac{c_{23}+c_{44}}{c_{12}+c_{66}}+\frac{c_{44}}{c_{11}}-\frac{c_{44}}{c_{55}}\nu,$$
$$a_3=\frac{c_{33}}{c_{11}}-\nu\left(\frac{c_{33}}{c_{55}}-\frac{c_{13}+c_{55}}{c_{55}}\frac{c_{23}+c_{44}}{c_{12}+c_{66}}\right),\quad a_4=\frac{c_{33}}{c_{55}}-\frac{c_{13}+c_{55}}{c_{55}}\frac{c_{23}+c_{44}}{c_{12}+c_{66}}. \tag{28a-d}$$

If we decouple Eq. (24) from Eq. (27), we can obtain the solution. We now rewrite Eqs. (24) and (27) by means of differential operators into

$$L_{11}\phi-\kappa\left[L_{12}\psi_1+\frac{1}{c_{66}}\left(b_y^*-\frac{c_{22}j}{c_{11}\mu}b_x^*\right)\right]+\frac{c_{12}+c_{66}}{c_{11}}\frac{\partial^2\psi_1}{\partial y^2}+\frac{b_x^*}{c_{11}}=0, \tag{29a}$$

$$\frac{\partial^2}{\partial z^2}L_{21}\phi+\kappa L_{22}\left[L_{12}\psi_1+\frac{1}{c_{66}}\left(b_y^*-\frac{c_{22}j}{c_{11}\mu}b_x^*\right)\right]+\frac{c_{13}+c_{55}}{c_{11}c_{55}}\frac{\partial^2}{\partial z^2}\left(\frac{c_{23}+c_{44}}{c_{12}+c_{66}}b_x^*-b_z^*\right)=0, \tag{29b}$$

where

$$L_{11}=\frac{\partial^2}{\partial x^2}+\mu\frac{\partial^2}{\partial y^2}+\nu\frac{\partial^2}{\partial z^2},\quad L_{12}=\frac{\partial^2}{\partial x^2}+\frac{c_{22}}{c_{11}\mu}\frac{\partial^2}{\partial y^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2}{\partial z^2}, \tag{30a,b}$$

$$L_{21}=a_1\frac{\partial^2}{\partial x^2}+a_2\frac{\partial^2}{\partial y^2}+a_3\frac{\partial^2}{\partial z^2},\quad L_{22}=\frac{\partial^2}{\partial x^2}+\frac{c_{44}}{c_{55}}\frac{\partial^2}{\partial y^2}+a_4\frac{\partial^2}{\partial z^2}. \tag{30c,d}$$

Multiplying Eq. (29a) by $L_{22}$, we obtain

$$L_{22}L_{11}\phi-\kappa L_{22}\left[L_{12}\psi_1+\frac{1}{c_{66}}\left(b_y^*-\frac{c_{22}j}{c_{11}\mu}b_x^*\right)\right]+\frac{c_{12}+c_{66}}{c_{11}}\frac{\partial^2}{\partial y^2}L_{22}\psi_1+\frac{1}{c_{11}}L_{22}b_x^*=0. \tag{31}$$

Adding Eq. (29b) and Eq. (31), we obtain

$$\frac{\partial^2}{\partial y^2}L_{22}\psi_1=-\frac{c_{11}}{c_{12}+c_{66}}\left[\left(L_{22}L_{11}+\frac{\partial^2}{\partial z^2}L_{21}\right)\phi+\frac{c_{13}+c_{55}}{c_{11}c_{55}}\frac{\partial^2}{\partial z^2}\left(\frac{c_{23}+c_{44}}{c_{12}+c_{66}}b_x^*-b_z^*\right)+\frac{1}{c_{11}}L_{22}b_x^*\right]. \tag{32}$$

