A Solution Expressed by Three Potential Functions for Cylindrically Orthotropic Solids with Body Forces

I. OKUMURA*

Abstract

A solution in the general form for cylindrically orthotropic solids with body forces is presented. Three displacement components are expressed by three potential functions. The decoupling of the displacement equilibrium equations expressed by the three potential functions, i.e., the governing equations of the potential functions is discussed. The governing equations of the potential functions obtained as the result are a system of partial differential equations with two unknowns. They are decoupled by means of three relations to nine independent elastic constants.

1. Introduction

With the development of anisotropic and composite materials, the number of studies on anisotropic solids have lately appeared. Although there are various classes of anisotropy, transversely isotropic, orthotropic and cylindrically orthotropic solids have been principally treated. Three-dimensional solutions for cylindrically orthotropic solids differ from those for transversely isotropic and orthotropic solids, which can be obtained in the general form with comparative facility, and have not been obtained in the general and definite form thus far. However, the special and approximate solutions to the displacement equilibrium equations of cylindrically orthotropic solids have already been obtained and have been applied to the analyses of shell structures. Mirsky [1] and Armenakas and Reitz [2] analyzed the propagation of harmonic waves in infinite cylindrical shells. Furthermore, Kardomateas [3,4] analyzed transient thermal stresses in infinite cylindrical shells, and Kim et al. [5] analyzed the buckling of thick cylindrical shells under torsion.

Although there are some studies applying the special solutions to boundary-value problems of shells as mentioned above, very few studies which have solved the displacement equilibrium equations by means of potential functions in the general form without fixing the form of solutions have appeared. As one of the reasons, it is considered that in the case of cylindrically orthotropic solids, the use of potential functions does not so effect on the solution method for the displacement equilibrium equations because of very irregular coefficients

*Department of Civil Engineering, Kitami Institute of Technology.

composed of elastic constants. In other words, it is difficult to decouple the displacement equilibrium equations by means of potential functions in the general form.

This paper is concerned with solutions in the general form for cylindrically orthotropic solids with body forces. Three displacement components are expressed by three potential functions. The decoupling of the displacement equilibrium equations expressed by the three potential functions, i.e., the governing equations of the potential functions is discussed. The governing equations of the potential functions obtained as the result are a system of partial differential equations with two unknowns. They are decoupled by means of three relations to nine independent elastic constants, because the exact decoupling of them is very difficult. Therefore, the solutions presented in this paper are for approximately orthotropic solids. However, they become exact solutions for transversely isotropic and isotropic solids, because the three relations to elastic constants are self-evidently held for such cases.

2. Displacement equilibrium equations

Using cylindrical coordinates (r, θ, z) such that the axes of r, θ and z are taken parallel to the three principal axes of elasticity, the equilibrium equations are

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta}+\frac{\partial \sigma_{zr}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+b_r=0,\\
&\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\partial \sigma_{z\theta}}{\partial z}+\frac{2\sigma_{r\theta}}{r}+b_\theta=0,\\
&\frac{\partial \sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}+b_z=0,
\end{aligned} \tag{1}$$

where σ_{ji} and b_i denote stress components and body forces, respectively. Furthermore, the strain-displacement relations are

$$\begin{aligned}
&\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\ \varepsilon_{\theta\theta}=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta},\ \varepsilon_{zz}=\frac{\partial u_z}{\partial z},\\
&\varepsilon_{\theta z}=\frac{1}{2}\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right),\ \varepsilon_{zr}=\frac{1}{2}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right),\\
&\varepsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right),
\end{aligned} \tag{2}$$

where ε_{ij} and u_i denote strain components and displacement components, respectively. The stress-strain relations of cylindrically orthotropic solids with nine independent elastic constants are expressed as

$$\begin{Bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \sigma_{\theta z}\\ \sigma_{zr}\\ \sigma_{r\theta}\end{Bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&0&0&0\\ c_{12}&c_{22}&c_{23}&0&0&0\\ c_{13}&c_{23}&c_{33}&0&0&0\\ 0&0&0&c_{44}&0&0\\ 0&0&0&0&c_{55}&0\\ 0&0&0&0&0&c_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{zz}\\ 2\varepsilon_{\theta z}\\ 2\varepsilon_{zr}\\ 2\varepsilon_{r\theta}\end{Bmatrix}, \tag{3}$$

where c_{ij} denotes the elastic constants of cylindrically orthotropic solids.

Representing the stress components by the displacement components by making use of Eqs. (2) and (3) and substituting them into Eq. (1), the displacement equilibrium equations of the cylindrically orthotropic solid are obtained in the form

$$\begin{aligned}
&c_{11}\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}\right)+c_{66}\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+c_{55}\frac{\partial^2 u_r}{\partial z^2}-c_{22}\frac{u_r}{r^2}+\left(c_{12}+c_{66}\right)\frac{1}{r}\frac{\partial^2 u_\theta}{\partial r\partial \theta}-\left(c_{22}+c_{66}\right)\frac{1}{r^2}\frac{\partial u_\theta}{\partial \theta}\\
&\quad+\left(c_{13}+c_{55}\right)\frac{\partial^2 u_z}{\partial r\partial z}+\left(c_{13}-c_{23}\right)\frac{1}{r}\frac{\partial u_z}{\partial z}+b_r=0,\\
&c_{66}\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}\right)+c_{22}\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+c_{44}\frac{\partial^2 u_\theta}{\partial z^2}-c_{66}\frac{u_\theta}{r^2}+\left(c_{66}+c_{12}\right)\frac{1}{r}\frac{\partial^2 u_r}{\partial r\partial \theta}+\left(c_{66}+c_{22}\right)\frac{1}{r^2}\frac{\partial u_r}{\partial \theta}\\
&\quad+\left(c_{23}+c_{44}\right)\frac{1}{r}\frac{\partial^2 u_z}{\partial \theta\partial z}+b_\theta=0,\\
&c_{55}\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)+c_{44}\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+c_{33}\frac{\partial^2 u_z}{\partial z^2}+\left(c_{13}+c_{55}\right)\frac{\partial^2 u_r}{\partial r\partial z}+\left(c_{23}+c_{55}\right)\frac{1}{r}\frac{\partial u_r}{\partial z}\\
&\quad+\left(c_{23}+c_{44}\right)\frac{1}{r}\frac{\partial^2 u_\theta}{\partial \theta\partial z}+b_z=0.
\end{aligned} \tag{4}$$

