On Neuber's and Galerkin's Solutions Taking Heat and the Curl of a Harmonic Vector into Account

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Abstract

Neuber's and Galerkin's solutions taking heat and the curl of a harmonic vector into account are derived from the Navier equation with the temperature field by means of the vector calculus. The solutions have one more vector potentials than those in Neuber's and Galerkin's solutions. When heat and body forces are neglected, and the curl of a harmonic vector is eliminated, the solutions are in agreement with Neuber's and Galerkin's solutions. The process of that is described in detail. Muki's and Love's solutions in cylindrical coordinates are extended to the case of the existence of heat and one body force, by making use of one of the solutions.

1. Introduction

Three-dimensional solutions of elasticity in isotropic solids have been found by many researchers to date. Among the various solutions, Boussinesq's [1], Galerkin's [2], Papkovich's [3] and Neuber's [4] solutions are considered to be typical solutions. Although Boussinesq's solution is insufficient for general, three-dimensional problems of elasticity, it is sufficient for axially symmetric problems in spherical coordinates. Other solutions can be applied to some three-dimensional problems of elasticity in every coordinate system belonging to orthogonal curvilinear coordinates, because they are expressed in the displacement vector obtained from the direct integration of the Navier equation, by means of potential functions called the displacement potentials. However, it takes place that they are not successfully applicable to three-dimensional problems of elasticity because of too many or too few displacement potentials according to any coordinate system. This is clear from the fact that the number of boundary conditions in boundary-value problems of finite solids depending on the use of rectangular Cartesian, cylindrical, spherical and so on coordinates differs from one another. From paying attention to this, Hata [5, 6] derived the generalized Neuber and generalized Galerkin solutions in which the curl of a harmonic vector is added to Neuber's and Galerkin's solutions, as a general solution to boundary-value problems in rectangular Cartesian coordinates. However, the solutions do not take heat and body forces into account. Mindlin [7] described the induction process of Galerkin's solution in the existence of body

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forces and Papkovich's solution in the absence of body forces from the Navier equation.

This paper is concerned with the induction of Neuber's and Galerkin's solutions by the direct integration of the Navier equation taking heat into account. The solutions presented are expressed in the form like the solutions given by Hata [5, 6], in which the curl of a harmonic vector is separated from the Neuber potentials and the Galerkin vector. Therefore, the vector potentials in the solutions are one more than those in Neuber's and Galerkin's solutions. When heat and body forces are neglected, and the curl of a harmonic vector is eliminated, the solutions are in agreement with Neuber's and Galerkin's solutions. The solutions are simply derived by means of the vector calculus using formulae for a scalar field and a vector field. By making use of one of the solutions, Muki's [8] and love's [9] solutions to axially asymmetric and symmetric problems of elasticity in cylindrical coordinates, respectively, are extended to the case of the existence of heat and one body force.

2. The extension of Neuber's solution

If we let the displacement, body force and temperature fields be $\boldsymbol{u}$, $\boldsymbol{b}$ and T, respectively, we obtain the Navier equation taking heat into account in the form

$$\nabla^2\boldsymbol{u}+\frac{1}{1-2\nu}\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u}-\frac{2\alpha(1+\nu)}{1-2\nu}\,\mathrm{grad}\,T+\frac{\boldsymbol{b}}{G}=\mathbf{0} \tag{1}$$

where ν, G and α denote Poisson's ratio, the shear modulus and the coefficient of linear thermal expansion, respectively, and ∇^2 denotes the Laplacian operator in orthogonal curvilinear coordinates.

From Helmholtz's theorem, we can set

$$\boldsymbol{u}=\mathrm{grad}\,\phi+\mathrm{curl}\,\boldsymbol{S}\quad(\mathrm{div}\,\boldsymbol{S}=0). \tag{2}$$

By substituting expression (2) into equation (1) and making use of the following formulae for a scalar field φ and a vector field $\boldsymbol{A}$:

$$\mathrm{div}\,\mathrm{grad}\,\varphi=\nabla^2\varphi,\quad \mathrm{div}\,\mathrm{curl}\,\boldsymbol{A}=0, \tag{3a, b}$$

we obtain

$$\nabla^2(\mathrm{grad}\,\phi+\mathrm{curl}\,\boldsymbol{S})+\frac{1}{1-2\nu}\,\mathrm{grad}\,\nabla^2\phi+\frac{\boldsymbol{b}}{G}-\frac{2\alpha(1+\nu)}{1-2\nu}\,\mathrm{grad}\,T=\mathbf{0}. \tag{4}$$

If we use the following formulae :

$$\nabla^2\,\mathrm{grad}\,\varphi=\mathrm{grad}\,\nabla^2\varphi,\quad \nabla^2\,\mathrm{div}\,\boldsymbol{A}=\mathrm{div}\,\nabla^2\boldsymbol{A},\quad \nabla^2\,\mathrm{curl}\,\boldsymbol{A}=\mathrm{curl}\,\nabla^2\boldsymbol{A}, \tag{5a−c}$$

we can rewrite equation (4) in the form

$$\nabla^2\left[\frac{2(1-\nu)}{1-2\nu}\,\mathrm{grad}\,\phi+\mathrm{curl}\,\boldsymbol{S}\right]=-\frac{\boldsymbol{b}}{G}+\frac{2\alpha(1+\nu)}{1-2\nu}\,\mathrm{grad}\,T. \tag{6}$$

Now, setting

$$\frac{2(1-\nu)}{1-2\nu}\operatorname{grad}\phi+\operatorname{curl}\boldsymbol{S}=\frac{2(1-\nu)}{G}\boldsymbol{\Phi}+\frac{1-\nu}{G(1-2\nu)}\operatorname{grad}\chi, \tag{7}$$

we will obtain Neuber's solution taking the heat and the body forces into account. However, we separate the curl of a harmonic vector from Φ and set the right-hand side of equation (7) as

$$\frac{2(1-\nu)}{1-2\nu}\operatorname{grad}\phi+\operatorname{curl}\boldsymbol{S}=\frac{2(1-\nu)}{G}\boldsymbol{\Phi}+\frac{\operatorname{curl}\boldsymbol{\vartheta}}{G}+\frac{1-\nu}{G(1-2\nu)}\operatorname{grad}\chi. \tag{8}$$

