On the Stress Field in Orthogonal Curvilinear Coordinates

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Abstract

Stress and strain tensors, the equilibrium equation in terms of the stress tensor and transformation laws for the tensors, in orthogonal curvilinear coordinates, are presented. The components of the tensors are defined as covariant components of second-order tensors, which are identical to stress and strain components. The invariants of stress and strain, and the principal stresses and strains in the orthogonal curvilinear coordinates are obtained from characteristic polynomials and eigenvalues of the tensors. The transformation expressions of stress and strain components among cylindrical, spherical and rectangular Cartesian coordinates are obtained from the transformation laws.

1. Introduction

Studies on two- and three-dimensional elasticity have a long history and have recently developed into studies on the nonlinear theory of elasticity. The linear theory of elasticity seems to be substantially established at the present time, as has been summarized by Gurtin[1]. The linear theory of elasticity in the future will turn to the construction of a simplified theory of the stress field in curvilinear coordinates and anisotropic solids.

Although theories of displacement and stress fields in curvilinear coordinates have long been studied in continuum mechanics, they may be simplified in orthogonal curvilinear coordinates which are a special case of curvilinear coordinates. The usual theories have left the indefiniteness that components of stress and strain tensors in orthogonal curvilinear coordinates do not directly correspond to stress and strain components, unlike components of the Cartesian tensors. This may be caused by the restriction that the equilibrium equation, in terms of the stress tensor in orthogonal curvilinear coordinates, must be formally identical to that in rectangular Cartesian coordinates. If the restriction is excluded, it may be possible to determine new stress and strain tensors in orthogonal curvilinear coordinates and to construct a unified theory of the stress field in orthogonal curvilinear coordinates. The possibility for the new stress and strain tensors is clear from observing that the stress-strain relations expressed by components of stress and strain are identical to those in every coordinate system belonging to orthogonal curvilinear coordinates. Furthermore, the transformation expressions of stress and strain components between orthogonal

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curvilinear coordinate systems demonstrate implicitly the existence of the new stress and strain tensors.

This paper is concerned with stress and strain tensors, the equilibrium equation in terms of the stress tensor and transformation laws for the stress and strain tensors, in orthogonal curvilinear coordinates. The components of the stress and strain tensors are defined as covariant components of second-order tensors, which are identical to stress and strain components. The equilibrium equation expressed by the stress tensor differs from that expressed by the Cartesian tensor, but the stress-strain relations are identical to those expressed by the Cartesian tensors. When orthogonal curvilinear coordinates are specified to rectangular Cartesian coordinates, the equilibrium equation expressed by the stress tensor in orthogonal curvilinear coordinates yields that in rectangular Cartesian coordinates. Furthermore, the invariants of stress and strain, and the principal stresses and strains in orthogonal curvilinear coordinates are obtained from characteristic polynomials and eigenvalues of the stress and strain tensors. The transformation expressions of stress and strain components among cylindrical, spherical and rectangular Cartesian coordinates are obtained from the transformation laws for the stress and strain tensors. The scalar representation of the equilibrium equations and the strain-displacement relations in orthogonal curvilinear coordinates depend on Saada[2], and mathematical formulae and the transformation laws for the stress and strain tensors depend on Iwahori[3].

2. Basic equations in orthogonal curvilinear coordinates

(1) Mathematical formulae[3]

We determine orthogonal curvilinear coordinates (α, β, γ) as shown in Fig.1 and let the basis vectors be $\boldsymbol{e}_\alpha$, $\boldsymbol{e}_\beta$ and $\boldsymbol{e}_\gamma$ in the orthogonal curvilinear coordinates and $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ in rectangular Cartesian coordinates (x, y, z).

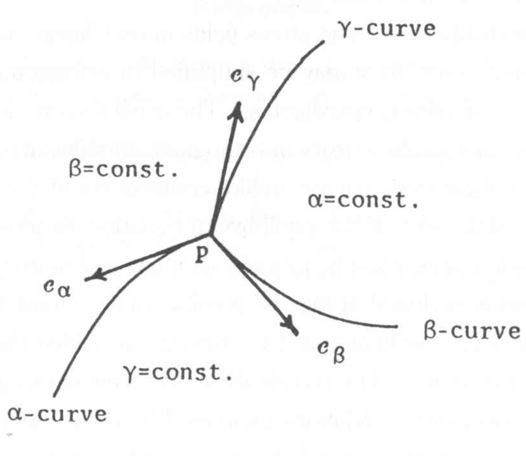

Fig. 1. Orthogonal curvilinear coordinates.

We consider that the two sets of coordinates correspond one to one and that there are the following relationships between them :

$$x = f_1(\alpha, \beta, \gamma),\ y = f_2(\alpha, \beta, \gamma),\ z = f_3(\alpha, \beta, \gamma) \tag{1a−c}$$

$$\alpha = \varphi_1(x, y, z),\ \beta = \varphi_2(x, y, z),\ \gamma = \varphi_3(x, y, z) \tag{2a−c}$$

If we let the first fundamental quantities in the orthogonal curvilinear coordinates be g_1, g_2 and g_3, they are expressed as

$$g_1 = \left[\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2\right]^{\frac{1}{2}},\ g_2 = \left[\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2 + \left(\frac{\partial z}{\partial \beta}\right)^2\right]^{\frac{1}{2}} \tag{3a, b}$$

$$g_3 = \left[\left(\frac{\partial x}{\partial \gamma}\right)^2 + \left(\frac{\partial y}{\partial \gamma}\right)^2 + \left(\frac{\partial z}{\partial \gamma}\right)^2\right]^{\frac{1}{2}} \tag{3c}$$

and the orthogonal conditions in the orthogonal curvilinear coordinates are

$$\frac{\partial x}{\partial \alpha}\frac{\partial x}{\partial \beta} + \frac{\partial y}{\partial \alpha}\frac{\partial y}{\partial \beta} + \frac{\partial z}{\partial \alpha}\frac{\partial z}{\partial \beta} = 0,\ \frac{\partial x}{\partial \beta}\frac{\partial x}{\partial \gamma} + \frac{\partial y}{\partial \beta}\frac{\partial y}{\partial \gamma} + \frac{\partial z}{\partial \beta}\frac{\partial z}{\partial \gamma} = 0 \tag{4a, b}$$

$$\frac{\partial x}{\partial \gamma}\frac{\partial x}{\partial \alpha} + \frac{\partial y}{\partial \gamma}\frac{\partial y}{\partial \alpha} + \frac{\partial z}{\partial \gamma}\frac{\partial z}{\partial \alpha} = 0 \tag{4c}$$

The transformation laws for the basis vectors between both coordinates are as follows :

$$\boldsymbol{e}_\alpha = \frac{1}{g_1}\left(\frac{\partial x}{\partial \alpha}\boldsymbol{e}_x + \frac{\partial y}{\partial \alpha}\boldsymbol{e}_y + \frac{\partial z}{\partial \alpha}\boldsymbol{e}_z\right),\ \boldsymbol{e}_\beta = \frac{1}{g_2}\left(\frac{\partial x}{\partial \beta}\boldsymbol{e}_x + \frac{\partial y}{\partial \beta}\boldsymbol{e}_y + \frac{\partial z}{\partial \beta}\boldsymbol{e}_z\right) \tag{5a, b}$$

$$\boldsymbol{e}_\gamma = \frac{1}{g_3}\left(\frac{\partial x}{\partial \gamma}\boldsymbol{e}_x + \frac{\partial y}{\partial \gamma}\boldsymbol{e}_y + \frac{\partial z}{\partial \gamma}\boldsymbol{e}_z\right) \tag{5c}$$

or

$$\boldsymbol{e}_x = \frac{1}{g_1}\frac{\partial x}{\partial \alpha}\boldsymbol{e}_\alpha + \frac{1}{g_2}\frac{\partial x}{\partial \beta}\boldsymbol{e}_\beta + \frac{1}{g_3}\frac{\partial x}{\partial \gamma}\boldsymbol{e}_\gamma,\ \boldsymbol{e}_y = \frac{1}{g_1}\frac{\partial y}{\partial \alpha}\boldsymbol{e}_\alpha + \frac{1}{g_2}\frac{\partial y}{\partial \beta}\boldsymbol{e}_\beta + \frac{1}{g_3}\frac{\partial y}{\partial \gamma}\boldsymbol{e}_\gamma \tag{6a, b}$$

$$\boldsymbol{e}_z = \frac{1}{g_1}\frac{\partial z}{\partial \alpha}\boldsymbol{e}_\alpha + \frac{1}{g_2}\frac{\partial z}{\partial \beta}\boldsymbol{e}_\beta + \frac{1}{g_3}\frac{\partial z}{\partial \gamma}\boldsymbol{e}_\gamma \tag{6c}$$