3. Governing equations of potential functions

We now express the displacement components by means of potential functions, i.e., ϕ , ψ_1 and ψ_2 as

$$u_r=\frac{\partial \phi}{\partial r},\ u_\theta=j\frac{1}{r}\frac{\partial \phi}{\partial \theta}+\frac{1}{r}\frac{\partial \psi_1}{\partial \theta},\ u_z=k\frac{\partial \phi}{\partial z}+\frac{\partial \psi_2}{\partial z}, \tag{5}$$

where j and k denote coefficients as determined below. Substituting the displacement components of Eq. (5) into Eq. (4), we consecutively obtain

$$\begin{aligned}
&c_{11}\frac{\partial}{\partial r}\left\{\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{c_{66}+j\left(c_{12}+c_{66}\right)}{c_{11}}\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}+\frac{c_{55}+k\left(c_{13}+c_{55}\right)}{c_{11}}\frac{\partial^2 \phi}{\partial z^2}+\frac{c_{12}+c_{66}}{c_{11}}\frac{1}{r^2}\frac{\partial^2 \psi_1}{\partial \theta^2}\right.\\
&\quad\left.+\frac{c_{13}+c_{55}}{c_{11}}\frac{\partial^2 \psi_2}{\partial z^2}\right\}+\frac{1}{r}\left\{\left(c_{11}-c_{22}\right)\frac{1}{r}\frac{\partial \phi}{\partial r}+\left[2c_{66}+j\left(c_{12}-c_{22}\right)\right]\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}+k\left(c_{13}-c_{23}\right)\frac{\partial^2 \phi}{\partial z^2}\right.\\
&\quad\left.+\left(c_{12}-c_{22}\right)\frac{1}{r^2}\frac{\partial^2 \psi_1}{\partial \theta^2}+\left(c_{13}-c_{23}\right)\frac{\partial^2 \psi_2}{\partial z^2}\right\}+b_r=0,
\end{aligned} \tag{6a}$$

$$\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial \theta}\left\{\left(c_{66}j+c_{66}+c_{12}\right)\frac{\partial^2 \phi}{\partial r^2}+\left(c_{66}j+c_{66}+c_{12}\right)\frac{1}{r}\frac{\partial \phi}{\partial r}+c_{22}j\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}+\left[c_{44}j+k\left(c_{23}+c_{44}\right)\right]\frac{\partial^2 \phi}{\partial z^2}+\left(c_{22}\right.\right.\\
&\quad\left.\left.-c_{12}-2c_{66}j\right)\frac{1}{r}\frac{\partial \phi}{\partial r}+c_{66}\left(\frac{\partial^2 \psi_1}{\partial r^2}-\frac{1}{r}\frac{\partial \psi_1}{\partial r}\right)+c_{22}\frac{1}{r^2}\frac{\partial^2 \psi_1}{\partial \theta^2}+c_{44}\frac{\partial^2 \psi_1}{\partial z^2}+\left(c_{23}+c_{44}\right)\frac{\partial^2 \psi_2}{\partial z^2}\right\}+b_\theta=0,
\end{aligned} \tag{6b}$$

$$\frac{\partial}{\partial z}\left\{(c_{55}k+c_{13}+c_{55})\frac{\partial^2\phi}{\partial r^2}+(c_{55}k+c_{13}+c_{55})\frac{1}{r}\frac{\partial\phi}{\partial r}+\left[c_{44}k+j(c_{44}+c_{23})\right]\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+c_{33}k\frac{\partial^2\phi}{\partial z^2}+(c_{23}-c_{13})\right.$$
$$\left.\times\frac{1}{r}\frac{\partial\phi}{\partial r}+c_{55}\left(\frac{\partial^2\psi_2}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_2}{\partial r}\right)+c_{44}\frac{1}{r^2}\frac{\partial^2\psi_2}{\partial\theta^2}+c_{33}\frac{\partial^2\psi_2}{\partial z^2}+(c_{23}+c_{44})\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}\right\}+b_z=0. \tag{6c}$$

To exclude $(1/r)\,\partial/\partial\theta$ and $\partial/\partial z$ from Eqs. (6b, c), we replace the body forces b_θ and b_z with

$$b_\theta=\frac{1}{r}\frac{\partial b_\theta^*}{\partial\theta},\ b_z=\frac{\partial b_z^*}{\partial z}. \tag{7}$$

Substituting Eq. (7) into Eqs. (6b,c), Eqs. (6b,c) are rewritten into

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{c_{22}j}{c_{66}j+c_{66}+c_{12}}\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{c_{44}j+k(c_{23}+c_{44})}{c_{66}j+c_{66}+c_{12}}\frac{\partial^2\phi}{\partial z^2}+\frac{1}{c_{66}j+c_{66}+c_{12}}\left[(c_{22}-c_{12}-2c_{66}j)\frac{1}{r}\frac{\partial\phi}{\partial r}\right.$$
$$\left.+c_{66}\left(\frac{\partial^2\psi_1}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right)+c_{22}\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}+c_{44}\frac{\partial^2\psi_1}{\partial z^2}+(c_{23}+c_{44})\frac{\partial^2\psi_2}{\partial z^2}+b_\theta^*\right]=0, \tag{8a}$$

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{c_{44}k+j(c_{44}+c_{23})}{c_{55}k+c_{13}+c_{55}}\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{c_{33}k}{c_{55}k+c_{13}+c_{55}}\frac{\partial^2\phi}{\partial z^2}+\frac{1}{c_{55}k+c_{13}+c_{55}}\left[(c_{23}-c_{13})\frac{1}{r}\frac{\partial\phi}{\partial r}\right.$$
$$\left.+c_{55}\left(\frac{\partial^2\psi_2}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_2}{\partial r}\right)+c_{44}\frac{1}{r^2}\frac{\partial^2\psi_2}{\partial\theta^2}+c_{33}\frac{\partial^2\psi_2}{\partial z^2}+(c_{23}+c_{44})\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}+b_z^*\right]=0. \tag{8b}$$

Now, we set the coefficients k and j in Eqs. (6a) and (8a,b) as

$$k=\frac{c_{11}\nu-c_{55}}{c_{13}+c_{55}},\ j=\frac{c_{11}\mu-c_{66}}{c_{12}+c_{66}}, \tag{9a, b}$$

where ν denotes the root of a quadratic equation as determined below, and μ is a constant to be determined from ν. Substituting Eqs. (9a,b) into Eq. (6a), we obtain

$$\frac{\partial}{\partial r}\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\mu\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\nu\frac{\partial^2\phi}{\partial z^2}+\frac{c_{12}+c_{66}}{c_{11}}\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}+\frac{c_{13}+c_{55}}{c_{11}}\frac{\partial^2\psi_2}{\partial z^2}\right)$$
$$+\frac{1}{c_{11}}\frac{1}{r}\left[(c_{11}-c_{22})\frac{1}{r}\frac{\partial\phi}{\partial r}+\left[2c_{66}+j(c_{12}-c_{22})\right]\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+k(c_{13}-c_{23})\frac{\partial^2\phi}{\partial z^2}\right.$$
$$\left.+(c_{12}-c_{22})\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}+(c_{13}-c_{23})\frac{\partial^2\psi_2}{\partial z^2}\right]+\frac{b_r}{c_{11}}=0. \tag{10}$$

Comparing Eq. (10) with Eq. (8b), it is convenient to set ν and μ as

$$\frac{c_{33}k}{c_{55}k+c_{13}+c_{55}}=\nu,\ \frac{c_{44}k+j(c_{44}+c_{23})}{c_{55}k+c_{13}+c_{55}}=\mu. \tag{11a, b}$$

Substituting Eq. (9a) into Eq. (11a), we obtain the following quadratic equation with respect to ν:

$$c_{11}c_{55}\nu^2+\left[c_{13}(c_{13}+2c_{55})-c_{11}c_{33}\right]\nu+c_{33}c_{55}=0. \tag{12}$$