By applying ∇^2 to both sides of equation (8) and making use of formulae (5a,c), we obtain

$$\begin{aligned}\nabla^2\left[\frac{2(1-\nu)}{1-2\nu}\operatorname{grad}\phi+\operatorname{curl}\boldsymbol{S}\right]&=\frac{2(1-\nu)}{G}\nabla^2\boldsymbol{\Phi}+\frac{\nabla^2\operatorname{curl}\boldsymbol{\vartheta}}{G}+\frac{1-\nu}{G(1-2\nu)}\nabla^2\operatorname{grad}\chi\\&=\frac{2(1-\nu)}{G}\nabla^2\boldsymbol{\Phi}+\frac{\operatorname{curl}\nabla^2\boldsymbol{\vartheta}}{G}+\frac{1-\nu}{G(1-2\nu)}\operatorname{grad}\nabla^2\chi.\end{aligned} \tag{9}$$

If we set

$$\nabla^2\boldsymbol{\vartheta}=0 \tag{10}$$

and rewrite equation (9), we obtain

$$\nabla^2\left[\frac{2(1-\nu)}{1-2\nu}\operatorname{grad}\phi+\operatorname{curl}\boldsymbol{S}\right]=\frac{2(1-\nu)}{G}\nabla^2\boldsymbol{\Phi}+\frac{1-\nu}{G(1-2\nu)}\operatorname{grad}\nabla^2\chi. \tag{11}$$

From equations (6) and (11), we obtain the following equation :

$$\frac{2(1-\nu)}{G}\nabla^2\boldsymbol{\Phi}+\frac{1-\nu}{G(1-2\nu)}\operatorname{grad}\nabla^2\chi=-\frac{\boldsymbol{b}}{G}+\frac{2\alpha(1+\nu)}{1-2\nu}\operatorname{grad}T. \tag{12}$$

By solving equation (12), we obtain

$$\nabla^2\boldsymbol{\Phi}=-\frac{\boldsymbol{b}}{2(1-\nu)},\quad \nabla^2\chi=\frac{\alpha E}{1-\nu}T \tag{13a, b}$$

where E denotes Young's modulus. By applying div to both sides of equation (8) and making use of formulae (3a, b), we obtain

$$\frac{2(1-\nu)}{1-2\nu}\nabla^2\phi=\frac{2(1-\nu)}{G}\operatorname{div}\boldsymbol{\Phi}+\frac{1-\nu}{G(1-2\nu)}\nabla^2\chi. \tag{14}$$

If we let the position vector in the orthogonal curvilinear coordinates be $\boldsymbol{r}$ and use the following formula :

$$\nabla^2(\boldsymbol{r}\cdot\boldsymbol{A})=\boldsymbol{r}\cdot\nabla^2\boldsymbol{A}+2\operatorname{div}\boldsymbol{A}, \tag{15}$$

we obtain

$$\operatorname{div}\boldsymbol{\Phi}=\frac{1}{2}\left[\nabla^2(\boldsymbol{r}\cdot\boldsymbol{\Phi})-\boldsymbol{r}\cdot\nabla^2\boldsymbol{\Phi}\right]. \tag{16}$$

By substituting equation (13a) into expression (16), we obtain

$$\operatorname{div}\boldsymbol{\Phi}=\frac{1}{2}\left[\nabla^2(\boldsymbol{r}\cdot\boldsymbol{\Phi})+\frac{\boldsymbol{r}\cdot\boldsymbol{b}}{2(1-\nu)}\right]. \tag{17}$$

By substituting expression (17) into the right-hand side of equation (14), we obtain

$$\frac{2(1-\nu)}{1-2\nu}\nabla^2\phi=\frac{1-\nu}{G}\left[\nabla^2(\boldsymbol{r}\cdot\boldsymbol{\Phi})+\frac{\boldsymbol{r}\cdot\boldsymbol{b}}{2(1-\nu)}\right]+\frac{1-\nu}{G(1-2\nu)}\nabla^2\chi\,. \tag{18}$$

If we rewrite equation (18), we have

$$\frac{2(1-\nu)}{1-2\nu}\nabla^2\left[\phi-\frac{\chi}{2G}-\frac{1-2\nu}{2G}(\boldsymbol{r}\cdot\boldsymbol{\Phi})\right]=\frac{\boldsymbol{r}\cdot\boldsymbol{b}}{2G}\,. \tag{19}$$

Now, setting

$$\phi-\frac{\chi}{2G}-\frac{1-2\nu}{2G}(\boldsymbol{r}\cdot\boldsymbol{\Phi})=\frac{1-2\nu}{2G}\Phi_0, \tag{20}$$

we obtain

$$\nabla^2\Phi_0=\frac{\boldsymbol{r}\cdot\boldsymbol{b}}{2(1-\nu)}\,. \tag{21}$$

From equation (20), we have

$$\phi=\frac{1-2\nu}{2G}(\Phi_0+\boldsymbol{r}\cdot\boldsymbol{\Phi})+\frac{\chi}{2G}\,. \tag{22}$$

By substituting equations (8) and (22) into expression (2), we obtain

$$\begin{aligned}\boldsymbol{u}&=\mathrm{grad}\,\phi+\mathrm{curl}\,\boldsymbol{S}=\frac{2(1-\nu)}{1-2\nu}\mathrm{grad}\,\phi+\mathrm{curl}\,\boldsymbol{S}-\frac{\mathrm{grad}\,\phi}{1-2\nu}\\&=\frac{2(1-\nu)}{G}\boldsymbol{\Phi}+\frac{\mathrm{curl}\,\boldsymbol{\vartheta}}{G}+\frac{1-\nu}{G(1-2\nu)}\mathrm{grad}\,\chi-\frac{1}{1-2\nu}\mathrm{grad}\left[\frac{1-2\nu}{2G}(\Phi_0+\boldsymbol{r}\cdot\boldsymbol{\Phi})+\frac{\chi}{2G}\right]\\&=\frac{2(1-\nu)}{G}\boldsymbol{\Phi}+\frac{\mathrm{curl}\,\boldsymbol{\vartheta}}{G}-\frac{1}{2G}\mathrm{grad}(\Phi_0+\boldsymbol{r}\cdot\boldsymbol{\Phi})+\frac{\mathrm{grad}\,\chi}{2G}\,.\end{aligned} \tag{23}$$