If we denote a scalar field by f and define a vector field as

$$\boldsymbol{A} = A_\alpha \boldsymbol{e}_\alpha + A_\beta \boldsymbol{e}_\beta + A_\gamma \boldsymbol{e}_\gamma \tag{7}$$

we obtain the gradient of f, the divergence of $\boldsymbol{A}$, the rotation of $\boldsymbol{A}$ and the Laplacian operator in the form

$$\operatorname{grad} f = \frac{1}{g_1}\frac{\partial f}{\partial \alpha}\boldsymbol{e}_\alpha + \frac{1}{g_2}\frac{\partial f}{\partial \beta}\boldsymbol{e}_\beta + \frac{1}{g_3}\frac{\partial f}{\partial \gamma}\boldsymbol{e}_\gamma \tag{8a}$$

$$\operatorname{div} \boldsymbol{A} = \frac{1}{g_1 g_2 g_3}\left[\frac{\partial}{\partial \alpha}(g_2 g_3 A_\alpha) + \frac{\partial}{\partial \beta}(g_3 g_1 A_\beta) + \frac{\partial}{\partial \gamma}(g_1 g_2 A_\gamma)\right] \tag{8b}$$

$$\begin{aligned}\operatorname{rot} \boldsymbol{A} = {}& \frac{1}{g_2 g_3}\left[\frac{\partial}{\partial \beta}(g_3 A_\gamma) - \frac{\partial}{\partial \gamma}(g_2 A_\beta)\right]\boldsymbol{e}_\alpha + \frac{1}{g_3 g_1}\left[\frac{\partial}{\partial \gamma}(g_1 A_\alpha) - \frac{\partial}{\partial \alpha}(g_3 A_\gamma)\right]\boldsymbol{e}_\beta \\ & + \frac{1}{g_1 g_2}\left[\frac{\partial}{\partial \alpha}(g_2 A_\beta) - \frac{\partial}{\partial \beta}(g_1 A_\alpha)\right]\boldsymbol{e}_\gamma\end{aligned} \tag{8c}$$

$$\nabla^2 \equiv \frac{1}{g_1 g_2 g_3}\left[\frac{\partial}{\partial \alpha}\left(\frac{g_2 g_3}{g_1}\frac{\partial}{\partial \alpha}\right)+\frac{\partial}{\partial \beta}\left(\frac{g_3 g_1}{g_2}\frac{\partial}{\partial \beta}\right)+\frac{\partial}{\partial \gamma}\left(\frac{g_1 g_2}{g_3}\frac{\partial}{\partial \gamma}\right)\right] \tag{8d}$$

(2) Displacement vector, strain and stress tensors

If we let the displacement vector be $\boldsymbol{u}$, it is defined as

$$\boldsymbol{u} = u_\alpha \boldsymbol{e}_\alpha + u_\beta \boldsymbol{e}_\beta + u_\gamma \boldsymbol{e}_\gamma = u_x \boldsymbol{e}_x + u_y \boldsymbol{e}_y + u_z \boldsymbol{e}_z \tag{9}$$

where u_i $(i=\alpha, \beta, \gamma)$ and u_k $(k=x, y, z)$ denote displacement components in the orthogonal curvilinear and rectangular Cartesian coordinates, respectively. The transformation law for the displacement components between both coordinates is as follows :

$$u_\alpha = \frac{1}{g_1}\left(u_x\frac{\partial x}{\partial \alpha}+u_y\frac{\partial y}{\partial \alpha}+u_z\frac{\partial z}{\partial \alpha}\right), \quad u_\beta = \frac{1}{g_2}\left(u_x\frac{\partial x}{\partial \beta}+u_y\frac{\partial y}{\partial \beta}+u_z\frac{\partial z}{\partial \beta}\right) \tag{10a, b}$$

$$u_\gamma = \frac{1}{g_3}\left(u_x\frac{\partial x}{\partial \gamma}+u_y\frac{\partial y}{\partial \gamma}+u_z\frac{\partial z}{\partial \gamma}\right) \tag{10c}$$

or

$$u_x = \frac{u_\alpha}{g_1}\frac{\partial x}{\partial \alpha}+\frac{u_\beta}{g_2}\frac{\partial x}{\partial \beta}+\frac{u_\gamma}{g_3}\frac{\partial x}{\partial \gamma}, \quad u_y = \frac{u_\alpha}{g_1}\frac{\partial y}{\partial \alpha}+\frac{u_\beta}{g_2}\frac{\partial y}{\partial \beta}+\frac{u_\gamma}{g_3}\frac{\partial y}{\partial \gamma} \tag{11a, b}$$

$$u_z = \frac{u_\alpha}{g_1}\frac{\partial z}{\partial \alpha}+\frac{u_\beta}{g_2}\frac{\partial z}{\partial \beta}+\frac{u_\gamma}{g_3}\frac{\partial z}{\partial \gamma} \tag{11c}$$

If we let the strain tensor be $\mathbf{E}$, it is defined as

$$\mathbf{E} = \begin{pmatrix} \varepsilon_{\alpha\alpha} & \varepsilon_{\alpha\beta} & \varepsilon_{\alpha\gamma} \\ \varepsilon_{\beta\alpha} & \varepsilon_{\beta\beta} & \varepsilon_{\beta\gamma} \\ \varepsilon_{\gamma\alpha} & \varepsilon_{\gamma\beta} & \varepsilon_{\gamma\gamma} \end{pmatrix} \tag{12}$$

where

$$\varepsilon_{\gamma\beta} = \varepsilon_{\beta\gamma}, \quad \varepsilon_{\alpha\gamma} = \varepsilon_{\gamma\alpha}, \quad \varepsilon_{\beta\alpha} = \varepsilon_{\alpha\beta} \tag{13a-c}$$

and $\varepsilon_{\alpha\alpha}$, $\varepsilon_{\beta\beta}$ and $\varepsilon_{\gamma\gamma}$ denote normal strain components, and $\varepsilon_{\beta\gamma}$, $\varepsilon_{\gamma\alpha}$ and $\varepsilon_{\alpha\beta}$ denote shearing strain components. They are expressed by the displacement components u_i in the form[2)]

$$\varepsilon_{\alpha\alpha} = \frac{1}{g_1}\frac{\partial u_\alpha}{\partial \alpha}+\frac{u_\beta}{g_1 g_2}\frac{\partial g_1}{\partial \beta}+\frac{u_\gamma}{g_3 g_1}\frac{\partial g_1}{\partial \gamma}, \quad \varepsilon_{\beta\beta} = \frac{1}{g_2}\frac{\partial u_\beta}{\partial \beta}+\frac{u_\gamma}{g_2 g_3}\frac{\partial g_2}{\partial \gamma}+\frac{u_\alpha}{g_1 g_2}\frac{\partial g_2}{\partial \alpha} \tag{14a, b}$$

$$\varepsilon_{\gamma\gamma} = \frac{1}{g_3}\frac{\partial u_\gamma}{\partial \gamma}+\frac{u_\alpha}{g_3 g_1}\frac{\partial g_3}{\partial \alpha}+\frac{u_\beta}{g_2 g_3}\frac{\partial g_3}{\partial \beta} \tag{14c}$$

$$\varepsilon_{\beta\gamma} = \frac{1}{2}\left[\frac{g_2}{g_3}\frac{\partial}{\partial \gamma}\left(\frac{u_\beta}{g_2}\right)+\frac{g_3}{g_2}\frac{\partial}{\partial \beta}\left(\frac{u_\gamma}{g_3}\right)\right], \quad \varepsilon_{\gamma\alpha} = \frac{1}{2}\left[\frac{g_3}{g_1}\frac{\partial}{\partial \alpha}\left(\frac{u_\gamma}{g_3}\right)+\frac{g_1}{g_3}\frac{\partial}{\partial \gamma}\left(\frac{u_\alpha}{g_1}\right)\right] \tag{14d, e}$$

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left[\frac{g_1}{g_2}\frac{\partial}{\partial \beta}\left(\frac{u_\alpha}{g_1}\right)+\frac{g_2}{g_1}\frac{\partial}{\partial \alpha}\left(\frac{u_\beta}{g_2}\right)\right] \tag{14f}$$

If we let the stress tensor be $\mathbf{T}$, it is defined as

$$\mathbf{T} = \begin{pmatrix} \sigma_{\alpha\alpha} & \sigma_{\alpha\beta} & \sigma_{\alpha\gamma} \\ \sigma_{\beta\alpha} & \sigma_{\beta\beta} & \sigma_{\beta\gamma} \\ \sigma_{\gamma\alpha} & \sigma_{\gamma\beta} & \sigma_{\gamma\gamma} \end{pmatrix} \tag{15}$$

where

$$\sigma_{\gamma\beta} = \sigma_{\beta\gamma}, \quad \sigma_{\alpha\gamma} = \sigma_{\gamma\alpha}, \quad \sigma_{\beta\alpha} = \sigma_{\alpha\beta} \tag{16a-c}$$

and $\sigma_{\alpha\alpha}$, $\sigma_{\beta\beta}$ and $\sigma_{\gamma\gamma}$ denote normal stress components, and $\sigma_{\beta\gamma}$, $\sigma_{\gamma\alpha}$ and $\sigma_{\alpha\beta}$ denote shearing stress components. If we use the stress tensor defined in Eq. (15), we obtain the stress vectors as

$$\boldsymbol{t}_\alpha = \sigma_{\alpha\alpha}\boldsymbol{e}_\alpha + \sigma_{\alpha\beta}\boldsymbol{e}_\beta + \sigma_{\alpha\gamma}\boldsymbol{e}_\gamma, \quad \boldsymbol{t}_\beta = \sigma_{\beta\alpha}\boldsymbol{e}_\alpha + \sigma_{\beta\beta}\boldsymbol{e}_\beta + \sigma_{\beta\gamma}\boldsymbol{e}_\gamma \tag{17a, b}$$

$$\boldsymbol{t}_\gamma = \sigma_{\gamma\alpha}\boldsymbol{e}_\alpha + \sigma_{\gamma\beta}\boldsymbol{e}_\beta + \sigma_{\gamma\gamma}\boldsymbol{e}_\gamma \tag{17c}$$

If we use the above stress vectors, we can express the stress tensor as

$$\mathbf{T} = \boldsymbol{n}_\alpha\boldsymbol{t}_\alpha + \boldsymbol{n}_\beta\boldsymbol{t}_\beta + \boldsymbol{n}_\gamma\boldsymbol{t}_\gamma \tag{18}$$

where $\boldsymbol{n}_\alpha$, $\boldsymbol{n}_\beta$ and $\boldsymbol{n}_\gamma$ denote the basis vectors.