Denoting two roots of Eq. (12) by ν_1 and ν_2 and using the root-coefficient relations, the following expressions hold :

$$\nu_1\nu_2 = \frac{c_{33}}{c_{11}},\ \nu_1+\nu_2 = -\frac{1}{c_{11}c_{55}}\left[c_{13}\left(c_{13}+2c_{55}\right)-c_{11}c_{33}\right]. \tag{13}$$

Using the two roots, Eq. (9a) is expressed as

$$k_1 = \frac{c_{11}\nu_1 - c_{55}}{c_{13}+c_{55}},\ k_2 = \frac{c_{11}\nu_2 - c_{55}}{c_{13}+c_{55}}. \tag{14}$$

Substituting Eq. (9b) into Eq. (11b), we obtain

$$\mu = \frac{c_{66}\left(c_{23}+c_{44}\right)-c_{44}k\left(c_{12}+c_{66}\right)}{c_{11}\left(c_{23}+c_{44}\right)-\left(c_{12}+c_{66}\right)\left(c_{55}k+c_{13}+c_{55}\right)}. \tag{15}$$

Since k takes either value of k_1 or k_2, μ in Eq. (15) takes also either value of μ_1 or μ_2 in the form

$$\mu_1 = \frac{c_{66}\left(c_{23}+c_{44}\right)-c_{44}k_1\left(c_{12}+c_{66}\right)}{c_{11}\left(c_{23}+c_{44}\right)-\left(c_{12}+c_{66}\right)\left(c_{55}k_1+c_{13}+c_{55}\right)},\ \mu_2 = \frac{c_{66}\left(c_{23}+c_{44}\right)-c_{44}k_2\left(c_{12}+c_{66}\right)}{c_{11}\left(c_{23}+c_{44}\right)-\left(c_{12}+c_{66}\right)\left(c_{55}k_2+c_{13}+c_{55}\right)}. \tag{16}$$

Since k_1 and k_2 are equal to one in the case of isotropic solids, μ_1 and μ_2 in Eq. (16) for the case of isotropic solids yield the following indeterminate form:

$$\mu_1 = \mu_2 = \frac{0}{0}. \tag{17}$$

However, we define μ_1 and μ_2 as one for isotropic solids. Using Eq. (16), Eq. (9b) is expressed as

$$j_1 = \frac{c_{11}\mu_1 - c_{66}}{c_{12}+c_{66}},\ j_2 = \frac{c_{11}\mu_2 - c_{66}}{c_{12}+c_{66}}. \tag{18}$$

Since we defined μ_1 and μ_2 as one for isotropic solids, j_1 and j_2 in Eq. (18) are equal to one for isotropic solids. Using Eqs. (9a,b), (11a,b) and

$$\Phi = \frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\mu\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\nu\frac{\partial^2\phi}{\partial z^2}, \tag{19}$$

and rewriting Eqs. (10) and (8a,b), we obtain

$$\begin{aligned}&\frac{\partial}{\partial r}\left(\Phi+\frac{c_{12}+c_{66}}{c_{11}}\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}+\frac{c_{13}+c_{55}}{c_{11}}\frac{\partial^2\psi_2}{\partial z^2}\right)+\frac{1}{c_{11}}\frac{1}{r}\left\{\left(c_{11}-c_{22}\right)\frac{1}{r}\frac{\partial\phi}{\partial r}+\left[2c_{66}+j\left(c_{12}-c_{22}\right)\right]\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right.\\&\left.+k\left(c_{13}-c_{23}\right)\frac{\partial^2\phi}{\partial z^2}+\left(c_{12}-c_{22}\right)\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}+\left(c_{13}-c_{23}\right)\frac{\partial^2\psi_2}{\partial z^2}\right\}+\frac{b_r}{c_{11}}=0,\end{aligned} \tag{20a}$$

$$\begin{aligned}&\Phi+\frac{1}{c_{66}j+c_{66}+c_{12}}\left\{\left[c_{22}j-\mu\left(c_{66}j+c_{66}+c_{12}\right)\right]\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\left[c_{44}j+k\left(c_{23}+c_{44}\right)-\nu\left(c_{66}j+c_{66}+c_{12}\right)\right]\frac{\partial^2\phi}{\partial z^2}\right.\\&+\left(c_{22}-c_{12}-2c_{66}j\right)\frac{1}{r}\frac{\partial\phi}{\partial r}+c_{66}\left(\frac{\partial^2\psi_1}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right)+c_{22}\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}\\&\left.+c_{44}\frac{\partial^2\psi_1}{\partial z^2}+\left(c_{23}+c_{44}\right)\frac{\partial^2\psi_2}{\partial z^2}+b_\theta^*\right\}=0,\end{aligned} \tag{20b}$$

$$\Phi+\frac{1}{c_{55}k+c_{13}+c_{55}}\left[\left(c_{23}-c_{13}\right)\frac{1}{r}\frac{\partial\phi}{\partial r}+c_{55}\left(\frac{\partial^2\psi_2}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_2}{\partial r}\right)+c_{44}\frac{1}{r^2}\frac{\partial^2\psi_2}{\partial\theta^2}\right.$$
$$\left.+c_{33}\frac{\partial^2\psi_2}{\partial z^2}+\left(c_{23}+c_{44}\right)\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}+b_z^*\right]=0. \tag{20c}$$

Finding ψ_2 from Eq. (20 b), we obtain

$$\frac{\partial^2\psi_2}{\partial z^2}=-\frac{c_{66}}{c_{23}+c_{44}}\left\{\frac{c_{66}j+c_{66}+c_{12}}{c_{66}}\Phi+\frac{1}{c_{66}}\left[c_{22}j-\mu\left(c_{66}j+c_{66}+c_{12}\right)\right]\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right.$$
$$\left.+\frac{1}{c_{66}}\left[c_{44}j+k\left(c_{23}+c_{44}\right)-\nu\left(c_{66}j+c_{66}+c_{12}\right)\right]\frac{\partial^2\phi}{\partial z^2}+\frac{c_{22}-c_{12}-2c_{66}j}{c_{66}}\frac{1}{r}\frac{\partial\phi}{\partial r}+\Psi+\frac{b_\theta^*}{c_{66}}\right\}, \tag{21}$$

where

$$\Psi=r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right)+\frac{c_{22}}{c_{66}}\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2\psi_1}{\partial z^2}. \tag{22}$$