From expression (23), we obtain a solution as the result :

$$2G\boldsymbol{u}=-\,\mathrm{grad}(\Phi_0+\boldsymbol{r}\cdot\boldsymbol{\Phi})+4(1-\nu)\boldsymbol{\Phi}+2\,\mathrm{curl}\,\boldsymbol{\vartheta}+\mathrm{grad}\,\chi \tag{24}$$

where

$$\nabla^2\Phi_0=\frac{\boldsymbol{r}\cdot\boldsymbol{b}}{2(1-\nu)},\ \ \nabla^2\boldsymbol{\Phi}=-\frac{\boldsymbol{b}}{2(1-\nu)},\ \ \nabla^2\boldsymbol{\vartheta}=\boldsymbol{0},\ \ \nabla^2\chi=\frac{\alpha E}{1-\nu}T. \tag{25a-d}$$

Solution (24) with (25a-d) is a certain extension of Neuber's solution [4].

Now, replacing Φ_0 and $\boldsymbol{\Phi}$ in solution (24) with

$$\Phi_0=-\lambda_0,\ \ \boldsymbol{\Phi}=-\boldsymbol{\lambda}, \tag{26a, b}$$

solution (24) with (25a-d) can be expressed as

$$2G\boldsymbol{u}=\mathrm{grad}\,(\lambda_0+\boldsymbol{r}\cdot\boldsymbol{\lambda})-4(1-\nu)\boldsymbol{\lambda}+2\,\mathrm{curl}\,\boldsymbol{\vartheta}+\mathrm{grad}\,\chi \tag{27}$$

where

$$\nabla^2\lambda_0=-\frac{\boldsymbol{r}\cdot\boldsymbol{b}}{2(1-\nu)},\ \ \nabla^2\boldsymbol{\lambda}=\frac{\boldsymbol{b}}{2(1-\nu)},\ \ \nabla^2\boldsymbol{\vartheta}=\boldsymbol{0},\ \ \nabla^2\chi=\frac{\alpha E}{1-\nu}T. \tag{28a-d}$$

Solution (27) with (28a-d) is a certain extension of Boussinesq's solution [1].

If we impose the following conditions on equations (25a-d) :

$$\boldsymbol{b}=\mathbf{0}\,,\ T=0\,, \tag{29a, b}$$

we can exclude χ from the solution because it is not independent of Φ_0. Then, solution (24) with (25a-d) yields

$$2G\boldsymbol{u}=-\mathrm{grad}\,(\Phi_0+\boldsymbol{r}\cdot\boldsymbol{\Phi})+4(1-\nu)\boldsymbol{\Phi}+2\,\mathrm{curl}\,\boldsymbol{\vartheta} \tag{30}$$

where

$$\nabla^2\Phi_0=0\,,\ \nabla^2\boldsymbol{\Phi}=\mathbf{0}\,,\ \nabla^2\boldsymbol{\vartheta}=\mathbf{0}\,. \tag{31a−c}$$

Solution (30) with (31a-c) is the generalized Neuber solution given by Hata [5]. By imposing conditions (29a, b) on equations (28a-d), solution (27) with (28a-d) becomes

$$2G\boldsymbol{u}=\mathrm{grad}\,(\lambda_0+\boldsymbol{r}\cdot\boldsymbol{\lambda})-4(1-\nu)\boldsymbol{\lambda}+2\,\mathrm{curl}\,\boldsymbol{\vartheta} \tag{32}$$

where

$$\nabla^2\lambda_0=0\,,\ \nabla^2\boldsymbol{\lambda}=\mathbf{0}\,,\ \nabla^2\boldsymbol{\vartheta}=\mathbf{0}\,. \tag{33a−c}$$

Solution (32) with (33a-c) is the generalized Boussinesq solution.

In the generalized Neuber solution (30) with (31a-c), we now set

$$\boldsymbol{\Phi}=\boldsymbol{\Phi}'-\frac{\mathrm{curl}\,\boldsymbol{\vartheta}}{2(1-\nu)}\,,\quad \Phi_0=\Phi_0'+\frac{\boldsymbol{r}\cdot\mathrm{curl}\,\boldsymbol{\vartheta}}{2(1-\nu)} \tag{34a, b}$$

where

$$\nabla^2\boldsymbol{\Phi}'=\mathbf{0}\,,\ \nabla^2\Phi_0'=0\,. \tag{35a, b}$$

Setting $\boldsymbol{\Phi}$ and Φ_0 as expressions (34a, b) is based on the following relations under equation (33c) :

$$\nabla^2\mathrm{curl}\,\boldsymbol{\vartheta}=\mathrm{curl}\,\nabla^2\boldsymbol{\vartheta}=\mathbf{0}\,,\ \nabla^2(\boldsymbol{r}\cdot\mathrm{curl}\,\boldsymbol{\vartheta})=0\,. \tag{36a, b}$$

By substituting expressions (34a, b) into solution (30), we obtain

$$\begin{aligned}2G\boldsymbol{u}&=-\mathrm{grad}\left[\Phi_0'+\frac{\boldsymbol{r}\cdot\mathrm{curl}\,\boldsymbol{\vartheta}}{2(1-\nu)}+\boldsymbol{r}\cdot\boldsymbol{\Phi}'-\frac{\boldsymbol{r}\cdot\mathrm{curl}\,\boldsymbol{\vartheta}}{2(1-\nu)}\right]+4(1-\nu)\left[\boldsymbol{\Phi}'-\frac{\mathrm{curl}\,\boldsymbol{\vartheta}}{2(1-\nu)}\right]+2\,\mathrm{curl}\,\boldsymbol{\vartheta}\\&=-\mathrm{grad}\,(\Phi_0'+\boldsymbol{r}\cdot\boldsymbol{\Phi}')+4(1-\nu)\,\boldsymbol{\Phi}'.\end{aligned} \tag{37}$$