(3) Stress-strain relations

The relationships between the stress tensor **T** and the strain tensor **E**, i. e., the stress-strain relations are

$$\mathbf{T} = 2G\left(\mathbf{E} + \frac{\nu}{1-2\nu}\mathbf{I}e\right), \quad \mathbf{E} = \frac{1}{2G}\left(\mathbf{T} - \frac{\nu}{1+\nu}\mathbf{I}\Theta\right) \tag{19a, b}$$

where e and Θ denote the volumetric dilatation and the stress invariant, respectively, which are expressed as

$$e = \mathrm{Tr}(\mathbf{E}) = \varepsilon_{\alpha\alpha} + \varepsilon_{\beta\beta} + \varepsilon_{\gamma\gamma}, \quad \Theta = \mathrm{Tr}(\mathbf{T}) = \sigma_{\alpha\alpha} + \sigma_{\beta\beta} + \sigma_{\gamma\gamma} \tag{20a, b}$$

and G, ν and **I** denote the shear modulus, Poisson's ratio and the unit tensor, respectively. Equations (19a, b) are also written in the components of the stress and strain tensors as

$$\sigma_{ij} = 2G\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}e\delta_{ij}\right), \quad \varepsilon_{ij} = \frac{1}{2G}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\Theta\delta_{ij}\right) \quad (i, j = \alpha, \beta, \gamma) \tag{21a, b}$$

where δ_{ij} denotes the Kronecker delta. By making use of Eqs. (21a, b), the stress-strain relations in the orthogonal curvilinear coordinates are concretely written in the form

$$\sigma_{\alpha\alpha} = 2G\left(\varepsilon_{\alpha\alpha} + \frac{\nu}{1-2\nu}e\right), \quad \sigma_{\beta\beta} = 2G\left(\varepsilon_{\beta\beta} + \frac{\nu}{1-2\nu}e\right), \quad \sigma_{\gamma\gamma} = 2G\left(\varepsilon_{\gamma\gamma} + \frac{\nu}{1-2\nu}e\right) \tag{22a-c}$$

$$\sigma_{\beta\gamma} = 2G\varepsilon_{\beta\gamma}, \sigma_{\gamma\alpha} = 2G\varepsilon_{\gamma\alpha}, \sigma_{\alpha\beta} = 2G\varepsilon_{\alpha\beta} \tag{22d-f}$$

$$\varepsilon_{\alpha\alpha} = \frac{1}{2G}\left(\sigma_{\alpha\alpha} - \frac{\nu}{1+\nu}\Theta\right), \quad \varepsilon_{\beta\beta} = \frac{1}{2G}\left(\sigma_{\beta\beta} - \frac{\nu}{1+\nu}\Theta\right), \quad \varepsilon_{\gamma\gamma} = \frac{1}{2G}\left(\sigma_{\gamma\gamma} - \frac{\nu}{1+\nu}\Theta\right) \tag{23a-c}$$

$$\varepsilon_{\beta\gamma} = \frac{\sigma_{\beta\gamma}}{2G}, \varepsilon_{\gamma\alpha} = \frac{\sigma_{\gamma\alpha}}{2G}, \varepsilon_{\alpha\beta} = \frac{\sigma_{\alpha\beta}}{2G} \tag{23d-f}$$

(4) Equilibrium equations

By making use of the components σ_{ij} $(i, j = \alpha, \beta, \gamma)$ of the stress tensor in Eq.(15), the equilibrium equations in the orthogonal curvilinear coordinates are expressed in the form[2)]

$$\frac{\partial}{\partial\alpha}\left(\sigma_{\alpha\alpha}g_2g_3\right)+\frac{\partial}{\partial\beta}\left(\sigma_{\beta\alpha}g_1g_3\right)+\frac{\partial}{\partial\gamma}\left(\sigma_{\gamma\alpha}g_1g_2\right)+\sigma_{\alpha\beta}g_3\frac{\partial g_1}{\partial\beta}+\sigma_{\alpha\gamma}g_2\frac{\partial g_1}{\partial\gamma}-\sigma_{\beta\beta}g_3\frac{\partial g_2}{\partial\alpha}$$
$$-\sigma_{\gamma\gamma}g_2\frac{\partial g_3}{\partial\alpha}+g_1g_2g_3b_\alpha=0 \tag{24a}$$
$$\frac{\partial}{\partial\alpha}\left(\sigma_{\alpha\beta}g_2g_3\right)+\frac{\partial}{\partial\beta}\left(\sigma_{\beta\beta}g_1g_3\right)+\frac{\partial}{\partial\gamma}\left(\sigma_{\gamma\beta}g_1g_2\right)+\sigma_{\beta\gamma}g_1\frac{\partial g_2}{\partial\gamma}+\sigma_{\beta\alpha}g_3\frac{\partial g_2}{\partial\alpha}-\sigma_{\gamma\gamma}g_1\frac{\partial g_3}{\partial\beta}$$
$$-\sigma_{\alpha\alpha}g_3\frac{\partial g_1}{\partial\beta}+g_1g_2g_3b_\beta=0 \tag{24b}$$
$$\frac{\partial}{\partial\alpha}\left(\sigma_{\alpha\gamma}g_2g_3\right)+\frac{\partial}{\partial\beta}\left(\sigma_{\beta\gamma}g_1g_3\right)+\frac{\partial}{\partial\gamma}\left(\sigma_{\gamma\gamma}g_1g_2\right)+\sigma_{\gamma\alpha}g_2\frac{\partial g_3}{\partial\alpha}+\sigma_{\gamma\beta}g_1\frac{\partial g_3}{\partial\beta}-\sigma_{\alpha\alpha}g_2\frac{\partial g_1}{\partial\gamma}$$
$$-\sigma_{\beta\beta}g_1\frac{\partial g_2}{\partial\gamma}+g_1g_2g_3b_\gamma=0 \tag{24c}$$

where b_α, b_β and b_γ denote body forces per unit volume. If we multiply Eqs. (24a-c) by the basis vectors $\boldsymbol{e}_\alpha$, $\boldsymbol{e}_\beta$ and $\boldsymbol{e}_\gamma$ and sum the results, we obtain

$$\frac{\partial}{\partial\alpha}\left[g_2g_3\left(\sigma_{\alpha\alpha}\boldsymbol{e}_\alpha+\sigma_{\alpha\beta}\boldsymbol{e}_\beta+\sigma_{\alpha\gamma}\boldsymbol{e}_\gamma\right)\right]+\frac{\partial}{\partial\beta}\left[g_1g_3\left(\sigma_{\beta\alpha}\boldsymbol{e}_\alpha+\sigma_{\beta\beta}\boldsymbol{e}_\beta+\sigma_{\beta\gamma}\boldsymbol{e}_\gamma\right)\right]$$
$$+\frac{\partial}{\partial\gamma}\left[g_1g_2\left(\sigma_{\gamma\alpha}\boldsymbol{e}_\alpha+\sigma_{\gamma\beta}\boldsymbol{e}_\beta+\sigma_{\gamma\gamma}\boldsymbol{e}_\gamma\right)\right]+g_1g_2g_3\boldsymbol{b}+\boldsymbol{F}=\boldsymbol{0} \tag{25}$$

where

$$\boldsymbol{F}=\left(\sigma_{\alpha\beta}g_3\frac{\partial g_1}{\partial\beta}+\sigma_{\alpha\gamma}g_2\frac{\partial g_1}{\partial\gamma}-\sigma_{\beta\beta}g_3\frac{\partial g_2}{\partial\alpha}-\sigma_{\gamma\gamma}g_2\frac{\partial g_3}{\partial\alpha}\right)\boldsymbol{e}_\alpha+\left(\sigma_{\beta\alpha}g_3\frac{\partial g_2}{\partial\alpha}+\sigma_{\beta\gamma}g_1\frac{\partial g_2}{\partial\gamma}\right.$$
$$\left.-\sigma_{\gamma\gamma}g_1\frac{\partial g_3}{\partial\beta}-\sigma_{\alpha\alpha}g_3\frac{\partial g_1}{\partial\beta}\right)\boldsymbol{e}_\beta+\left(\sigma_{\gamma\alpha}g_2\frac{\partial g_3}{\partial\alpha}+\sigma_{\gamma\beta}g_1\frac{\partial g_3}{\partial\beta}-\sigma_{\alpha\alpha}g_2\frac{\partial g_1}{\partial\gamma}-\sigma_{\beta\beta}g_1\frac{\partial g_2}{\partial\gamma}\right)\boldsymbol{e}_\gamma \tag{26}$$

and $\boldsymbol{b}$ denotes the body force vector, which is expressed as

$$\boldsymbol{b}=b_\alpha\boldsymbol{e}_\alpha+b_\beta\boldsymbol{e}_\beta+b_\gamma\boldsymbol{e}_\gamma \tag{27}$$