Substituting Eq. (21) into Eq. (20a) and eliminating ψ_2, we obtain

$$\left(\frac{\partial}{\partial r}+\frac{c_{13}-c_{23}}{c_{13}+c_{55}}\frac{1}{r}\right)\Psi-\frac{c_{12}+c_{66}}{c_{66}}\frac{c_{23}+c_{44}}{c_{13}+c_{55}}\left(\frac{\partial}{\partial r}+\frac{c_{12}-c_{22}}{c_{12}+c_{66}}\frac{1}{r}\right)\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}=F_1, \tag{23a}$$

where

$$F_1=\frac{c_{11}}{c_{66}}\frac{c_{23}+c_{44}}{c_{13}+c_{55}}\left\{\left[\left(1-\frac{c_{13}+c_{55}}{c_{23}+c_{44}}\frac{c_{66}j+c_{66}+c_{12}}{c_{11}}\right)\frac{\partial}{\partial r}-\frac{c_{13}-c_{23}}{c_{23}+c_{44}}\frac{c_{66}j+c_{66}+c_{12}}{c_{11}}\frac{1}{r}\right]\Phi\right.$$
$$-\left[\frac{c_{13}+c_{55}}{c_{23}+c_{44}}\frac{1}{c_{11}}\left[c_{22}j-\mu\left(c_{66}j+c_{66}+c_{12}\right)\right]\frac{\partial}{\partial r}+\left(\frac{c_{13}-c_{23}}{c_{23}+c_{44}}\frac{1}{c_{11}}\left[c_{22}j-\mu\left(c_{66}j+c_{66}+c_{12}\right)\right]\right.\right.$$
$$\left.\left.-\frac{2c_{66}+j\left(c_{12}-c_{22}\right)}{c_{11}}\right)\frac{1}{r}\right]\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}-\left[\frac{c_{13}+c_{55}}{c_{23}+c_{44}}\frac{1}{c_{11}}\left[c_{44}j+k\left(c_{23}+c_{44}\right)-\nu\left(c_{66}j+c_{66}+c_{12}\right)\right]\frac{\partial}{\partial r}\right.$$
$$\left.+\left(\frac{c_{13}-c_{23}}{c_{23}+c_{44}}\frac{1}{c_{11}}\left[c_{44}j+k\left(c_{23}+c_{44}\right)-\nu\left(c_{66}j+c_{66}+c_{12}\right)\right]-\frac{k\left(c_{13}-c_{23}\right)}{c_{11}}\right)\frac{1}{r}\right]\frac{\partial^2\phi}{\partial z^2}$$
$$\left.-\left[\frac{c_{13}+c_{55}}{c_{23}+c_{44}}\frac{c_{22}-c_{12}-2c_{66}j}{c_{11}}\frac{\partial}{\partial r}+\left(\frac{c_{13}-c_{23}}{c_{23}+c_{44}}\frac{c_{22}-c_{12}-2c_{66}j}{c_{11}}-\frac{c_{11}-c_{22}}{c_{11}}\right)\frac{1}{r}\right]\frac{1}{r}\frac{\partial\phi}{\partial r}\right\}$$
$$-\frac{1}{c_{66}}\left(\frac{\partial b_\theta^*}{\partial r}+\frac{c_{13}-c_{23}}{c_{13}+c_{55}}\frac{b_\theta^*}{r}-\frac{c_{23}+c_{44}}{c_{13}+c_{55}}b_r\right). \tag{24}$$

Differentiating Eq. (20c) by z twice and substituting (21) into the result, we can eliminate ψ_2 from Eq. (20c) as

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{c_{44}}{c_{55}}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{c_{33}}{c_{55}}\frac{\partial^2}{\partial z^2}\right)\Psi-\frac{\left(c_{23}+c_{44}\right)^2}{c_{55}c_{66}}\frac{1}{r^2}\frac{\partial^4\psi_1}{\partial\theta^2\partial z^2}=F_2', \tag{23b}$$

where

$$F_2'=-\left[\frac{c_{66}j+c_{66}+c_{12}}{c_{66}}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{c_{44}}{c_{55}}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{c_{33}}{c_{55}}\frac{\partial^2}{\partial z^2}\right)-\frac{c_{23}+c_{44}}{c_{55}c_{66}}\left(c_{55}k+c_{13}+c_{55}\right)\frac{\partial^2}{\partial z^2}\right]\Phi$$

$$
-\left[\frac{c_{22}-c_{12}-2c_{66}j}{c_{66}}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{c_{44}}{c_{55}}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{c_{33}}{c_{55}}\frac{\partial^2}{\partial z^2}\right)-\frac{c_{23}+c_{44}}{c_{55}c_{66}}\left(c_{23}-c_{13}\right)\frac{\partial^2}{\partial z^2}\right]\frac{1}{r}\frac{\partial \phi}{\partial r}
$$

$$
-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{c_{44}}{c_{55}}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{c_{33}}{c_{55}}\frac{\partial^2}{\partial z^2}\right)\left\{\frac{1}{c_{66}}\left[c_{22}j-\mu\left(c_{66}j+c_{66}+c_{12}\right)\right]\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right.
$$

$$
\left.+\frac{1}{c_{66}}\left[c_{44}j+k\left(c_{23}+c_{44}\right)-\nu\left(c_{66}j+c_{66}+c_{12}\right)\right]\frac{\partial^2 \phi}{\partial z^2}\right\}-\frac{1}{c_{66}}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{c_{44}}{c_{55}}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right.
$$

$$
\left.+\frac{c_{33}}{c_{55}}\frac{\partial^2}{\partial z^2}\right)b_\theta^*+\frac{c_{23}+c_{44}}{c_{55}c_{66}}\frac{\partial^2 b_z^*}{\partial z^2}. \tag{25}
$$

Modifying Eqs. (23a,b), we obtain

$$
L_1\Psi'-\alpha\beta\frac{1}{r^3}\frac{\partial^2\psi_1}{\partial\theta^2}=F_1,\ L_2\Psi'+\alpha L_2\left(\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}\right)+\gamma\frac{\partial^2\Psi'}{\partial z^2}=F_2, \tag{26a, b}
$$

where

$$
\alpha=\frac{c_{12}+c_{66}}{c_{66}}\frac{c_{23}+c_{44}}{c_{13}+c_{55}},\ \beta=\frac{c_{12}-c_{22}}{c_{12}+c_{66}}-\frac{c_{13}-c_{23}}{c_{13}+c_{55}},\ \gamma=\frac{c_{13}+c_{55}}{c_{12}+c_{66}}\frac{c_{23}+c_{44}}{c_{55}}, \tag{27}
$$

$$
\Psi'=r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right)+\left(\frac{c_{22}}{c_{66}}-\alpha\right)\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2\psi_1}{\partial z^2}, \tag{28}
$$

$$
L_1=\frac{\partial}{\partial r}+\frac{c_{13}-c_{23}}{c_{13}+c_{55}}\frac{1}{r},\ L_2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{c_{44}}{c_{55}}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\left(\frac{c_{33}}{c_{55}}-\gamma\right)\frac{\partial^2}{\partial z^2}, \tag{29a, b}
$$

$$
F_2=-\left\{\frac{c_{66}j+c_{66}+c_{12}}{c_{66}}L_2+\frac{c_{23}+c_{44}}{c_{55}c_{66}}\left[\frac{c_{13}+c_{55}}{c_{12}+c_{66}}\left(c_{66}j+c_{66}+c_{12}\right)-\left(c_{55}k+c_{13}+c_{55}\right)\right]\frac{\partial^2}{\partial z^2}\right\}\Phi
$$

$$
-\left\{\frac{c_{22}-c_{12}-2c_{66}j}{c_{66}}L_2+\frac{c_{23}+c_{44}}{c_{55}c_{66}}\left[\frac{c_{13}+c_{55}}{c_{12}+c_{66}}\left(c_{22}-c_{12}-2c_{66}j\right)-\left(c_{23}-c_{13}\right)\right]\frac{\partial^2}{\partial z^2}\right\}\frac{1}{r}\frac{\partial\phi}{\partial r}
$$

$$
-\left(L_2+\gamma\frac{\partial^2}{\partial z^2}\right)\left\{\frac{1}{c_{66}}\left[c_{22}j-\mu\left(c_{66}j+c_{66}+c_{12}\right)\right]\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{1}{c_{66}}\left[c_{44}j+k\left(c_{23}+c_{44}\right)\right.\right.
$$

$$
\left.\left.-\nu\left(c_{66}j+c_{66}+c_{12}\right)\right]\frac{\partial^2\phi}{\partial z^2}\right\}-\frac{1}{c_{66}}\left[\left(L_2+\gamma\frac{\partial^2}{\partial z^2}\right)b_\theta^*-\frac{c_{23}+c_{44}}{c_{55}}\frac{\partial^2 b_z^*}{\partial z^2}\right]. \tag{30}
$$