Differentiating Eq. (29b) by $y$ twice and substituting Eq. (32) into the result, we can eliminate $\psi_1$ from Eq. (29b) as

$$\left[L_{12}\left(L_{22}L_{11}+\frac{\partial^2}{\partial z^2}L_{21}\right)-\frac{c_{12}+c_{66}}{c_{11}\kappa}\frac{\partial^4}{\partial y^2\partial z^2}L_{21}\right]\phi$$
$$=\frac{c_{12}+c_{66}}{c_{11}c_{66}}\frac{\partial^2}{\partial y^2}L_{22}\left(b_y^*-\frac{c_{22}j}{c_{11}\mu}b_x^*\right)-\frac{c_{13}+c_{55}}{c_{11}c_{55}}\frac{\partial^2}{\partial z^2}\left(L_{12}-\frac{c_{12}+c_{66}}{c_{11}\kappa}\frac{\partial^2}{\partial y^2}\right)\left(\frac{c_{23}+c_{44}}{c_{12}+c_{66}}b_x^*-b_z^*\right)-\frac{1}{c_{11}}$$
$$\times L_{12}L_{22}b_x^*. \tag{33}$$

Equation (33) is the governing equation of $\phi$. If we determine $\phi$ from Eq. (33) and substitute it into Eq. (32), we can obtain $\psi_1$. Thus, the solution with body forces for the orthotropic solid is obtained.

We now consider specializing Eqs. (33) and (32) into isotropic solids. In the case of isotropic solids, the following relations hold:

$$c_{22}=c_{33}=c_{11},\quad c_{13}=c_{23}=c_{12},\quad c_{44}=c_{55}=c_{66}=(c_{11}-c_{12})/2,\quad \mu=j=\nu=k=1,\quad 1/\kappa=0,$$
$$a_1=a_2=a_3=0,\quad a_4=1, \tag{34}$$

then, Eqs. (30a-d) become

$$L_{21}=0,\quad L_{11}=L_{12}=L_{22}=\nabla^2, \tag{35a,b}$$

where

$$\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

By making use of Eqs. (35a,b) and (34), Eqs. (33) and (32) are expressed as

$$\nabla^2\nabla^2\phi=\frac{c_{12}+c_{66}}{c_{11}c_{66}}\left[\frac{\partial^2}{\partial y^2}(b_y^*-b_x^*)-\frac{\partial^2}{\partial z^2}(b_x^*-b_z^*)\right]-\frac{1}{c_{11}}\nabla^2b_x^*$$
$$=\frac{1}{c_{11}(1-2\nu)}\left[\frac{\partial^2}{\partial y^2}(b_y^*-b_x^*)-\frac{\partial^2}{\partial z^2}(b_x^*-b_z^*)\right]-\frac{1}{c_{11}}\nabla^2b_x^*, \tag{36a}$$

$$\nabla^2\psi_1=-\frac{1}{c_{66}}(b_y^*-b_x^*)=-\frac{2(1-\nu)}{c_{11}(1-2\nu)}(b_y^*-b_x^*), \tag{36b}$$

where $\nu$ denotes Poisson's ratio of isotropic solids. Since Eq. (22) does not hold in the case of isotropic solids, we use Eqs. (10) and (8c) and obtain

$$\nabla^2\psi_2=\frac{1}{c_{66}}(b_x^*-b_z^*)=\frac{2(1-\nu)}{c_{11}(1-2\nu)}(b_x^*-b_z^*). \tag{36c}$$

Equations (36a-c) are a particular solution for the body forces in isotropic solids.

## 4. A three-dimensional elasticity solution

Although $\mu$ and $\nu$ in Eqs. (33) and (32) can take either value of $\mu_1$ or $\mu_2$ and of $\nu_1$ or $\nu_2$, respectively, we now set

$$\mu=\mu_1,\quad j=j_1,\quad \nu=\nu_1,\quad k=k_1 \tag{37}$$

and exclude the differential operators from Eqs. (33) and (32), we obtain the solution as the result:

$$u_x=\frac{\partial\phi}{\partial x},\quad u_y=\frac{\partial}{\partial y}(j_1\phi+\psi_1),\quad u_z=\frac{\partial}{\partial z}(k_1\phi+\psi_2), \tag{38a-c}$$