By substituting Eqs. (17a-c) into Eq. (25), we obtain

$$\frac{1}{g_1g_2g_3}\left[\frac{\partial}{\partial\alpha}\left(g_2g_3\boldsymbol{t}_\alpha\right)+\frac{\partial}{\partial\beta}\left(g_1g_3\boldsymbol{t}_\beta\right)+\frac{\partial}{\partial\gamma}\left(g_1g_2\boldsymbol{t}_\gamma\right)\right]+\boldsymbol{b}+\frac{\boldsymbol{F}}{g_1g_2g_3}=\boldsymbol{0} \tag{28}$$

By substituting Eq. (18) into Eq. (28) and making use of Eq. (8b), we obtain

$$\operatorname{div}\mathbf{T}+\boldsymbol{b}+\frac{\boldsymbol{F}}{g_1g_2g_3}=\boldsymbol{0} \tag{29}$$

The vector $\boldsymbol{F}$ in Eq. (26) is changed to the following expression:

$$\boldsymbol{F}=-\left(\sigma_{\alpha\alpha}g_2g_3\,\mathrm{grad}\,g_1+\sigma_{\beta\beta}g_1g_3\,\mathrm{grad}\,g_2+\sigma_{\gamma\gamma}g_2g_1\,\mathrm{grad}\,g_3\right)+\sigma_{\alpha\alpha}\frac{g_2g_3}{g_1}\frac{\partial g_1}{\partial\alpha}\boldsymbol{e}_\alpha$$
$$+\sigma_{\beta\beta}\frac{g_1g_3}{g_2}\frac{\partial g_2}{\partial\beta}\boldsymbol{e}_\beta+\sigma_{\gamma\gamma}\frac{g_1g_2}{g_3}\frac{\partial g_3}{\partial\gamma}\boldsymbol{e}_\gamma+\left(\sigma_{\alpha\beta}g_3\frac{\partial g_1}{\partial\beta}+\sigma_{\alpha\gamma}g_2\frac{\partial g_1}{\partial\gamma}\right)\boldsymbol{e}_\alpha$$
$$+\left(\sigma_{\beta\alpha}g_3\frac{\partial g_2}{\partial\alpha}+\sigma_{\beta\gamma}g_1\frac{\partial g_2}{\partial\gamma}\right)\boldsymbol{e}_\beta+\left(\sigma_{\gamma\alpha}g_2\frac{\partial g_3}{\partial\alpha}+\sigma_{\gamma\beta}g_1\frac{\partial g_3}{\partial\beta}\right)\boldsymbol{e}_\gamma$$
$$=-\left(\boldsymbol{t}_\alpha\cdot\boldsymbol{e}_\alpha g_2g_3\,\mathrm{grad}\,g_1+\boldsymbol{t}_\beta\cdot\boldsymbol{e}_\beta g_1g_3\,\mathrm{grad}\,g_2+\boldsymbol{t}_\gamma\cdot\boldsymbol{e}_\gamma g_2g_1\,\mathrm{grad}\,g_3\right)+g_2g_3\left(\sigma_{\alpha\alpha}\boldsymbol{e}_\alpha+\sigma_{\alpha\beta}\boldsymbol{e}_\beta+\sigma_{\alpha\gamma}\boldsymbol{e}_\gamma\right)$$

$$
\begin{aligned}
&\cdot \operatorname{grad} g_1 \boldsymbol{e}_\alpha + g_1 g_3 \left(\sigma_{\beta\alpha}\boldsymbol{e}_\alpha + \sigma_{\beta\beta}\boldsymbol{e}_\beta + \sigma_{\beta\gamma}\boldsymbol{e}_\gamma\right)\cdot \operatorname{grad} g_2 \boldsymbol{e}_\beta + g_1 g_2 \left(\sigma_{\gamma\alpha}\boldsymbol{e}_\alpha + \sigma_{\gamma\beta}\boldsymbol{e}_\beta + \sigma_{\gamma\gamma}\boldsymbol{e}_\gamma\right)\cdot \operatorname{grad} g_3 \boldsymbol{e}_\gamma \\
&= g_2 g_3 \boldsymbol{t}_\alpha \cdot \left(\operatorname{grad} g_1 \boldsymbol{e}_\alpha - \boldsymbol{e}_\alpha \operatorname{grad} g_1\right) + g_1 g_3 \boldsymbol{t}_\beta \cdot \left(\operatorname{grad} g_2 \boldsymbol{e}_\beta - \boldsymbol{e}_\beta \operatorname{grad} g_2\right) \\
&\quad + g_1 g_2 \boldsymbol{t}_\gamma \cdot \left(\operatorname{grad} g_3 \boldsymbol{e}_\gamma - \boldsymbol{e}_\gamma \operatorname{grad} g_3\right) \\
&= g_2 g_3 \left(\boldsymbol{n}_\alpha \cdot \mathbf{T}\right)\cdot \left(\operatorname{grad} g_1 \boldsymbol{e}_\alpha - \boldsymbol{e}_\alpha \operatorname{grad} g_1\right) + g_1 g_3 \left(\boldsymbol{n}_\beta \cdot \mathbf{T}\right)\cdot \left(\operatorname{grad} g_2 \boldsymbol{e}_\beta - \boldsymbol{e}_\beta \operatorname{grad} g_2\right) \\
&\quad + g_1 g_2 \left(\boldsymbol{n}_\gamma \cdot \mathbf{T}\right)\cdot \left(\operatorname{grad} g_3 \boldsymbol{e}_\gamma - \boldsymbol{e}_\gamma \operatorname{grad} g_3\right)
\end{aligned} \tag{30}
$$

By substituting Eq. (30) into Eq. (29), the equilibrium equation in terms of the stress tensor is obtained in the form

$$
\begin{aligned}
&\operatorname{div}\mathbf{T} + \boldsymbol{b} + \frac{1}{g_1}\left(\boldsymbol{n}_\alpha \cdot \mathbf{T}\right)\cdot \left(\operatorname{grad} g_1 \boldsymbol{e}_\alpha - \boldsymbol{e}_\alpha \operatorname{grad} g_1\right) + \frac{1}{g_2}\left(\boldsymbol{n}_\beta \cdot \mathbf{T}\right)\cdot \left(\operatorname{grad} g_2 \boldsymbol{e}_\beta - \boldsymbol{e}_\beta \operatorname{grad} g_2\right) \\
&\quad + \frac{1}{g_3}\left(\boldsymbol{n}_\gamma \cdot \mathbf{T}\right)\cdot \left(\operatorname{grad} g_3 \boldsymbol{e}_\gamma - \boldsymbol{e}_\gamma \operatorname{grad} g_3\right) = \mathbf{0}
\end{aligned} \tag{31}
$$

where the three dyadics in Eq. (31) become the three alternate tensors, which are expressed as

$$
\operatorname{grad} g_1 \boldsymbol{e}_\alpha - \boldsymbol{e}_\alpha \operatorname{grad} g_1 = \begin{pmatrix} 0 & -\dfrac{1}{g_2}\dfrac{\partial g_1}{\partial \beta} & -\dfrac{1}{g_3}\dfrac{\partial g_1}{\partial \gamma} \\ \dfrac{1}{g_2}\dfrac{\partial g_1}{\partial \beta} & 0 & 0 \\ \dfrac{1}{g_3}\dfrac{\partial g_1}{\partial \gamma} & 0 & 0 \end{pmatrix} \tag{32a}
$$

$$
\operatorname{grad} g_2 \boldsymbol{e}_\beta - \boldsymbol{e}_\beta \operatorname{grad} g_2 = \begin{pmatrix} 0 & \dfrac{1}{g_1}\dfrac{\partial g_2}{\partial \alpha} & 0 \\ -\dfrac{1}{g_1}\dfrac{\partial g_2}{\partial \alpha} & 0 & -\dfrac{1}{g_3}\dfrac{\partial g_2}{\partial \gamma} \\ 0 & \dfrac{1}{g_3}\dfrac{\partial g_2}{\partial \gamma} & 0 \end{pmatrix} \tag{32b}
$$

$$
\operatorname{grad} g_3 \boldsymbol{e}_\gamma - \boldsymbol{e}_\gamma \operatorname{grad} g_3 = \begin{pmatrix} 0 & 0 & \dfrac{1}{g_1}\dfrac{\partial g_3}{\partial \alpha} \\ 0 & 0 & \dfrac{1}{g_2}\dfrac{\partial g_3}{\partial \beta} \\ -\dfrac{1}{g_1}\dfrac{\partial g_3}{\partial \alpha} & -\dfrac{1}{g_2}\dfrac{\partial g_3}{\partial \beta} & 0 \end{pmatrix} \tag{32c}
$$