From Eq. (26a), we have

$$
\frac{\alpha}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}=\frac{1}{\beta}\left(rL_1\Psi'-rF_1\right). \tag{31}
$$

Substituting Eq. (31) into Eq. (26b) and eliminating $(\alpha/r^2)\ \partial^2\psi_1/\partial\theta^2$, Eq. (26b) yields

$$
\left(2+\beta+L_4\right)L_2\Psi'-\left[\frac{2c_{33}}{c_{55}}-\gamma\left(2+\beta\right)\right]\frac{\partial^2\Psi'}{\partial z^2}=\beta F_2+L_2\left(rF_1\right), \tag{32}
$$

where

$$
L_4=rL_1=r\frac{\partial}{\partial r}+\frac{c_{13}-c_{23}}{c_{13}+c_{55}}. \tag{33}
$$

To eliminate $(1/r^3)\ \partial^2\psi_1/\partial\theta^2$ from Eq. (26a), we use the following means:

Rewriting Eq. (31), we obtain

$$\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}-\frac{1}{\alpha\beta}\left(r\frac{\partial}{\partial r}+\frac{c_{13}-c_{23}}{c_{13}+c_{55}}\right)\Psi'=-\frac{1}{\alpha\beta}(rF_1). \tag{34}$$

Differentiating Eq. (34) by r and dividing the result by r, we obtain

$$\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(-\frac{2}{r^2}\psi_1+\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right)-\frac{1}{\alpha\beta}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{c_{13}-c_{23}}{c_{13}+c_{55}}\frac{1}{r}\frac{\partial}{\partial r}\right)\Psi'=-\frac{1}{\alpha\beta}\frac{1}{r}\frac{\partial}{\partial r}(rF_1).$$

Differentiating the above equation by r and multiplying the result by r, we obtain

$$\frac{4}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\frac{2}{r^2}\psi_1-\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right)\right]-\frac{1}{\alpha\beta}\left[\left(r\frac{\partial}{\partial r}+\frac{c_{13}-c_{23}}{c_{13}+c_{55}}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\right.$$
$$\left.-\frac{2(c_{13}-c_{23})}{c_{13}+c_{55}}\frac{1}{r}\frac{\partial}{\partial r}\right]\Psi'=-\frac{1}{\alpha\beta}r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rF_1)\right]. \tag{a}$$

Differentiating Eq. (34) by θ twice and multiplying the result by $(c_{22}/c_{66}-\alpha)/r^2$, we obtain

$$\left(\frac{c_{22}}{c_{66}}-\alpha\right)\frac{1}{r^4}\frac{\partial^4\psi_1}{\partial\theta^4}-\frac{1}{\alpha\beta}\left(\frac{c_{22}}{c_{66}}-\alpha\right)\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(r\frac{\partial}{\partial r}+\frac{c_{13}-c_{23}}{c_{13}+c_{55}}\right)\Psi'=-\frac{1}{\alpha\beta}\left(\frac{c_{22}}{c_{66}}-\alpha\right)\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(rF_1). \tag{b}$$

Differentiating Eq. (34) by z twice and multiplying the result by c_{44}/c_{66}, we obtain

$$\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\frac{c_{44}}{c_{66}}\frac{\partial^2\psi_1}{\partial z^2}\right)-\frac{1}{\alpha\beta}\frac{c_{44}}{c_{66}}\frac{\partial^2}{\partial z^2}\left(r\frac{\partial}{\partial r}+\frac{c_{13}-c_{23}}{c_{13}+c_{55}}\right)\Psi'=-\frac{1}{\alpha\beta}\frac{c_{44}}{c_{66}}\frac{\partial^2}{\partial z^2}(rF_1). \tag{c}$$

Adding Eqs. (a), (b) and (c) and using Eq. (34), we obtain

$$(4+L_4)L_3\Psi'-\frac{4c_{44}}{c_{66}}\frac{\partial^2\Psi'}{\partial z^2}+\frac{2(c_{13}-c_{23})}{c_{13}+c_{55}}\frac{1}{r}\frac{\partial\Psi'}{\partial r}-\left[2\left(\frac{c_{22}}{c_{66}}-\alpha\right)+\alpha\beta\right]\frac{1}{r^2}\frac{\partial^2\Psi'}{\partial\theta^2}=\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1), \tag{35}$$

where

$$L_3=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\left(\frac{c_{22}}{c_{66}}-\alpha\right)\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{c_{44}}{c_{66}}\frac{\partial^2}{\partial z^2}. \tag{36}$$

Equations (32) and (35) constitute a system of partial differential equations with Ψ' and ϕ. However, it seems to be very difficult to decouple Eqs. (32) from (35).

4. Solutions for approximately orthotropic, transversely isotropic and isotropic solids

4.1 Approximately orthotropic solids

Changing L_2 in Eq. (32) into L_3 and rewriting Eqs. (35) and (32), we obtain

$$(4+L_4)L_3\Psi'-\frac{4c_{44}}{c_{66}}\frac{\partial^2\Psi'}{\partial z^2}-\left[2\left(\frac{c_{22}}{c_{66}}-\alpha\right)+\alpha\beta\right]\frac{1}{r^2}\frac{\partial^2\Psi'}{\partial\theta^2}+\frac{2(c_{13}-c_{23})}{c_{13}+c_{55}}\frac{1}{r}\frac{\partial\Psi'}{\partial r}$$
$$=\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1), \tag{37a}$$

$$(2+\beta+L_4)L_3\Psi' - \left[\left(\frac{c_{44}}{c_{66}} - \frac{c_{33}}{c_{55}} + \gamma\right)(2+\beta+L_4) + \frac{2c_{33}}{c_{55}} - \gamma(2+\beta)\right]\frac{\partial^2\Psi'}{\partial z^2}$$
$$-\left(\frac{c_{22}}{c_{66}} - \alpha - \frac{c_{44}}{c_{55}}\right)(2+\beta+L_4)\frac{1}{r^2}\frac{\partial^2\Psi'}{\partial\theta^2} = \beta F_2 + L_2(rF_1). \tag{37b}$$

Now, we set up the following relations to the elastic constants:

$$2\left(\frac{c_{22}}{c_{66}} - \alpha\right) + \alpha\beta = 0, \quad \frac{2(c_{13} - c_{23})}{c_{13} + c_{55}} = 0, \quad \frac{c_{22}}{c_{66}} - \alpha - \frac{c_{44}}{c_{55}} = 0. \tag{38a–c}$$