Since removing the prime affix in solution (37) does not spoil the generality of the solution, solution (37) is expressed as

$$2G\boldsymbol{u}=-\mathrm{grad}\,(\Phi_0+\boldsymbol{r}\cdot\boldsymbol{\Phi})+4(1-\nu)\,\boldsymbol{\Phi} \tag{38}$$

where

$$\nabla^2\Phi_0=0\,,\ \nabla^2\boldsymbol{\Phi}=\mathbf{0}\,. \tag{39a, b}$$

Solution (38) with (39a, b) is Neuber's solution [4].

Now, replacing Φ_0 and $\boldsymbol{\Phi}$ in solution (38) with

$$\Phi_0=\frac{G}{2(1-\nu)}B_0,\quad \boldsymbol{\Phi}=\frac{G}{2(1-\nu)}\boldsymbol{B},\tag{40a, b}$$

solution (38) with (39a, b) can be expressed as

$$\boldsymbol{u}=-\frac{1}{4(1-\nu)}\operatorname{grad}(B_0+\boldsymbol{r}\cdot\boldsymbol{B})+\boldsymbol{B}\tag{41}$$

where

$$\nabla^2 B_0=0,\quad \nabla^2\boldsymbol{B}=\mathbf{0}.\tag{42a, b}$$

Solution (41) with (42a, b) is Papkovich's solution [3].

We consider the generalized Boussinesq solution (32) with (33a-c) in rectangular Cartesian coordinates (x, y, z) and let the basis vectors be $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$. If we define the vectors in the solution as

$$\boldsymbol{\lambda}=\lambda_1\boldsymbol{e}_x+\lambda_2\boldsymbol{e}_y+\lambda_3\boldsymbol{e}_z=(\lambda_1,\ \lambda_2,\ \lambda_3),\tag{43a}$$

$$\boldsymbol{\vartheta}=\vartheta_1\boldsymbol{e}_x+\vartheta_2\boldsymbol{e}_y+\vartheta_3\boldsymbol{e}_z=(\vartheta_1,\ \vartheta_2,\ \vartheta_3),\tag{43b}$$

$$\boldsymbol{r}=x\boldsymbol{e}_x+y\boldsymbol{e}_y+z\boldsymbol{e}_z=(x, y, z)\tag{43c}$$

and set

$$\lambda_1=\lambda_2=0,\quad \lambda_3=\lambda_3,\quad \vartheta_1=\vartheta_2=0,\quad \vartheta_3=\vartheta_3,\tag{44a–d}$$

solution (32) with (33a-c) is expressed as

$$2G\boldsymbol{u}=\operatorname{grad}(\lambda_0+z\lambda_3)-4(1-\nu)\lambda_3\boldsymbol{e}_z+2\operatorname{curl}(\vartheta_3\boldsymbol{e}_z)\tag{45}$$

where

$$\nabla^2\lambda_0=0,\quad \nabla^2\lambda_3=0,\quad \nabla^2\vartheta_3=0\tag{46a–c}$$

with

$$\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

Solution (45) with (46a-c) is Boussinesq's solution [1]. If we divide solution (45) into three solutions, we obtain

the first basic solution : $2G\boldsymbol{u}^{(1)}=\operatorname{grad}\lambda_0$, (47a)

the second basic solution : $2G\boldsymbol{u}^{(2)}=2\operatorname{curl}(\vartheta_3\boldsymbol{e}_z)$, (47b)

the third basic solution : $2G\boldsymbol{u}^{(3)}=\operatorname{grad}(z\lambda_3)-4(1-\nu)\lambda_3\boldsymbol{e}_z$. (47c)

3. The extension of Galerkin's solution

If we use formula (3b) and set $\boldsymbol{S}$ as

$$\boldsymbol{S} = -\frac{2(1-\nu)}{2G}\,\mathrm{curl}\,\boldsymbol{W}, \tag{48}$$

we will obtain Galerkin's solution taking the heat and the body forces into account. However, we separate a harmonic vector from curl $\boldsymbol{W}$ and set the right-hand side of expression (48) as

$$\boldsymbol{S} = -\frac{2(1-\nu)}{2G}\,\mathrm{curl}\,\boldsymbol{W} + \frac{1}{2G}\left[2\boldsymbol{\vartheta} - \mathrm{grad}\,(\boldsymbol{r}\cdot\boldsymbol{\vartheta})\right] \tag{49}$$

where

$$\nabla^2\boldsymbol{\vartheta} = \boldsymbol{0}. \tag{50}$$

Expression (49) is based on the following relation using formula (15) and equation (50) :

$$\mathrm{div}\left[2\boldsymbol{\vartheta} - \mathrm{grad}(\boldsymbol{r}\cdot\boldsymbol{\vartheta})\right] = 2\,\mathrm{div}\boldsymbol{\vartheta} - \nabla^2(\boldsymbol{r}\cdot\boldsymbol{\vartheta}) = 2\,\mathrm{div}\boldsymbol{\vartheta} - 2\,\mathrm{div}\boldsymbol{\vartheta} = 0. \tag{51}$$