If we replace α, β and γ in Eqs. (3a-c) with x, y and z, respectively, we obtain

$$
g_1 = g_2 = g_3 = 1 \tag{33}
$$

By substituting Eq. (33) into Eq. (31), we obtain the equilibrium equation in the rectangular Cartesian coordinates in the form

$$
\operatorname{div}\mathbf{T} + \boldsymbol{b} = \mathbf{0} \tag{34}
$$

where $\mathbf{T}$ and $\boldsymbol{b}$ are the stress tensor and the body force vector in the rectangular Cartesian coordinates, respectively, in which α, β and γ in Eqs. (15) and (27) are replaced with x, y and z. In the case of the rectangular Cartesian coordinates, the stress tensor $\mathbf{T}$ becomes the Cartesian tensor.

(5) Transformation of stress and strain tensors[3)]

We consider the rectangular Cartesian coordinates (x, y , z) and the orthogonal curvilinear coordinates (α, β, γ) having the same origin. If we let the components of the stress tensors in both coordinates be σ_{ij} and $\tilde{\sigma}_{ij}$ and the basis vectors be $\boldsymbol{e}_i$ and $\tilde{\boldsymbol{e}}_i$, the components σ_{ij} and $\tilde{\sigma}_{ij}$ are expressed as

$$\sigma_{ij} = \mathbf{T}(\boldsymbol{e}_i, \boldsymbol{e}_j), \quad \tilde{\sigma}_{ij} = \mathbf{T}(\tilde{\boldsymbol{e}}_i, \tilde{\boldsymbol{e}}_j) \\ (i, j = \alpha, \beta, \gamma = x, y, z = 1,2,3) \tag{35a, b}$$

Then, we investigate the relationship between σ_{ij} and $\tilde{\sigma}_{ij}$. If we set

$$\tilde{\boldsymbol{e}}_i = \sum_{j=1}^{3} p_i^j \boldsymbol{e}_j, \quad \boldsymbol{e}_i = \sum_{j=1}^{3} q_i^j \tilde{\boldsymbol{e}}_j \tag{36a, b}$$

and substitute them into Eqs. (35a, b), we obtain

$$\sigma_{ij} = \mathbf{T}\left(\sum_k q_i^k \tilde{\boldsymbol{e}}_k, \sum_l q_j^l \tilde{\boldsymbol{e}}_l\right) = \sum_{k,l} q_i^k q_j^l \tilde{\sigma}_{kl}, \quad \tilde{\sigma}_{ij} = \mathbf{T}\left(\sum_k p_i^k \boldsymbol{e}_k, \sum_l p_j^l \boldsymbol{e}_l\right) = \sum_{k,l} p_i^k p_j^l \sigma_{kl} \tag{37a, b}$$

Namely, the transformation laws for the components of the stress tensors are as follows :

$$\sigma_{ij} = \sum_{k,l=1}^{3} q_i^k q_j^l \tilde{\sigma}_{kl}, \quad \tilde{\sigma}_{ij} = \sum_{k,l=1}^{3} p_i^k p_j^l \sigma_{kl} \tag{38a, b}$$

If we use the following matrices :

$$\boldsymbol{P} = (p_i^j), \quad \boldsymbol{Q} = (q_i^j), \quad \boldsymbol{T}_0 = (\sigma_{ij}), \quad \tilde{\boldsymbol{T}}_0 = (\tilde{\sigma}_{ij}) \tag{39a–d}$$

we can express Eqs. (38a, b) as

$$\boldsymbol{T}_0 = \boldsymbol{Q}^T \tilde{\boldsymbol{T}}_0 \boldsymbol{Q}, \quad \tilde{\boldsymbol{T}}_0 = \boldsymbol{P}^T \boldsymbol{T}_0 \boldsymbol{P} \tag{40a, b}$$

Since, from Eqs. (5a–c) and (6a–c), $\boldsymbol{P} = (p_i^j)$ and $\boldsymbol{Q} = (q_i^j)$ become

$$\boldsymbol{P} = (p_i^j) = \begin{bmatrix} p_1^1 & p_2^1 & p_3^1 \\ p_1^2 & p_2^2 & p_3^2 \\ p_1^3 & p_2^3 & p_3^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{g_1}\frac{\partial x}{\partial \alpha} & \frac{1}{g_2}\frac{\partial x}{\partial \beta} & \frac{1}{g_3}\frac{\partial x}{\partial \gamma} \\ \frac{1}{g_1}\frac{\partial y}{\partial \alpha} & \frac{1}{g_2}\frac{\partial y}{\partial \beta} & \frac{1}{g_3}\frac{\partial y}{\partial \gamma} \\ \frac{1}{g_1}\frac{\partial z}{\partial \alpha} & \frac{1}{g_2}\frac{\partial z}{\partial \beta} & \frac{1}{g_3}\frac{\partial z}{\partial \gamma} \end{bmatrix} \tag{41a}$$

$$\boldsymbol{Q} = (q_i^j) = \begin{bmatrix} q_1^1 & q_2^1 & q_3^1 \\ q_1^2 & q_2^2 & q_3^2 \\ q_1^3 & q_2^3 & q_3^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{g_1}\frac{\partial x}{\partial \alpha} & \frac{1}{g_1}\frac{\partial y}{\partial \alpha} & \frac{1}{g_1}\frac{\partial z}{\partial \alpha} \\ \frac{1}{g_2}\frac{\partial x}{\partial \beta} & \frac{1}{g_2}\frac{\partial y}{\partial \beta} & \frac{1}{g_2}\frac{\partial z}{\partial \beta} \\ \frac{1}{g_3}\frac{\partial x}{\partial \gamma} & \frac{1}{g_3}\frac{\partial y}{\partial \gamma} & \frac{1}{g_3}\frac{\partial z}{\partial \gamma} \end{bmatrix} = \boldsymbol{P}^T = (p_j^i) \tag{41b}$$

the following equation is held from Eqs. (3a–c) and (4a–c) :

$$\boldsymbol{QP} = \boldsymbol{P}^T \boldsymbol{P} = \boldsymbol{I} \tag{42}$$

where $\boldsymbol{I}$ denotes the unit matrix. Equation (42) indicates that $\boldsymbol{P}$ is the orthogonal matrix, and yields

$$\boldsymbol{Q} = \boldsymbol{P}^T = \boldsymbol{P}^{-1} \tag{43}$$

By making use of Eqs. (41b) and (43), Eqs. (38a, b) and (40a, b) are expressed as

$$\sigma_{ij} = \sum_{k,l=1}^{3} p_k^i p_l^j \widetilde{\sigma}_{kl}, \quad \widetilde{\sigma}_{ij} = \sum_{k,l=1}^{3} p_i^k p_j^l \sigma_{kl} \tag{44a,b}$$

$$\boldsymbol{T}_0 = \boldsymbol{P}\widetilde{\boldsymbol{T}}_0\boldsymbol{P}^{-1}, \quad \widetilde{\boldsymbol{T}}_0 = \boldsymbol{P}^{-1}\boldsymbol{T}_0\boldsymbol{P} \tag{45a, b}$$

Equations (44a, b) or (45a, b) are the transformation laws for the stress tensors between the rectangular Cartesian and orthogonal curvilinear coordinates. Since, from Eqs. (45a, b), the relationship of the characteristic polynomial det ($\lambda \boldsymbol{I} - \widetilde{\boldsymbol{T}}_0$) of the matrix $\widetilde{\boldsymbol{T}}_0$ to the characteristic polynomial det ($\lambda \boldsymbol{I} - \boldsymbol{T}_0$) of the matrix $\boldsymbol{T}_0$ is

$$\det\left(\lambda\ \boldsymbol{I} - \widetilde{\boldsymbol{T}}_0\right) = \det \boldsymbol{P} \det\left(\lambda\ \boldsymbol{I} - \widetilde{\boldsymbol{T}}_0\right)\det \boldsymbol{P}^{-1} = \det\left[\boldsymbol{P}\left(\lambda\ \boldsymbol{I} - \widetilde{\boldsymbol{T}}_0\right)\boldsymbol{P}^{-1}\right] = \det\left(\lambda\ \boldsymbol{I} - \boldsymbol{T}_0\right) \tag{46}$$

the following equation is held :

$$\begin{vmatrix} \lambda - \sigma_{\alpha\alpha} & -\sigma_{\alpha\beta} & -\sigma_{\alpha\gamma} \\ -\sigma_{\beta\alpha} & \lambda - \sigma_{\beta\beta} & -\sigma_{\beta\gamma} \\ -\sigma_{\gamma\alpha} & -\sigma_{\gamma\beta} & \lambda - \sigma_{\gamma\gamma} \end{vmatrix} = \begin{vmatrix} \lambda - \sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ -\sigma_{yx} & \lambda - \sigma_{yy} & -\sigma_{yz} \\ -\sigma_{zx} & -\sigma_{zy} & \lambda - \sigma_{zz} \end{vmatrix} \tag{47}$$