From Eq. (38b), we have

$$c_{23} = c_{13}. \tag{39a}$$

Using Eq. (39a), the coefficients α, β and γ in Eq. (27) are expressed as

$$\alpha = \frac{c_{12}+c_{66}}{c_{66}}\frac{c_{13}+c_{44}}{c_{13}+c_{55}}, \quad \beta = \frac{c_{12}-c_{22}}{c_{12}+c_{66}}, \quad \gamma = \frac{c_{13}+c_{55}}{c_{12}+c_{66}}\frac{c_{13}+c_{44}}{c_{55}}. \tag{40}$$

Using Eqs. (38a) and (40), we obtain

$$c_{55} = \frac{1}{2c_{22}}\left[c_{44}(c_{12}+2c_{66}+c_{22}) - c_{13}(c_{22}-c_{12}-2c_{66})\right]. \tag{39b}$$

Using Eqs. (38c), (40) and (39b), we obtain

$$(c_{22}-c_{12}-2c_{66})\left[(c_{12}+2c_{66}+c_{22})c_{44}^2 + 2c_{13}(c_{12}+c_{66})c_{44} - c_{13}^2(c_{22}-c_{12})\right] = 0. \tag{39c}$$

Since three elastic constants were replaced with other elastic constants in Eqs. (39a-c), nine independent elastic constants of the orthotropic solid decrease into six independent elastic constants. We call such anisotropic solids with six independent elastic constants approximately orthotropic solids here.

Using Eqs. (39a-c), Eqs. (37a,b) can be rewritten into

$$(4+L_4)L_3\Psi' - \frac{4c_{44}}{c_{66}}\frac{\partial^2\Psi'}{\partial z^2} = \left(\frac{2}{r}\frac{\partial}{\partial r} + L_3\right)(rF_1), \tag{41a}$$

$$(2+\beta+L_4)L_3\Psi' - (a_1 + a_2L_4)\frac{\partial^2\Psi'}{\partial z^2} = \beta F_2 + L_2(rF_1), \tag{41b}$$

where

$$a_1 = \frac{2c_{44}}{c_{66}} + \beta\left(\frac{c_{44}}{c_{66}} - \frac{c_{33}}{c_{55}}\right), \qquad a_2 = \frac{c_{44}}{c_{66}} - \frac{c_{33}}{c_{55}} + \gamma, \tag{42}$$

$$L_4 = r\frac{\partial}{\partial r}, \qquad L_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{c_{44}}{c_{55}}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \left(\frac{c_{33}}{c_{55}} - \gamma\right)\frac{\partial^2}{\partial z^2}, \tag{43a, b}$$

$$L_3 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{c_{44}}{c_{55}}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{c_{44}}{c_{66}}\frac{\partial^2}{\partial z^2}, \tag{43c}$$

$$\Psi' = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right) + \frac{c_{44}}{c_{55}}\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2} + \frac{c_{44}}{c_{66}}\frac{\partial^2\psi_1}{\partial z^2}, \tag{44}$$

$$F_1=\frac{c_{11}}{c_{66}}\frac{c_{13}+c_{44}}{c_{13}+c_{55}}\left\{\left(1-\frac{c_{13}+c_{55}}{c_{13}+c_{44}}\frac{c_{66}j+c_{66}+c_{12}}{c_{11}}\right)\frac{\partial\Phi}{\partial r}-\left(\frac{c_{13}+c_{55}}{c_{13}+c_{44}}\frac{c_{22}j-\mu(c_{66}j+c_{66}+c_{12})}{c_{11}}\frac{\partial}{\partial r}\right.\right.$$
$$\left.-\frac{2c_{66}+j(c_{12}-c_{22})}{c_{11}}\frac{1}{r}\right)\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}-\frac{c_{13}+c_{55}}{c_{13}+c_{44}}\frac{c_{44}j+k(c_{13}+c_{44})-\nu(c_{66}j+c_{66}+c_{12})}{c_{11}}\frac{\partial^3\phi}{\partial r\partial z^2}$$
$$\left.-\left(\frac{c_{13}+c_{55}}{c_{13}+c_{44}}\frac{c_{22}-c_{12}-2c_{66}j}{c_{11}}\frac{\partial}{\partial r}-\frac{c_{11}-c_{22}}{c_{11}}\frac{1}{r}\right)\frac{1}{r}\frac{\partial\phi}{\partial r}-\frac{1}{c_{11}}\left(\frac{c_{13}+c_{55}}{c_{13}+c_{44}}\frac{\partial b_\theta^*}{\partial r}-b_r\right)\right\},\tag{45a}$$

$$F_2=-\left\{\frac{c_{66}j+c_{66}+c_{12}}{c_{66}}L_2+\frac{c_{13}+c_{44}}{c_{55}c_{66}}\left[\frac{c_{13}+c_{55}}{c_{12}+c_{66}}(c_{66}j+c_{66}+c_{12})-(c_{55}k+c_{13}+c_{55})\right]\frac{\partial^2}{\partial z^2}\right\}\Phi$$
$$-\left[\frac{c_{22}-c_{12}-2c_{66}j}{c_{66}}L_2+\frac{c_{13}+c_{44}}{c_{55}c_{66}}\frac{c_{13}+c_{55}}{c_{12}+c_{66}}(c_{22}-c_{12}-2c_{66}j)\frac{\partial^2}{\partial z^2}\right]\frac{1}{r}\frac{\partial\phi}{\partial r}-\left(L_2+\gamma\frac{\partial^2}{\partial z^2}\right)$$
$$\times\left[\frac{c_{22}j-\mu(c_{66}j+c_{66}+c_{12})}{c_{66}}\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{c_{44}j+k(c_{13}+c_{44})-\nu(c_{66}j+c_{66}+c_{12})}{c_{66}}\frac{\partial^2\phi}{\partial z^2}+\frac{b_\theta^*}{c_{66}}\right]$$
$$+\frac{c_{13}+c_{44}}{c_{55}c_{66}}\frac{\partial^2 b_z^*}{\partial z^2}.\tag{45b}$$

If we decouple Ψ' from ϕ in Eqs. (41a,b), we can find a three-dimensional elasticity solution for the approximately orthotropic solid. Multiplying Eq. (41a) by $(a_1+a_2L_4)$, we obtain

$$(a_1+a_2L_4)(4+L_4)L_3\Psi'-\frac{4c_{44}}{c_{66}}(a_1+a_2L_4)\frac{\partial^2\Psi'}{\partial z^2}=(a_1+a_2L_4)\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1).\tag{d}$$

Multiplying Eq. (41b) by $4c_{44}/c_{66}$, we obtain

$$\frac{4c_{44}}{c_{66}}(2+\beta+L_4)L_3\Psi'-\frac{4c_{44}}{c_{66}}(a_1+a_2L_4)\frac{\partial^2\Psi'}{\partial z^2}=\frac{4c_{44}}{c_{66}}\left[\beta F_2+L_2(rF_1)\right].\tag{e}$$

Subtracting Eq. (e) from Eq. (d), we obtain

$$\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]L_3\Psi'$$
$$=(a_1+a_2L_4)\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1)-\frac{4c_{44}}{c_{66}}\left[\beta F_2+L_2(rF_1)\right].\tag{46}$$