By applying curl to both sides of expression (49) and making use of the following formulae :

$$\nabla^2\boldsymbol{A} = \mathrm{grad}\,\mathrm{div}\,\boldsymbol{A} - \mathrm{curl}\,\mathrm{curl}\,\boldsymbol{A},\ \mathrm{curl}\,\mathrm{grad}\,\varphi = \boldsymbol{0}, \tag{52a, b}$$

we obtain

$$\begin{aligned}\mathrm{curl}\,\boldsymbol{S} &= -\frac{2(1-\nu)}{2G}\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{W} + \frac{2\mathrm{curl}\,\boldsymbol{\vartheta}}{2G} - \frac{1}{2G}\,\mathrm{curl}\,\mathrm{grad}(\boldsymbol{r}\cdot\boldsymbol{\vartheta})\\ &= -\frac{2(1-\nu)}{2G}(\mathrm{grad}\,\mathrm{div}\,\boldsymbol{W} - \nabla^2\boldsymbol{W}) + \frac{2\mathrm{curl}\,\boldsymbol{\vartheta}}{2G}.\end{aligned} \tag{53}$$

By substituting expression (53) into equation (6), we obtain

$$\begin{aligned}&\nabla^2\left[\frac{2(1-\nu)}{1-2\nu}\,\mathrm{grad}\,\phi - \frac{2(1-\nu)}{2G}(\mathrm{grad}\,\mathrm{div}\,\boldsymbol{W} - \nabla^2\boldsymbol{W}) + \frac{2\mathrm{curl}\,\boldsymbol{\vartheta}}{2G}\right]\\ &\quad = -\frac{\boldsymbol{b}}{G} + \frac{2\alpha(1+\nu)}{1-2\nu}\,\mathrm{grad}\,T.\end{aligned} \tag{54}$$

Rewriting equation (54) yields

$$\begin{aligned}&\frac{2(1-\nu)}{1-2\nu}\nabla^2\,\mathrm{grad}\left(\phi - \frac{1-2\nu}{2G}\,\mathrm{div}\,\boldsymbol{W}\right) + \frac{2(1-\nu)}{2G}\nabla^2\nabla^2\boldsymbol{W} + \frac{2\nabla^2\,\mathrm{curl}\,\boldsymbol{\vartheta}}{2G}\\ &= -\frac{\boldsymbol{b}}{G} + \frac{2\alpha(1+\nu)}{1-2\nu}\,\mathrm{grad}\,T.\end{aligned} \tag{55}$$

Now, setting

$$\nabla^2\nabla^2\boldsymbol{W} = -\frac{\boldsymbol{b}}{1-\nu} \tag{56}$$

and using relation (36a), equation (55) yields

$$\frac{2(1-\nu)}{1-2\nu}\nabla^2\,\mathrm{grad}\left(\phi - \frac{1-2\nu}{2G}\,\mathrm{div}\,\boldsymbol{W}\right) = \frac{2\alpha(1+\nu)}{1-2\nu}\,\mathrm{grad}\,T. \tag{57}$$

In equation (57), setting

$$\phi - \frac{1-2\nu}{2G}\,\mathrm{div}\,\boldsymbol{W} = \frac{\chi}{2G}, \tag{58}$$

we obtain

$$\nabla^2 \chi = \frac{\alpha E}{1-\nu} T. \tag{13b}$$

From equation (58), we have

$$\phi = \frac{1-2\nu}{2G} \operatorname{div} \boldsymbol{W} + \frac{\chi}{2G}. \tag{59}$$

By substituting expressions (53) and (59) into expression (2), we obtain

$$\begin{aligned} \boldsymbol{u} &= \frac{1-2\nu}{2G} \operatorname{grad} \operatorname{div} \boldsymbol{W} - \frac{2(1-\nu)}{2G} \left(\operatorname{grad} \operatorname{div} \boldsymbol{W} - \nabla^2 \boldsymbol{W} \right) + \frac{2\operatorname{curl} \boldsymbol{\vartheta}}{2G} + \frac{\operatorname{grad} \chi}{2G} \\ &= -\frac{1}{2G} \operatorname{grad} \operatorname{div} \boldsymbol{W} + \frac{2(1-\nu)}{2G} \nabla^2 \boldsymbol{W} + + \frac{2\operatorname{curl} \boldsymbol{\vartheta}}{2G} + \frac{\operatorname{grad} \chi}{2G}. \end{aligned} \tag{60}$$

From expression (60), we obtain a solution as the result :

$$2G\boldsymbol{u} = -\operatorname{grad} \operatorname{div} \boldsymbol{W} + 2(1-\nu) \nabla^2 \boldsymbol{W} + 2\operatorname{curl} \boldsymbol{\vartheta} + \operatorname{grad} \chi \tag{61}$$

where

$$\nabla^2 \nabla^2 \boldsymbol{W} = -\frac{\boldsymbol{b}}{1-\nu}, \quad \nabla^2 \boldsymbol{\vartheta} = \boldsymbol{0}, \quad \nabla^2 \chi = \frac{\alpha E}{1-\nu} T. \tag{62a-c}$$

Solution (61) with (62a-c) is a certain extension of Galerkin's solution [2]. If we impose conditions (29a, b) on equations (62a-c), we can exclude χ from the solution because it is transfered to biharmonic function div $\boldsymbol{W}$. Then, solution (61) with (62a-c) yields

$$2G\boldsymbol{u} = -\operatorname{grad} \operatorname{div} \boldsymbol{W} + 2(1-\nu) \nabla^2 \boldsymbol{W} + 2\operatorname{curl} \boldsymbol{\vartheta} \tag{63}$$

where

$$\nabla^2 \nabla^2 \boldsymbol{W} = \boldsymbol{0}, \quad \nabla^2 \boldsymbol{\vartheta} = \boldsymbol{0}. \tag{64a, b}$$

Solution (63) with (64a, b) is the generalized Galerkin solution given by Hata [6].