Equation (47) gives the characteristic polynomials of the stress tensors $\widetilde{\mathbf{T}}$ and $\mathbf{T}$ whose three roots are called eigenvalues. If we expand both sides of Eq. (47) into algebraic equations, we obtain

$$\begin{aligned} &\lambda^3 - \left(\sigma_{\alpha\alpha} + \sigma_{\beta\beta} + \sigma_{\gamma\gamma}\right)\lambda^2 + \left(\sigma_{\beta\beta}\sigma_{\gamma\gamma} + \sigma_{\gamma\gamma}\sigma_{\alpha\alpha} + \sigma_{\alpha\alpha}\sigma_{\beta\beta} - \sigma_{\beta\gamma}^2 - \sigma_{\gamma\alpha}^2 - \sigma_{\alpha\beta}^2\right)\lambda - \det\widetilde{\sigma}_{ij} \\ &= \lambda^3 - \left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)\lambda^2 + \left(\sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} + \sigma_{xx}\sigma_{yy} - \sigma_{yz}^2 - \sigma_{zx}^2 - \sigma_{xy}^2\right)\lambda - \det\sigma_{ij} \end{aligned} \tag{48}$$

By comparing the coefficients of λ^3, λ^2, λ and 1 on the left-hand side of Eq. (48) with those on the right-hand side, the following stress invariants are obtained :

$$\sigma_{\alpha\alpha} + \sigma_{\beta\beta} + \sigma_{\gamma\gamma} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{49a}$$

$$\sigma_{\beta\beta}\sigma_{\gamma\gamma} + \sigma_{\gamma\gamma}\sigma_{\alpha\alpha} + \sigma_{\alpha\alpha}\sigma_{\beta\beta} - \sigma_{\beta\gamma}^2 - \sigma_{\gamma\alpha}^2 - \sigma_{\alpha\beta}^2 = \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} + \sigma_{xx}\sigma_{yy} - \sigma_{yz}^2 - \sigma_{zx}^2 - \sigma_{xy}^2 \tag{49b}$$

$$\begin{aligned} &\sigma_{\alpha\alpha}\sigma_{\beta\beta}\sigma_{\gamma\gamma} + 2\sigma_{\alpha\beta}\sigma_{\gamma\alpha}\sigma_{\beta\gamma} - \sigma_{\alpha\alpha}\sigma_{\beta\gamma}^2 - \sigma_{\beta\beta}\sigma_{\gamma\alpha}^2 - \sigma_{\gamma\gamma}\sigma_{\alpha\beta}^2 \\ &= \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{zx}\sigma_{yz} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2 \end{aligned} \tag{49c}$$

If we set both sides of Eq. (48) to

$$\begin{aligned} &\lambda^3 - \left(\sigma_{\alpha\alpha} + \sigma_{\beta\beta} + \sigma_{\gamma\gamma}\right)\lambda^2 + \left(\sigma_{\beta\beta}\sigma_{\gamma\gamma} + \sigma_{\gamma\gamma}\sigma_{\alpha\alpha} + \sigma_{\alpha\alpha}\sigma_{\beta\beta} - \sigma_{\beta\gamma}^2 - \sigma_{\gamma\alpha}^2 - \sigma_{\alpha\beta}^2\right)\lambda - \sigma_{\alpha\alpha}\sigma_{\beta\beta}\sigma_{\gamma\gamma} \\ &- 2\sigma_{\alpha\beta}\sigma_{\gamma\alpha}\sigma_{\beta\gamma} + \sigma_{\alpha\alpha}\sigma_{\beta\gamma}^2 + \sigma_{\beta\beta}\sigma_{\gamma\alpha}^2 + \sigma_{\gamma\gamma}\sigma_{\alpha\beta}^2 = 0 \end{aligned} \tag{50}$$

we obtain the principal stresses σ_1, σ_2 and σ_3 in the orthogonal curvilinear coordinates, which are three roots of λ. If we let the components of the strain tensors in the rectangular Cartesian and orthogonal curvilinear coordinates be ε_{ij} and $\widetilde{\varepsilon}_{ij}$, the transformation laws for the strain tensors are identical to those for the stress tensors. Therefore, from Eqs. (44a, b) and (45a, b), we obtain

$$\varepsilon_{ij} = \sum_{k,l=1}^{3} p_k^i p_l^j \widetilde{\varepsilon}_{kl}, \quad \widetilde{\varepsilon}_{ij} = \sum_{k,l=1}^{3} p_i^k p_j^l \varepsilon_{kl} \tag{51a, b}$$

$$\boldsymbol{E}_0 = \boldsymbol{P}\widetilde{\boldsymbol{E}}_0\boldsymbol{P}^{-1}, \quad \widetilde{\boldsymbol{E}}_0 = \boldsymbol{P}^{-1}\boldsymbol{E}_0\boldsymbol{P} \tag{52a, b}$$

If we replace the components of the stress tensors in Eqs. (49a-c) with those of the strain tensors, we can obtain the strain invariants. If we replace the components of the stress tensor in Eq. (50) with those of the strain tensor, we can obtain the principal strains ε_1, ε_2 and ε_3.

If we determine the direction cosines between the orthogonal curvilinear and rectangular

Cartesian coordinates, as given in Table 1,

Table 1. Direction cosines.

	x	y	z
α	l_1	m_1	n_1
β	l_2	m_2	n_2
γ	l_3	m_3	n_3

we obtain the following relationships :

$$\boldsymbol{P} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}, \quad \boldsymbol{P}^{-1} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \tag{53a, b}$$

If we substitute Eqs. (53a, b) into (45a, b) and arrange the results properly, we can obtain the transformation laws for the components of the stress tensors in another form

$$\begin{Bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{\gamma\gamma} \\ \sigma_{\beta\gamma} \\ \sigma_{\gamma\alpha} \\ \sigma_{\alpha\beta} \end{Bmatrix} = \boldsymbol{M} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \boldsymbol{M}^{-1} \begin{Bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{\gamma\gamma} \\ \sigma_{\beta\gamma} \\ \sigma_{\gamma\alpha} \\ \sigma_{\alpha\beta} \end{Bmatrix} \tag{54a, b}$$

where

$$\boldsymbol{M} = \begin{bmatrix} l_1^2 & m_1^2 & n_1^2 & 2m_1n_1 & 2n_1l_1 & 2l_1m_1 \\ l_2^2 & m_2^2 & n_2^2 & 2m_2n_2 & 2n_2l_2 & 2l_2m_2 \\ l_3^2 & m_3^2 & n_3^2 & 2m_3n_3 & 2n_3l_3 & 2l_3m_3 \\ l_2l_3 & m_2m_3 & n_2n_3 & m_2n_3+m_3n_2 & n_2l_3+n_3l_2 & l_2m_3+l_3m_2 \\ l_3l_1 & m_3m_1 & n_3n_1 & m_3n_1+m_1n_3 & n_3l_1+n_1l_3 & l_3m_1+l_1m_3 \\ l_1l_2 & m_1m_2 & n_1n_2 & m_1n_2+m_2n_1 & n_1l_2+n_2l_1 & l_1m_2+l_2m_1 \end{bmatrix} \tag{55a}$$

$$\boldsymbol{M}^{-1} = \begin{bmatrix} l_1^2 & l_2^2 & l_3^2 & 2l_2l_3 & 2l_3l_1 & 2l_1l_2 \\ m_1^2 & m_2^2 & m_3^2 & 2m_2m_3 & 2m_3m_1 & 2m_1m_2 \\ n_1^2 & n_2^2 & n_3^2 & 2n_2n_3 & 2n_3n_1 & 2n_1n_2 \\ m_1n_1 & m_2n_2 & m_3n_3 & m_2n_3+m_3n_2 & m_3n_1+m_1n_3 & m_1n_2+m_2n_1 \\ n_1l_1 & n_2l_2 & n_3l_3 & n_2l_3+n_3l_2 & n_3l_1+n_1l_3 & n_1l_2+n_2l_1 \\ l_1m_1 & l_2m_2 & l_3m_3 & l_2m_3+l_3m_2 & l_3m_1+l_1m_3 & l_1m_2+l_2m_1 \end{bmatrix} \tag{55b}$$

We also obtain the transformation laws for the components of the strain tensors from Eqs. (52a, b) in another form

$$\begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{\gamma\gamma} \\ \varepsilon_{\beta\gamma} \\ \varepsilon_{\gamma\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} = \boldsymbol{M} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{Bmatrix} = \boldsymbol{M}^{-1} \begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{\gamma\gamma} \\ \varepsilon_{\beta\gamma} \\ \varepsilon_{\gamma\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} \tag{56a, b}$$

3. Specified coordinate systems of orthogonal curvilinear coordinates

(1) Cylindrical coordinates

We replace α , β and γ in the orthogonal curvilinear coordinates with r, θ and z, respectively, and determine cylindrical coordinates (r, θ, z) as shown in Fig.2.