Multiplying Eq. (41a) by $[(a_1+a_2L_4)(4+L_4)-4(c_{44}/c_{66})(2+\beta+L_4)]$, we obtain

$$\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right](4+L_4)L_3\Psi'$$
$$-\frac{4c_{44}}{c_{66}}\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]\frac{\partial^2\Psi'}{\partial z^2}$$
$$=\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1).\tag{f}$$

Multiplying Eq. (46) by $(4+L_4)$, we obtain

$$(4+L_4)\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]L_3\Psi'$$

$$=(4+L_4)\left\{(a_1+a_2L_4)\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1)-\frac{4c_{44}}{c_{66}}\left[\beta F_2+L_2(rF_1)\right]\right\}. \tag{g}$$

Since the following expression holds:

$$(4+L_4)\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]=\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right](4+L_4),$$

subtracting Eq. (g) from Eq. (f), we obtain

$$\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]\frac{\partial^2\Psi'}{\partial z^2}$$
$$=(2+\beta+L_4)\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1)-(4+L_4)\left[\beta F_2+L_2(rF_1)\right]. \tag{47}$$

Multiplying Eq. (47) by L_3, we obtain

$$L_3\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]\frac{\partial^2\Psi'}{\partial z^2}$$
$$=L_3\left\{(2+\beta+L_4)\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1)-(4+L_4)\left[\beta F_2+L_2(rF_1)\right]\right\}. \tag{h}$$

Rewriting Eq. (h) using the following expression :

$$L_3L_4=L_4L_3+2\left(L_3-\frac{c_{44}}{c_{66}}\frac{\partial^2}{\partial z^2}\right),$$

we obtain

$$\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]L_3\frac{\partial^2\Psi'}{\partial z^2}+2\left[a_1-\frac{4c_{44}}{c_{66}}+2a_2(3+L_4)\right]\left(L_3-\frac{c_{44}}{c_{66}}\frac{\partial^2}{\partial z^2}\right)\frac{\partial^2\Psi'}{\partial z^2}$$
$$=L_3\left\{(2+\beta+L_4)\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1)-(4+L_4)\left[\beta F_2+L_2(rF_1)\right]\right\}. \tag{i}$$

Differentiating Eq. (46) by z twice, we obtain

$$\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]L_3\frac{\partial^2\Psi'}{\partial z^2}$$
$$=\frac{\partial^2}{\partial z^2}\left\{(a_1+a_2L_4)\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1)-\frac{4c_{44}}{c_{66}}\left[\beta F_2+L_2(rF_1)\right]\right\}. \tag{j}$$

Subtracting Eq. (j) from Eq. (i), we obtain

$$2\left[a_1-\frac{4c_{44}}{c_{66}}+2a_2(3+L_4)\right]\left(L_3-\frac{c_{44}}{c_{66}}\frac{\partial^2}{\partial z^2}\right)\frac{\partial^2\Psi'}{\partial z^2}$$
$$=L_3\left\{(2+\beta+L_4)\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1)-(4+L_4)\left[\beta F_2+L_2(rF_1)\right]\right\}$$
$$-\frac{\partial^2}{\partial z^2}\left\{(a_1+a_2L_4)\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1)-\frac{4c_{44}}{c_{66}}\left[\beta F_2+L_2(rF_1)\right]\right\}. \tag{48}$$

Subtracting Eq. (47) multiplied by c_{44}/c_{66} from Eq. (46) and differentiating the result by z twice, we obtain

$$\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]\left(L_3-\frac{c_{44}}{c_{66}}\frac{\partial^2}{\partial z^2}\right)\frac{\partial^2\Psi'}{\partial z^2}$$

$$=\frac{\partial^2}{\partial z^2}\left\{\left[a_1+a_2L_4-\frac{c_{44}}{c_{66}}(2+\beta+L_4)\right]\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1)+\frac{c_{44}}{c_{66}}L_4\left[\beta F_2+L_2(rF_1)\right]\right\}. \tag{49}$$

Subtracting Eq. (49) multiplied by $2[a_1-4c_{44}/c_{66}+2a_2(3+L_4)]$ from Eq. (48) multiplied by $[(a_1+a_2L_4)(4+L_4)-4(c_{44}/c_{66})(2+\beta+L_4)]$, we obtain

$$\begin{aligned}&\left\langle\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]L_3(2+\beta+L_4)\right.\\&-\frac{\partial^2}{\partial z^2}\left\{(a_1+a_2L_4)\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]\right.\\&\left.\left.+2\left[a_1-\frac{4c_{44}}{c_{66}}+2a_2(3+L_4)\right]\left[a_1+a_2L_4-\frac{c_{44}}{c_{66}}(2+\beta+L_4)\right]\right\}\right\rangle\left(\frac{2}{r}\frac{\partial}{\partial r}+L_3\right)(rF_1)\\&-\left\langle\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]L_3(4+L_4)\right.\\&-\frac{2c_{44}}{c_{66}}\frac{\partial^2}{\partial z^2}\left\{2\left[(a_1+a_2L_4)(4+L_4)-\frac{4c_{44}}{c_{66}}(2+\beta+L_4)\right]\right.\\&\left.\left.-L_4\left[a_1-\frac{4c_{44}}{c_{66}}+2a_2(3+L_4)\right]\right\}\right\rangle\left[\beta F_2+L_2(rF_1)\right]=0.\end{aligned} \tag{50}$$

Equation (50) is the differential equation only for ϕ and is a three-dimensional elasticity solution for the approximately orthotropic solid. If ϕ is obtained from Eq. (50), the function Ψ' can be obtained by substituting it into the right-hand side of Eq. (41a).

4. 2 Transversely isotropic solids

In the case of transversely isotropic solids, there are the following relations to the elastic constants:

$$c_{22}=c_{11},\ c_{66}=(c_{11}-c_{12})/2,\ c_{23}=c_{13},\ c_{55}=c_{44}, \tag{51}$$

then, the relations of Eqs. (39a-c) self-evidently hold. Therefore, Eq. (50) is the exact solution for the transversely isotropic solid. Equation (40) and Eqs. (42) to (45b) for the transversely isotropic solid are as follows:

$$\beta=-\frac{2(c_{11}-c_{12})}{c_{11}+c_{12}},\ \gamma=\frac{2(c_{13}+c_{44})^2}{c_{44}(c_{11}+c_{12})}, \tag{52}$$

$$a_1=\frac{2}{c_{11}+c_{12}}\left[\frac{4c_{12}c_{44}}{c_{11}-c_{12}}+\frac{c_{33}(c_{11}-c_{12})}{c_{44}}\right],\quad a_2=\frac{2c_{44}}{c_{11}-c_{12}}-\frac{c_{33}}{c_{44}}+\frac{2(c_{13}+c_{44})^2}{c_{44}(c_{11}+c_{12})}, \tag{53a, b}$$

$$\frac{c_{33}}{c_{44}}-\gamma=\frac{c_{33}}{c_{44}}-\frac{2(c_{13}+c_{44})^2}{c_{44}(c_{11}+c_{12})}, \tag{53c}$$