In the generalized Galerkin solution (63) with (64a, b), we now set

$$\boldsymbol{W} = \frac{\boldsymbol{\vartheta} \times \boldsymbol{r}}{2(1-\nu)} + \boldsymbol{W}' + \operatorname{grad} \Phi \tag{65}$$

where

$$\nabla^2 \nabla^2 \boldsymbol{W}' = \boldsymbol{0}, \quad \nabla^2 \boldsymbol{\vartheta} = \boldsymbol{0}, \quad \nabla^2 \Phi = \frac{\boldsymbol{r} \cdot \operatorname{curl} \boldsymbol{\vartheta}}{2(1-2\nu)(1-\nu)}. \tag{66a-c}$$

Setting biharmonic vector $\boldsymbol{W}$ as expression (65) is based on equation (64b) and formulae (3b), (52a, b) and

$$\operatorname{div} (\boldsymbol{A} \times \boldsymbol{C}) = \boldsymbol{C} \cdot \operatorname{curl} \boldsymbol{A} - \boldsymbol{A} \cdot \operatorname{curl} \boldsymbol{C}, \quad \operatorname{curl} \boldsymbol{r} = \boldsymbol{0}, \quad \operatorname{div} \boldsymbol{r} = 3, \tag{67a-c}$$

$$\begin{aligned} &\operatorname{curl} (\boldsymbol{A} \times \boldsymbol{C}) + \operatorname{grad} (\boldsymbol{A} \cdot \boldsymbol{C}) \\ &= 2(\boldsymbol{C} \cdot \operatorname{grad}) \boldsymbol{A} + \boldsymbol{A} \operatorname{div} \boldsymbol{C} - \boldsymbol{C} \operatorname{div} \boldsymbol{A} + \boldsymbol{C} \times \operatorname{curl} \boldsymbol{A} + \boldsymbol{A} \times \operatorname{curl} \boldsymbol{C} \end{aligned} \tag{67d}$$

and the following relation :

$$\nabla^2(\boldsymbol{\vartheta}\times\boldsymbol{r})=-\nabla^2(\boldsymbol{r}\times\boldsymbol{\vartheta})=\mathrm{curl\,curl}(\boldsymbol{r}\times\boldsymbol{\vartheta})-\mathrm{grad\,div}(\boldsymbol{r}\times\boldsymbol{\vartheta})$$
$$=\mathrm{curl}\left[-\mathrm{grad}(\boldsymbol{r}\cdot\boldsymbol{\vartheta})+2(\boldsymbol{\vartheta}\cdot\mathrm{grad})\,\boldsymbol{r}+\boldsymbol{r}\ \mathrm{div}\,\boldsymbol{\vartheta}-\boldsymbol{\vartheta}\ \mathrm{div}\boldsymbol{r}+\boldsymbol{\vartheta}\times\mathrm{curl}\boldsymbol{r}+\boldsymbol{r}\times\mathrm{curl}\boldsymbol{\vartheta}\right]$$
$$-\mathrm{grad}(\boldsymbol{\vartheta}\cdot\mathrm{curl}\boldsymbol{r}-\boldsymbol{r}\cdot\mathrm{curl}\boldsymbol{\vartheta})=-2\,\mathrm{curl}\boldsymbol{\vartheta}. \tag{68}$$

By applying ∇^2 to both sides of expression (65), we obtain

$$\nabla^2\boldsymbol{W}=\frac{\nabla^2(\boldsymbol{\vartheta}\times\boldsymbol{r})}{2(1-\nu)}+\nabla^2\boldsymbol{W}'+\nabla^2\,\mathrm{grad}\,\Phi=-\frac{2\,\mathrm{curl}\boldsymbol{\vartheta}}{2(1-\nu)}+\nabla^2\boldsymbol{W}'+\mathrm{grad}\,\nabla^2\Phi. \tag{69}$$

Furthermore, applying ∇^2 to both sides of expression (69) and using formulae (5a, c), we obtain

$$\nabla^2\nabla^2\boldsymbol{W}=-\frac{2\nabla^2\,\mathrm{curl}\boldsymbol{\vartheta}}{2(1-\nu)}+\nabla^2\nabla^2\boldsymbol{W}'+\nabla^2\,\mathrm{grad}\,\nabla^2\Phi$$
$$=-\frac{2\,\mathrm{curl}\,\nabla^2\boldsymbol{\vartheta}}{2(1-\nu)}+\nabla^2\nabla^2\boldsymbol{W}'+\mathrm{grad}\,\nabla^2\nabla^2\Phi=\boldsymbol{0}. \tag{70}$$

Expression (70) indicates that $\boldsymbol{W}$ in expression (65) satisfies the condition of the biharmonic vector. By applying div to both sides of expression (65) and making use of formula (3a), we obtain

$$\mathrm{div}\,\boldsymbol{W}=\frac{1}{2(1-\nu)}\,\mathrm{div}(\boldsymbol{\vartheta}\times\boldsymbol{r})+\mathrm{div}\,\boldsymbol{W}'+\nabla^2\Phi. \tag{71}$$

Since using formulae (67a, b) yields

$$\mathrm{div}(\boldsymbol{\vartheta}\times\boldsymbol{r})=-\mathrm{div}(\boldsymbol{r}\times\boldsymbol{\vartheta})=-\boldsymbol{\vartheta}\cdot\mathrm{curl}\ \boldsymbol{r}+\boldsymbol{r}\cdot\mathrm{curl}\boldsymbol{\vartheta}=\boldsymbol{r}\cdot\mathrm{curl}\boldsymbol{\vartheta}, \tag{72}$$

expression (71) becomes

$$\mathrm{div}\,\boldsymbol{W}=\frac{\boldsymbol{r}\cdot\mathrm{curl}\ \boldsymbol{\vartheta}}{2(1-\nu)}+\mathrm{div}\,\boldsymbol{W}'+\nabla^2\Phi. \tag{73}$$

By substituting expressions (69) and (73) into solution (63), we obtain

$$2G\boldsymbol{u}=-\mathrm{grad\,div}\,\boldsymbol{W}+2(1-\nu)\,\nabla^2\boldsymbol{W}+2\,\mathrm{curl}\boldsymbol{\vartheta}$$
$$=-\mathrm{grad}\left[\frac{\boldsymbol{r}\cdot\mathrm{curl}\ \boldsymbol{\vartheta}}{2(1-\nu)}+\mathrm{div}\,\boldsymbol{W}'+\nabla^2\Phi\right]$$
$$+2(1-\nu)\left[-\frac{2\,\mathrm{curl}\ \boldsymbol{\vartheta}}{2(1-\nu)}+\nabla^2\boldsymbol{W}'+\mathrm{grad}\ \nabla^2\Phi\right]+2\,\mathrm{curl}\boldsymbol{\vartheta}$$
$$=-\mathrm{grad\,div}\,\boldsymbol{W}'+2(1-\nu)\,\nabla^2\boldsymbol{W}'+(1-2\nu)\,\mathrm{grad}\left[\nabla^2\Phi-\frac{\boldsymbol{r}\cdot\mathrm{curl}\ \boldsymbol{\vartheta}}{2(1-2\nu)(1-\nu)}\right]$$
$$=-\mathrm{grad\,div}\,\boldsymbol{W}'+2(1-\nu)\,\nabla^2\boldsymbol{W}'. \tag{74}$$