where

$$\begin{aligned}&\left\{\left(\frac{\partial^2}{\partial x^2}+\mu_2\frac{\partial^2}{\partial y^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2}{\partial z^2}\right)\left[\left(\frac{\partial^2}{\partial x^2}+\frac{c_{44}}{c_{55}}\frac{\partial^2}{\partial y^2}+a_4\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial x^2}+\mu_1\frac{\partial^2}{\partial y^2}+\nu_1\frac{\partial^2}{\partial z^2}\right)\right.\right.\\&\left.\left.\quad+\frac{\partial^2}{\partial z^2}\left(a_1\frac{\partial^2}{\partial x^2}+a_2\frac{\partial^2}{\partial y^2}+a_3\frac{\partial^2}{\partial z^2}\right)\right]-\frac{c_{12}+c_{66}}{c_{11}\kappa_1}\frac{\partial^4}{\partial y^2\partial z^2}\left(a_1\frac{\partial^2}{\partial x^2}+a_2\frac{\partial^2}{\partial y^2}+a_3\frac{\partial^2}{\partial z^2}\right)\right\}\phi\\&\quad=\frac{c_{12}+c_{66}}{c_{11}c_{66}}\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2}{\partial x^2}+\frac{c_{44}}{c_{55}}\frac{\partial^2}{\partial y^2}+a_4\frac{\partial^2}{\partial z^2}\right)\left(b_y^*-\mu_2j_1b_x^*\right)-\frac{c_{13}+c_{55}}{c_{11}c_{55}}\frac{\partial^2}{\partial z^2}\left[\frac{\partial^2}{\partial x^2}+\left(\mu_2-\frac{c_{12}+c_{66}}{c_{11}\kappa_1}\right)\right.\\&\quad\left.\times\frac{\partial^2}{\partial y^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2}{\partial z^2}\right]\left(\frac{c_{23}+c_{44}}{c_{12}+c_{66}}b_x^*-b_z^*\right)-\frac{1}{c_{11}}\left(\frac{\partial^2}{\partial x^2}+\mu_2\frac{\partial^2}{\partial y^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2b_x^*}{\partial x^2}+\frac{c_{44}}{c_{55}}\frac{\partial^2b_x^*}{\partial y^2}\right.\\&\quad\left.+a_4\frac{\partial^2b_x^*}{\partial z^2}\right),\end{aligned} \tag{39a}$$

$$\begin{aligned}&\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\psi_1}{\partial x^2}+\frac{c_{44}}{c_{55}}\frac{\partial^2\psi_1}{\partial y^2}+a_4\frac{\partial^2\psi_1}{\partial z^2}\right)\\&\quad=-\frac{c_{11}}{c_{12}+c_{66}}\left\{\left[\left(\frac{\partial^2}{\partial x^2}+\frac{c_{44}}{c_{55}}\frac{\partial^2}{\partial y^2}+a_4\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial x^2}+\mu_1\frac{\partial^2}{\partial y^2}+\nu_1\frac{\partial^2}{\partial z^2}\right)+\frac{\partial^2}{\partial z^2}\left(a_1\frac{\partial^2}{\partial x^2}+a_2\frac{\partial^2}{\partial y^2}\right.\right.\right.\\&\quad\left.\left.\left.+a_3\frac{\partial^2}{\partial z^2}\right)\right]\phi+\frac{c_{13}+c_{55}}{c_{11}c_{55}}\frac{\partial^2}{\partial z^2}\left(\frac{c_{23}+c_{44}}{c_{12}+c_{66}}b_x^*-b_z^*\right)+\frac{1}{c_{11}}\left(\frac{\partial^2b_x^*}{\partial x^2}+\frac{c_{44}}{c_{55}}\frac{\partial^2b_x^*}{\partial y^2}+a_4\frac{\partial^2b_x^*}{\partial z^2}\right)\right\},\end{aligned} \tag{39b}$$