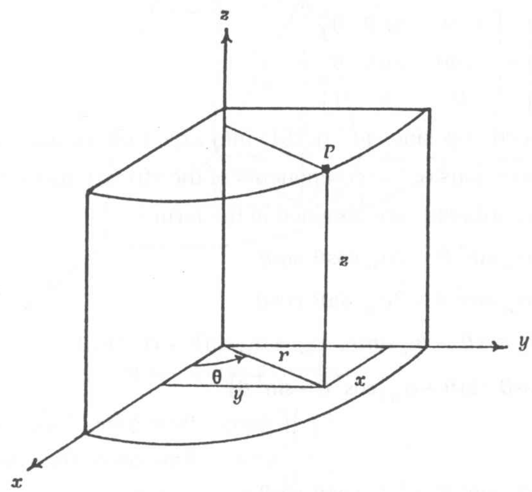

Fig 2. Cylindrical coordinates.

From Fig. 2, the relationships between the rectangular Cartesian and cylindrical coordinates are determined as

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z \tag{57a-c}$$

By substituting Eqs. (57a-c) into Eqs. (3a-c), we obtain

$$g_1 = 1,\quad g_2 = r,\quad g_3 = 1 \tag{58a-c}$$

If we let the basis vectors in the cylindrical coordinates be $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_z$ and substitute Eqs. (57a-c) and (58a-c) into Eqs. (5a-c) or (6a-c), we obtain the transformation laws for the basis vectors between the cylindrical and rectangular Cartesian coordinates in the form

$$\begin{Bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_z \end{Bmatrix} = \boldsymbol{P}^{-1} \begin{Bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{Bmatrix} \tag{59a}$$

or

$$\begin{Bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{Bmatrix} = \boldsymbol{P} \begin{Bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_z \end{Bmatrix} \tag{59b}$$

The stress and strain tensors in the cylindrical coordinates are obtained by replacing α, β and γ in Eqs. (15) and (12) with r, θ and z, respectively. Furthermore, the equilibrium equations are obtained by substituting Eqs. (58a-c) into (31) and making use of the divergence and gradient in the cylindrical coordinates. The results are

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta}+\frac{\partial\sigma_{zr}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+b_r=0,\ \frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\partial\sigma_{z\theta}}{\partial z}+\frac{2\sigma_{r\theta}}{r}+b_\theta=0 \quad (60a,\ b)$$

$$\frac{\partial\sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}+b_z=0 \quad (60c)$$

where b_r, b_θ and b_z denote the components of the body force vector $\boldsymbol{b}$. If we replace α, β and γ in Table 1 with r, θ and z, respectively and use Eq. (59a), we obtain

$$\boldsymbol{P}^{-1}=\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}=\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (61)$$

By substituting the direction cosines in Eq. (61) into Eqs. (55a, b) and making use of Eqs. (54a, b), the transformation expressions of the components of the stress tensors between the cylindrical and rectangular Cartesian coordinates are obtained in the form

$$\sigma_{rr}=\sigma_{xx}\cos^2\theta+\sigma_{yy}\sin^2\theta+2\sigma_{xy}\cos\theta\,\sin\theta \quad (62a)$$

$$\sigma_{\theta\theta}=\sigma_{xx}\sin^2\theta+\sigma_{yy}\cos^2\theta-2\sigma_{xy}\sin\theta\,\cos\theta \quad (62b)$$

$$\sigma_{zz}=\sigma_{zz}\ ,\ \sigma_{\theta z}=\sigma_{yz}\cos\theta-\sigma_{zx}\sin\theta\ ,\ \sigma_{zr}=\sigma_{yz}\sin\theta+\sigma_{zx}\cos\theta \quad (62c-e)$$

$$\sigma_{r\theta}=\left(\sigma_{yy}-\sigma_{xx}\right)\cos\theta\,\sin\theta+\sigma_{xy}\left(\cos^2\theta-\sin^2\theta\right) \quad (62f)$$

or

$$\sigma_{xx}=\sigma_{rr}\cos^2\theta+\sigma_{\theta\theta}\sin^2\theta-2\sigma_{r\theta}\cos\theta\,\sin\theta \quad (63a)$$

$$\sigma_{yy}=\sigma_{rr}\sin^2\theta+\sigma_{\theta\theta}\cos^2\theta+2\sigma_{r\theta}\sin\theta\,\cos\theta \quad (63b)$$

$$\sigma_{zz}=\sigma_{zz}\ ,\ \sigma_{yz}=\sigma_{\theta z}\cos\theta+\sigma_{zr}\sin\theta\ ,\ \sigma_{zx}=-\sigma_{\theta z}\sin\theta+\sigma_{zr}\cos\theta \quad (63c-e)$$

$$\sigma_{xy}=\left(\sigma_{rr}-\sigma_{\theta\theta}\right)\cos\theta\,\sin\theta+\sigma_{r\theta}\left(\cos^2\theta-\sin^2\theta\right) \quad (63f)$$

If we replace the stress components in Eqs. (62a-f) and (63a-f) with the strain components, we can obtain the transformation expressions of the components of the strain tensors between the cylindrical and rectangular Cartesian coordinates.

(2) Spherical coordinates

We replace α, β and γ in the orthogonal curvilinear coordinates with ρ, ϕ and θ, respectively, and determine spherical coordinates (ρ, ϕ, θ) as shown in Fig. 3.
From Fig. 3, the relationships between the rectangular Cartesian and spherical coordinates are determined as

$$x=\rho\,\sin\phi\,\cos\theta\ ,\quad y=\rho\,\sin\phi\,\sin\theta\ ,\ z=\rho\,\cos\phi \quad (64a-c)$$

By substituting Eqs. (64a-c) into Eqs. (3a-c), we obtain

$$g_1=1\ ,\ g_2=\rho\ ,\ g_3=\rho\,\sin\phi \quad (65a-c)$$

If we let the basis vectors in the spherical coordinates be $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_\theta$ and substitute Eqs. (64a-c) and (65a-c) into Eqs. (5a-c) or (6a-c), we obtain the transformation laws for the basis vectors between the spherical and rectangular Cartesian coordinates in the form

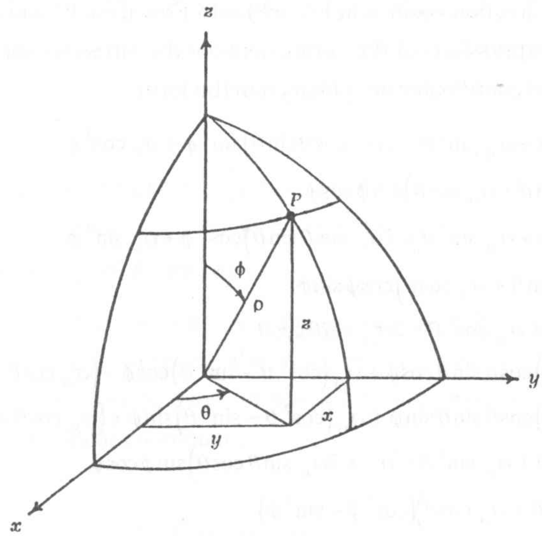

Fig 3. Spherical coordinates.

$$\begin{Bmatrix} e_\rho \\ e_\phi \\ e_\theta \end{Bmatrix} = P^{-1} \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} = \begin{bmatrix} \sin\phi\, \cos\theta & \sin\phi\, \sin\theta & \cos\phi \\ \cos\phi\, \cos\theta & \cos\phi\, \sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} \tag{66a}$$

or

$$\begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} = P \begin{Bmatrix} e_\rho \\ e_\phi \\ e_\theta \end{Bmatrix} = \begin{bmatrix} \sin\phi\, \cos\theta & \cos\phi\, \cos\theta & -\sin\theta \\ \sin\phi\, \sin\theta & \cos\phi\, \sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{Bmatrix} e_\rho \\ e_\phi \\ e_\theta \end{Bmatrix} \tag{66b}$$