Since removing the prime affix in solution (74) does not spoil the generality of the solution, solution (74) is expressed as

$$2G\boldsymbol{u}=-\mathrm{grad\,div}\,\boldsymbol{W}+2(1-\nu)\,\nabla^2\boldsymbol{W} \tag{75}$$

where

$$\nabla^2\nabla^2\boldsymbol{W}=\boldsymbol{0}. \tag{76}$$

Solution (75) with (76) is Galerkin's solution [2] in which $\boldsymbol{W}$ is the biharmonic vector called the Galerkin vector.

In Galerkin's solution (75) with (76), we now set

$$2\operatorname{div}\boldsymbol{W}-\boldsymbol{r}\cdot\nabla^2\boldsymbol{W}=2\Phi_0,\quad \nabla^2\boldsymbol{W}=2\boldsymbol{\Phi} \tag{77a, b}$$

where

$$\nabla^2\Phi_0=0,\quad \nabla^2\boldsymbol{\Phi}=\boldsymbol{0}. \tag{39a, b}$$

Expression (77a) is based on the following relation using formulae (5b) and (15) and equation (76):

$$\nabla^2\left(2\operatorname{div}\boldsymbol{W}-\boldsymbol{r}\cdot\nabla^2\boldsymbol{W}\right)=2\operatorname{div}\nabla^2\boldsymbol{W}-\left(\boldsymbol{r}\cdot\nabla^2\nabla^2\boldsymbol{W}+2\operatorname{div}\nabla^2\boldsymbol{W}\right)$$

$$=2\operatorname{div}\nabla^2\boldsymbol{W}-2\operatorname{div}\nabla^2\boldsymbol{W}=0. \tag{78}$$

From expressions (77a,b), we have

$$2\operatorname{div}\boldsymbol{W}=2\Phi_0+\boldsymbol{r}\cdot\nabla^2\boldsymbol{W}=2(\Phi_0+\boldsymbol{r}\cdot\boldsymbol{\Phi}) \tag{79}$$

and so

$$\operatorname{div}\boldsymbol{W}=\Phi_0+\boldsymbol{r}\cdot\boldsymbol{\Phi}. \tag{80}$$

By substituting expressions (77b) and (80) into Galerkin's solution (75), we obtain

$$2G\boldsymbol{u}=-\operatorname{grad}(\Phi_0+\boldsymbol{r}\cdot\boldsymbol{\Phi})+4(1-\nu)\boldsymbol{\Phi}. \tag{38}$$

Namely, Galerkin's solution is formally changed to Neuber's solution.

4. The application of solution (61) with (62a-c)

Now, we consider the application of solution (61) with (62a-c) to Muki's [8] and Love's [9] solutions to axially asymmetric and symmetric problems of elasticity in cylindrical coordinates, respectively. We let the basis vectors be $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_z$ in cylindrical coordinates (r, θ, z). If we define the vectors in the solution as

$$\boldsymbol{u}=u_r\boldsymbol{e}_r+u_\theta\boldsymbol{e}_\theta+u_z\boldsymbol{e}_z=(u_r,\ u_\theta,\ u_z), \tag{81a}$$

$$\boldsymbol{W}=W_r\boldsymbol{e}_r+W_\theta\boldsymbol{e}_\theta+W_z\boldsymbol{e}_z=(W_r,\ W_\theta,\ W_z), \tag{81b}$$

$$\boldsymbol{\vartheta}=\vartheta_r\boldsymbol{e}_r+\vartheta_\theta\boldsymbol{e}_\theta+\vartheta_z\boldsymbol{e}_z=(\vartheta_r,\ \vartheta_\theta,\ \vartheta_z), \tag{81c}$$

$$\boldsymbol{b}=b_r\boldsymbol{e}_r+b_\theta\boldsymbol{e}_\theta+b_z\boldsymbol{e}_z=(b_r,\ b_\theta,\ b_z) \tag{81d}$$

and set

$$W_r=W_\theta=0,\ W_z=2G\Phi,\quad \vartheta_r=\vartheta_\theta=0,\ \vartheta_z=2G\psi,\quad \chi=2G\chi', \tag{82a-e}$$

$$b_r=b_\theta=0,\ b_z=2Gb'_z, \tag{83a, b}$$

equations (62a-c) yield

$$\nabla^2\nabla^2\Phi = -\frac{b'z}{1-\nu}, \quad \nabla^2\psi = 0, \quad \nabla^2\chi' = \frac{1+\nu}{1-\nu}\alpha T \tag{84a-c}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}.$$

By making use of formula (52a), solution (61) is changed to

$$2G\boldsymbol{u} = -\mathrm{grad\,div}\,\boldsymbol{W} + 2(1-\nu)\nabla^2\boldsymbol{W} + 2\mathrm{curl}\,\boldsymbol{\vartheta} + \mathrm{grad}\,\chi$$

$$= (1-2\nu)\mathrm{grad\,div}\,\boldsymbol{W} - 2(1-\nu)\mathrm{curl\,curl}\,\boldsymbol{W} + 2\mathrm{curl}\,\boldsymbol{\vartheta} + \mathrm{grad}\,\chi. \tag{85}$$