$$\frac{\partial^2\psi_2}{\partial z^2}=-\frac{c_{11}\kappa_1}{c_{13}+c_{55}}\left[\frac{\partial^2\psi_1}{\partial x^2}+\mu_2\frac{\partial^2\psi_1}{\partial y^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2\psi_1}{\partial z^2}+\frac{1}{c_{66}}\left(b_y^*-\mu_2j_1b_x^*\right)\right], \tag{39c}$$

$$\begin{aligned}&a_1=\frac{c_{13}+c_{55}}{c_{55}}\frac{c_{23}+c_{44}}{c_{12}+c_{66}}-\frac{c_{13}(c_{13}+2c_{55})}{c_{11}c_{55}}-\nu_1,\quad a_2=\frac{c_{66}(c_{13}+c_{55})}{c_{11}c_{55}}\frac{c_{23}+c_{44}}{c_{12}+c_{66}}+\frac{c_{44}}{c_{11}}-\frac{c_{44}}{c_{55}}\nu_1,\\&a_3=\frac{c_{33}}{c_{11}}-\nu_1\left(\frac{c_{33}}{c_{55}}-\frac{c_{13}+c_{55}}{c_{55}}\frac{c_{23}+c_{44}}{c_{12}+c_{66}}\right),\quad a_4=\frac{c_{33}}{c_{55}}-\frac{c_{13}+c_{55}}{c_{55}}\frac{c_{23}+c_{44}}{c_{12}+c_{66}},\end{aligned} \tag{40a-d}$$

$$j_1=\frac{c_{11}\mu_1-c_{66}}{c_{12}+c_{66}},\quad k_1=\frac{c_{11}\nu_1-c_{55}}{c_{13}+c_{55}}, \tag{41a,b}$$

$$\begin{aligned}&\nu_1=\frac{c_{55}(c_{23}+c_{44})-c_{44}(c_{13}+c_{55})j_1}{c_{11}(c_{23}+c_{44})-(c_{13}+c_{55})(c_{66}j_1+c_{12}+c_{66})},\\&\kappa_1=\frac{c_{13}+c_{55}}{c_{11}}\frac{c_{11}c_{66}\mu_1}{c_{11}\mu_1(c_{23}+c_{44})-c_{22}j_1(c_{13}+c_{55})},\end{aligned} \tag{42a,b}$$

$$c_{11}c_{66}\mu^2+\left[c_{12}(c_{12}+2c_{66})-c_{11}c_{22}\right]\mu+c_{22}c_{66}=0, \tag{43}$$

$$b_x^*=\int b_x\,dx+f_1(y,z),\quad b_y^*=\int b_y\,dy+f_2(x,z),\quad b_z^*=\int b_z\,dz+f_3(x,y) \tag{44a-c}$$

and $\mu_1$ and $\mu_2$ denote the two roots of Eq. (43). The functions $f_1(y,z)$, $f_2(x,z)$ and $f_3(x,y)$ are arbitrary functions, if they are necessary for the body forces. Equations (38a-c) to (44a-c) are a three-dimensional elasticity solution with the body forces for the orthotropic solid.

## 5. Conclusion

Since we can mechanically apply differential operators to the displacement equilibrium equations for orthotropic solids by the best use of merits in rectangular Cartesian coordinates, we may directly obtain displcacement components by decoupling the equations, with no use of potential functions. However, the solution in this paper was derived by making use of three potential functions and two coefficients ($j$ and $k$) as a direct extension of the derivation process of Elliott's solution for transversely isotropic solids. Although the decoupling of governing equations of potential functions is very difficult in general, it was simply settled by means of the differential operators. The two governing equations of potential functions obtained as the result are the partial differential equations of sixth and fourth orders and can be solved by the separation of variables set up with the products of trigonometric functions and exponential functions. Since the governing equation of $\phi$ includes two roots together, it may be difficult to decompose the equation into some differential equations of fourth or second order. Although the solution presented in this paper is not yet applied to boundary-value problems, the application should be left to future subjects as a problem of stress analysis.

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