The stress and strain tensors in the spherical coordinates are obtained by replacing α, β and γ in Eqs. (15) and (12) with ρ, ϕ and θ, respectively. Furthermore, the equilibrium equations are obtained by substituting Eqs. (65a-c) into (31) and making use of the divergence and gradient in the spherical coordinates. The results are

$$\frac{\partial \sigma_{\rho\rho}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_{\phi\rho}}{\partial \phi} + \frac{1}{\rho \sin\phi}\frac{\partial \sigma_{\theta\rho}}{\partial \theta} + \frac{1}{\rho}\left(2\sigma_{\rho\rho} - \sigma_{\phi\phi} - \sigma_{\theta\theta} + \sigma_{\phi\rho}\cot\phi\right) + b_\rho = 0 \tag{67a}$$

$$\frac{\partial \sigma_{\rho\phi}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{\rho \sin\phi}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{\rho}\left[3\sigma_{\rho\phi} + \left(\sigma_{\phi\phi} - \sigma_{\theta\theta}\right)\cot\phi\right] + b_\phi = 0 \tag{67b}$$

$$\frac{\partial \sigma_{\rho\theta}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{1}{\rho \sin\phi}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{\rho}\left(3\sigma_{\rho\theta} + 2\sigma_{\phi\theta}\cot\phi\right) + b_\theta = 0 \tag{67c}$$

where b_ρ, b_ϕ and b_θ denote the components of the body force vector $\boldsymbol{b}$. If we replace α, β and γ in Table 1 with ρ, ϕ and θ, respectively and use Eq. (66a), we obtain

$$P^{-1} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} = \begin{bmatrix} \sin\phi \cos\theta & \sin\phi \sin\theta & \cos\phi \\ \cos\phi \cos\theta & \cos\phi \sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \tag{68}$$

By substituting the direction cosines in Eq. (68) into Eqs. (55a, b) and making use of Eqs. (54a, b), the transformation expressions of the components of the stress tensors between the spherical and rectangular Cartesian coordinates are obtained in the form

$$\sigma_{\rho\rho} = \left(\sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\cos\theta\sin\theta\right)\sin^2\phi + \sigma_{zz}\cos^2\phi + 2\left(\sigma_{yz}\sin\theta + \sigma_{zx}\cos\theta\right)\sin\phi\cos\phi \tag{69a}$$

$$\sigma_{\phi\phi} = \left(\sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\cos\theta\sin\theta\right)\cos^2\phi + \sigma_{zz}\sin^2\phi - 2\left(\sigma_{yz}\sin\theta + \sigma_{zx}\cos\theta\right)\cos\phi\sin\phi \tag{69b}$$

$$\sigma_{\theta\theta} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta \tag{69c}$$

$$\sigma_{\phi\theta} = \left(\sigma_{yy} - \sigma_{xx}\right)\cos\theta\sin\theta\cos\phi + \sigma_{xy}\left(\cos^2\theta - \sin^2\theta\right)\cos\phi - \left(\sigma_{yz}\cos\theta - \sigma_{zx}\sin\theta\right)\sin\phi \tag{69d}$$

$$\sigma_{\theta\rho} = \left(\sigma_{yy} - \sigma_{xx}\right)\cos\theta\sin\theta\sin\phi + \sigma_{xy}\left(\cos^2\theta - \sin^2\theta\right)\sin\phi + \left(\sigma_{yz}\cos\theta - \sigma_{zx}\sin\theta\right)\cos\phi \tag{69e}$$

$$\sigma_{\rho\phi} = \left(\sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta - \sigma_{zz} + 2\sigma_{xy}\sin\theta\cos\theta\right)\sin\phi\cos\phi + \left(\sigma_{yz}\sin\theta + \sigma_{zx}\cos\theta\right)\left(\cos^2\phi - \sin^2\phi\right) \tag{69f}$$

or

$$\sigma_{xx} = \left(\sigma_{\rho\rho}\sin^2\phi + \sigma_{\phi\phi}\cos^2\phi + 2\sigma_{\rho\phi}\sin\phi\cos\phi\right)\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - 2\left(\sigma_{\phi\theta}\cos\phi + \sigma_{\theta\rho}\sin\phi\right)\cos\theta\sin\theta \tag{70a}$$

$$\sigma_{yy} = \left(\sigma_{\rho\rho}\sin^2\phi + \sigma_{\phi\phi}\cos^2\phi + 2\sigma_{\rho\phi}\sin\phi\cos\phi\right)\sin^2\theta + \sigma_{\theta\theta}\cos^2\theta + 2\left(\sigma_{\phi\theta}\cos\phi + \sigma_{\theta\rho}\sin\phi\right)\cos\theta\sin\theta \tag{70b}$$

$$\sigma_{zz} = \sigma_{\rho\rho}\cos^2\phi + \sigma_{\phi\phi}\sin^2\phi - 2\sigma_{\rho\phi}\cos\phi\sin\phi \tag{70c}$$

$$\sigma_{yz} = \left(\sigma_{\rho\rho} - \sigma_{\phi\phi}\right)\cos\phi\sin\phi\sin\theta + \sigma_{\rho\phi}\left(\cos^2\phi - \sin^2\phi\right)\sin\theta - \left(\sigma_{\phi\theta}\sin\phi - \sigma_{\theta\rho}\cos\phi\right)\cos\theta \tag{70d}$$

$$\sigma_{zx} = \left(\sigma_{\rho\rho} - \sigma_{\phi\phi}\right)\cos\phi\sin\phi\cos\theta + \sigma_{\rho\phi}\left(\cos^2\phi - \sin^2\phi\right)\cos\theta + \left(\sigma_{\phi\theta}\sin\phi - \sigma_{\theta\rho}\cos\phi\right)\sin\theta \tag{70e}$$

$$\sigma_{xy} = \left(\sigma_{\rho\rho}\sin^2\phi + \sigma_{\phi\phi}\cos^2\phi - \sigma_{\theta\theta} + 2\sigma_{\rho\phi}\sin\phi\cos\phi\right)\cos\theta\sin\theta + \left(\sigma_{\phi\theta}\cos\phi + \sigma_{\theta\rho}\sin\phi\right)\left(\cos^2\theta - \sin^2\theta\right) \tag{70f}$$

If we replace the stress components in Eqs. (69a-f) and (70a-f) with the strain components, we can obtain the transformation expressions of the components of the strain tensors between the spherical and rectangular Cartesian coordinates.

If we substitute Eq. (59b) into Eq. (66a), we obtain the transformation laws for the basis vectors between the spherical and cylindrical coordinates in the form

$$\begin{Bmatrix} \boldsymbol{e}_\rho \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{Bmatrix} = \boldsymbol{P}^{-1}\begin{Bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_z \end{Bmatrix} = \begin{bmatrix} \sin\phi & 0 & \cos\phi \\ \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \end{bmatrix}\begin{Bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_z \end{Bmatrix} \tag{71}$$

By replacing α, β and γ with ρ, ϕ and θ, and x, y and z with r, θ and z in Table 1, respectively and making use of Eq. (71), we obtain

$$\boldsymbol{P}^{-1} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} = \begin{bmatrix} \sin\phi & 0 & \cos\phi \\ \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \end{bmatrix} \tag{72}$$

By substituting the direction cosines in Eq. (72) into Eq. (55a) and making use of Eq. (54a) in which α, β and γ are replaced with ρ, ϕ and θ, and x, y and z with r, θ and z, respectively, the transformation expressions of the components of the stress tensors between the spherical and cylindrical coordinates are obtained in the form

$$\sigma_{\rho\rho} = \sigma_{rr}\sin^2\phi + \sigma_{zz}\cos^2\phi + 2\sigma_{zr}\sin\phi\cos\phi\ ,\ \sigma_{\phi\phi} = \sigma_{rr}\cos^2\phi + \sigma_{zz}\sin^2\phi - 2\sigma_{zr}\cos\phi\sin\phi \quad \text{(73a, b)}$$

$$\sigma_{\theta\theta} = \sigma_{\theta\theta}\ ,\ \sigma_{\phi\theta} = \sigma_{r\theta}\cos\phi - \sigma_{\theta z}\sin\phi\ ,\ \sigma_{\theta\rho} = \sigma_{r\theta}\sin\phi + \sigma_{\theta z}\cos\phi \quad \text{(73c−e)}$$

$$\sigma_{\rho\phi} = (\sigma_{rr} - \sigma_{zz})\sin\phi\cos\phi + \sigma_{zr}(\cos^2\phi - \sin^2\phi) \quad \text{(73f)}$$

The transformation expressions of the components of the stress tensors in Eqs. (73a-f) can be also obtained from Eqs. (44b) or (45b). Therefore, the transformation laws for the stress tensors in Eqs. (44a, b), (45a, b) or (54a, b) can be used for those between arbitrary coordinate systems belonging to the orthogonal curvilinear coordinates.

4. Conclusions

The strain and stress tensors in orthogonal curvilinear coordinates were proposed. They were defined as the covariant components of the second-order tensors, which are identical to the stress and strain components. The tensors yielded the same stress-strain relations as those in the Cartesian tensors. However, the equilibrium equation in terms of the stress tensor was expressed in the form in which the three dyadics were added to that in terms of the usual tensor. By making use of the transformation laws for the tensors, the stress and strain invariants, and the principal stresses and strains were defined as the characteristic polynomials and the eigenvalues of the tensors. Furthermore, the transformation expressions of the stress components among rectangular Cartesian, cylindrical and spherical coordinates, which are useful for three-dimensional problems of elasticity, were obtained from the transformation laws for the stress tensors. The stress field in the orthogonal curvilinear coordinates which are a special case of curvilinear coordinates is not so minutely stated in books on solid mechanics. However, the stress and strain tensors presented in this paper systematized considerably the stress field in the orthogonal curvilinear coordinates.

For the reasons mentioned above, the author may conclude that the basic equations of elasticity in this paper should be useful for the linear theory of elasticity.

References

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