Since, in solution (85), there is the following relations :

$$\mathrm{grad\,div}\,\boldsymbol{W} = \mathrm{grad\,div}(2G\Phi\boldsymbol{e}_z) = 2G\left(\frac{\partial^2\Phi}{\partial r\partial z}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial^2\Phi}{\partial\theta\partial z}\boldsymbol{e}_\theta + \frac{\partial^2\Phi}{\partial z^2}\boldsymbol{e}_z\right), \tag{86a}$$

$$\mathrm{curl\,curl}\,\boldsymbol{W} = \mathrm{curl\,curl}(2G\Phi\boldsymbol{e}_z)$$

$$= 2G\left[\frac{\partial^2\Phi}{\partial r\partial z}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial^2\Phi}{\partial\theta\,\partial z}\boldsymbol{e}_\theta + \left(-\frac{1}{r}\frac{\partial\Phi}{\partial r} - \frac{\partial^2\Phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right)\boldsymbol{e}_z\right], \tag{86b}$$

$$2\,\mathrm{curl}\,\boldsymbol{\vartheta} = 2\mathrm{curl}(2G\psi\boldsymbol{e}_z) = 4G\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\boldsymbol{e}_r - \frac{\partial\psi}{\partial r}\boldsymbol{e}_\theta\right), \tag{86c}$$

$$\mathrm{grad}\,\chi = \mathrm{grad}(2G\chi') = 2G\left(\frac{\partial\chi'}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\chi'}{\partial\theta}\boldsymbol{e}_\theta + \frac{\partial\chi'}{\partial z}\boldsymbol{e}_z\right), \tag{86d}$$

we obtain a solution by substituting relations (86a-d) into solution (85), in the form

$$u_r = -\frac{\partial^2\Phi}{\partial r\partial z} + \frac{2}{r}\frac{\partial\psi}{\partial\theta} + \frac{\partial\chi'}{\partial r}, \tag{87a}$$

$$u_\theta = -\frac{1}{r}\frac{\partial^2\Phi}{\partial\theta\partial z} - 2\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial\chi'}{\partial\theta}, \tag{87b}$$

$$u_z = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2} + \frac{\partial\chi'}{\partial z} \tag{87c}$$

where

$$\nabla^2\nabla^2\Phi = -\frac{b'z}{1-\nu}, \quad \nabla^2\psi = 0, \quad \nabla^2\chi' = \frac{1+\nu}{1-\nu}\alpha T. \tag{84a-c}$$

Solution (87a-c) with (84a-c) is a certain extension of Muki's solution [8] to the heat and one body force.

Next, if we set

$$W_r = W_\theta = 0, \quad W_z = \Phi, \quad \boldsymbol{\vartheta} = \boldsymbol{0}, \tag{88a-c}$$

$$b_r = b_\theta = 0, \quad b_z = b_z \tag{89a, b}$$

and consider that Φ, χ and T do not depend on θ, equations (62a,c) yield

$$\nabla_1^2\nabla_1^2\Phi = -\frac{bz}{1-\nu}, \quad \nabla_1^2\chi = \frac{\alpha E}{1-\nu}T \tag{90a, b}$$

where

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

Since relations (86a, b, d) are changed to

$$\text{grav div}\, \boldsymbol{W} = \text{grav div}(\Phi \boldsymbol{e}_z) = \frac{\partial^2 \Phi}{\partial r \partial z}\boldsymbol{e}_r + \frac{\partial^2 \Phi}{\partial z^2}\boldsymbol{e}_z, \tag{91a}$$

$$\text{curl curl}\, \boldsymbol{W} = \text{curl curl}(\Phi \boldsymbol{e}_z) = \frac{\partial^2 \Phi}{\partial r \partial z}\boldsymbol{e}_r + \left(-\frac{1}{r}\frac{\partial \Phi}{\partial r} - \frac{\partial^2 \Phi}{\partial r^2}\right)\boldsymbol{e}_z, \tag{91b}$$

$$\text{grad}\, \chi = \frac{\partial \chi}{\partial r}\boldsymbol{e}_r + \frac{\partial \chi}{\partial z}\boldsymbol{e}_z, \tag{91c}$$

we obtain a solution by substituting relations (91a-c) and expression (88c) into solution (85), in the form

$$2Gu_r = -\frac{\partial^2 \Phi}{\partial r \partial z} + \frac{\partial \chi}{\partial r}, \tag{92a}$$

$$2Gu_\theta = 0, \tag{92b}$$

$$2Gu_z = 2(1-\nu)\nabla_1^2 \Phi - \frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial \chi}{\partial z} \tag{92c}$$

where

$$\nabla_1^2 \nabla_1^2 \Phi = -\frac{bz}{1-\nu}, \quad \nabla_1^2 \chi = \frac{\alpha E}{1-\nu} T. \tag{90a, b}$$

Solution (92a-c) with (90a, b) is a certain extension of Love's solution [9] to the heat and one body force.

5. Conclusions

Neuber's and Galerkin's solutions taking heat and the curl of a harmonic vector into account were derived from the Navier equation with the temperature field by means of the vector calculus. The solution to the heat was in agreement with Goodier's thermoelastic potential. Although Goodier's thermoelastic potential has been derived differently from three-dimensional solutions of elasticity thus far, it was unified into three-dimensional solutions of elasticity. Since the vector potentials in the solutions presented are one more than those in Neuber's and Galerkin's solutions, the solutions are suitable for boundary-value problems of finite solids, with many boundary conditions. The process of transfering the curl of a harmonic vector in the solutions to the Neuber potentials or the Galerkin vector was described in detail and yielded Neuber's and Galerkin's solutions as the result. The solution of the curl of a harmonic vector corresponds to the second basic solution given by Boussinesq when only the third component of a harmonic vector is picked out. Therefore, the solution may be important to axially asymmetric problems of elasticity in cylindrical, spherical and so on coordinates besides problems in rectangular Cartesian coordinates.

For the reasons mentioned above, the author may conclude that the induction process of Neuber's and Galerkin's solutions taking the heat and the curl of a harmonic vector into account and the formulation of the equivalence among their solutions and other solutions obtained already should be useful for the linear theory of elasticity.

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