$$L_4=r\frac{\partial}{\partial r},\quad L_2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\left[\frac{c_{33}}{c_{44}}-\frac{2(c_{13}+c_{44})^2}{c_{44}(c_{11}+c_{12})}\right]\frac{\partial^2}{\partial z^2}, \tag{54a, b}$$

$$L_3 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{2c_{44}}{c_{11} - c_{12}}\frac{\partial^2}{\partial z^2}, \tag{54c}$$

$$\Psi' = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi_1}{\partial \theta^2} + \frac{2c_{44}}{c_{11} - c_{12}}\frac{\partial^2 \psi_1}{\partial z^2}, \quad \Phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \nu\frac{\partial^2 \phi}{\partial z^2}, \tag{55a, b}$$

$$F_1 = -\frac{2}{c_{11} - c_{12}}\left(\frac{\partial b_\theta^*}{\partial r} - b_r\right), \tag{56a}$$

$$F_2 = -\frac{2c_{11}}{c_{11} - c_{12}}\left\{\left[L_2 + \frac{c_{13} + c_{44}}{c_{44}}\left(\frac{2(c_{13} + c_{44})}{c_{11} + c_{12}} - \frac{c_{44}k + c_{13} + c_{44}}{c_{11}}\right)\frac{\partial^2}{\partial z^2}\right]\Phi \right.$$
$$\left. + \frac{1}{c_{11}}\left[\left(L_2 + \frac{2(c_{13} + c_{44})^2}{c_{44}(c_{11} + c_{12})}\frac{\partial^2}{\partial z^2}\right)b_\theta^* - \frac{c_{13} + c_{44}}{c_{44}}\frac{\partial^2 b_z^*}{\partial z^2}\right]\right\}, \tag{56b}$$

$$\mu = j = 1. \tag{57}$$

The governing equations of the potential functions are shown in Eqs. (50) and (41a).

4. 3 Isotropic solids

In the case of isotropic solids, there are the following relations to the elastic constants:

$$c_{23} = c_{13} = c_{12} = \frac{c_{11}\nu}{1 - \nu}, \quad c_{22} = c_{33} = c_{11}, \quad c_{55} = c_{44} = c_{66} = \frac{c_{11}(1 - 2\nu)}{2(1 - \nu)},$$
$$\beta = -2(1 - 2\nu), \quad \gamma = \frac{1}{1 - 2\nu} = -\frac{2}{\beta}, \tag{58}$$

where ν denotes Poisson's ratio of isotropic solids. From Eq. (58), we have

$$a_1 = 4, \quad a_2 = 0, \quad c_{33}/c_{44} - \gamma = 1, \quad \nu = \mu = k = j = 1. \tag{59}$$

Furthermore, we obtain the following expressions from Eqs. (58) and (59) :

$$L_4 = r\frac{\partial}{\partial r}, \quad L_2 = L_3 = \nabla^2, \tag{60}$$

$$\Psi' = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi_1}{\partial \theta^2} + \frac{\partial^2 \psi_1}{\partial z^2}, \quad \Phi = \nabla^2 \phi, \tag{61}$$

$$F_1 = -\frac{2}{c_{11} - c_{12}}\left(\frac{\partial b_\theta^*}{\partial r} - b_r\right), \quad F_2 = -\frac{2c_{11}}{c_{11} - c_{12}}\left\{\nabla^2 \Phi + \frac{1}{c_{11}}\left[\left(\nabla^2 - \frac{2}{\beta}\frac{\partial^2}{\partial z^2}\right)b_\theta^* + \frac{2}{\beta}\frac{\partial^2 b_z^*}{\partial z^2}\right]\right\}, \tag{62a, b}$$

where ∇^2 denotes the Laplacian operator in the cylindrical coordinates. Using Eqs. (58) to (62b), Eqs. (50) and (41a) yield

$$\left[\nabla^2\left(4 + r\frac{\partial}{\partial r}\right) - 4\frac{\partial^2}{\partial z^2}\right]\nabla^2 \Phi$$
$$= \frac{1}{c_{11}\beta}\left\{\nabla^2\left[2\left(2 + \beta + r\frac{\partial}{\partial r}\right)\frac{1}{r}\frac{\partial}{\partial r} - (2 - \beta)\nabla^2\right] - \frac{8}{r}\frac{\partial^3}{\partial r \partial z^2}\right\}\left[r\left(\frac{\partial b_\theta^*}{\partial r} - b_r\right)\right]$$

$$-\frac{1}{c_{11}}\left[\nabla^2\left(4+r\frac{\partial}{\partial r}\right)-4\frac{\partial^2}{\partial z^2}\right]\left[\left(\nabla^2-\frac{2}{\beta}\frac{\partial^2}{\partial z^2}\right)b_\theta^*+\frac{2}{\beta}\frac{\partial^2 b_z^*}{\partial z^2}\right], \tag{63a}$$

$$\left[\left(4+r\frac{\partial}{\partial r}\right)\nabla^2-4\frac{\partial^2}{\partial z^2}\right]\Psi'=-\frac{2(1-\nu)}{c_{11}(1-2\nu)}\left(\frac{2}{r}\frac{\partial}{\partial r}+\nabla^2\right)\left[r\left(\frac{\partial b_\theta^*}{\partial r}-b_r\right)\right]. \tag{63b}$$

Equations (63a,b) are three-dimensional elasticity solutions for the isotropic solid.

5. Conclusions

In the case of cylindrically orthotropic solids, the coefficients composed of the elastic constants in the displacement equilibrium equations are very irregular. Therefore, even though the potential functions and the coefficients (j and k) based on the roots of the quadratic equation are used, the governing equations of the potential functions do not become simple. The governing equations (32) and (35) which were obtained as the result were very difficult for exactly decoupling and were approximately solved by means of the three relations to the elastic constants. Although the solution obtained by this means is correspondent to that for anisotropic solids with six independent elastic constants, it is exact for transversely isotropic and isotropic solids. The governing equation (50) of ϕ for the approximately orthotropic solid is a very complicated partial differential equation of eleventh order. However, if the governing equation can be separated into Euler's form in an ordinary differential equation only for r by the separation of variables, it may be simply solved. The solutions presented in this paper give the guidelines on general solution methods for cylindrically orthotropic and transversely isotropic solids with body forces and should be useful for anisotropic solids from the standpoint that general solution methods for cylindrically orthotropic solids by means of the potential functions are very few and difficult.

References

1 Mirsky, I. : Vibrations of orthotropic, thick, cylindrical shells. J. Acous. Soc. America 36 (1964), 41-51.

2 Armenakas, A. E. ; Reitz, E. S. : Propagation of harmonic waves in orthotropic circular cylindrical shells. J. Appl. Mech. 40 (1973), 168-174.

3 Kardomateas, G. A. : Transient thermal stresses in cylindrically orthotropic composite tubes. J. Appl. Mech. 56 (1989), 411-417.

4 Kardomateas, G. A. : The initial phase of transient ther2al stresses due to general boundary thermal loads in orthotropic hollow cylinders. J. Appl. Mech. 57 (1990), 719-724.

5 Kim, Y. S. ; Kardomateas, G. A. ; Zureick, A. : Buckling of thick orthotropic cylindrical shells under torsion. J. Appl. Mech. 66 (1999), 41-50.