Axi-Symmetric Solutions to States of Plane Stress and Generalized Plane Stress in Transversely Isotropic, Thick Annular Plates and Their Application to Bending

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Abstract

Axi-symmetric solutions to a state of plane stress and to a state of generalized plane stress in transversely isotropic, thick annular plates are obtained. The two solutions are deduced from the generalized Elliott solution which has five independent potential functions. Expressions for components of displacement and stress are explicitly presented. Axi-symmetric bending of the plate subjected to a uniformly distributed annular load is analyzed by one method of solution in that a homogeneous solution consisting of the two solutions and a particular solution deduced from a part of the generalized Elliott solution are used to satisfy boundary and loading conditions. The effect of anisotropy on the stresses is investigated through a comparison with the stresses in an isotropic material.

1. Introduction

The latest studies on two-dimensional or three-dimensional elasticity problems have turned to those of anisotropic solids. There are two reasons for this trend. The first is that studies on isotropic solids have been virtually accomplished in theory. The second is that the elucidation of the mechanical properties of anisotropic solids has grown in importance due to the recent increase in the use of composite materials. Although there are various classes of anisotropy, Elliott [1] and Lodge [2] have found distinguished solutions to transversely isotropic solids. In a recent paper [3], one of the authors proposed the generalized Elliott solution as a solution to make up for the deficiency in Elliott's solution.

Although the three-dimensional elasticity solutions stated above are expected to be theoretically applicable to analyses of the stretching and bending of transversely isotropic thick plates, they are not practically applicable to analyses of moderately thick plates which are usually called thick plates. This is because, except for special boundary conditions, the use of the solutions entails some difficulty with numerical calculations and cannot yield numerical results with rapid convergences. The difficulty of the numerical calculations in the application of the solutions provides a stronger motive for finding a simplified and extensively practicable theory of moderately thick plates than the analytical complexity of the three-dimen-

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sional elasticity solutions does. Several studies on the stretching and bending of anisotropic thick plates have lately appeared, because studies on isotropic thick plates have passed their peak. The bending of orthotropic, thick rectangular plates was studied by Sonoda and Horikawa [4] and Fan and Ye [5]. The bending of transversely isotopic, thick rectangular plates was studied by Reissner [6], Voyiadjis and Baluch [7] and Wang [8], and the stretching was studied by Clark and Reissner [9] and Wang [10]. However, a number of those studies are concerned with formal theories of thick rectangular plates and some of Navier-type rectangular plates which are practically analyzed. Studies on transversely isotropic, thick circular or annular plates seem to be few except for Okumura's work [11]. Okumura extended the theory of isotropic, thick rectangular plates by Love [12] to one of transversely isotropic, thick circular plates and obtained exact elasticity solutions to states of plane stress and generalized plane stress.

This paper is concerned with axi-symmetric solutions to states of plane stress and generalized plane stress in transversely isotropic, thick annular plates and their application to an analysis of the axi-symmetric bending. Although the two solutions in Love's theory were derived through the use of stress functions, they are deduced from the generalized Elliott solution in this paper because it is very difficult to derive the solutions to transversely isotropic solids directly from the equations of equilibrium and compatibility. Therefore, in this paper, four independent potential functions included in the generalized Elliott solution restricted to axi-symmetric problems are determined, and the relationships among arbitrary constants included in the potential functions are determined from certain conditions that the stress components in the solutions must satisfy. The determination of the potential functions is very important to the derivation of the solutions and is a point to require deliberation. As an application of the two solutions, the axi-symmetric bending of the plate subjected to a uniformly distributed annular load is analyzed by one method of solution in that a homogeneous solution consisting of the two solutions and a particular solution deduced from a part of the generalized Elliott solution are used for satisfying boundary conditions at the edges and loading conditions at the top and bottom faces, respectively.

2. The generalized Elliott solution

Using cylindrical coordinates r, θ, z such that the z-axis is taken parallel to the axis of elastic symmetry, the generalized Elliott solution [3] is expressed in terms of displacement components, i.e., u_r, u_θ and u_z as

$$
\begin{aligned}
u_r &= \frac{\partial}{\partial r}\left[\phi_{01}+\phi_{03}+\gamma_1\left(r\frac{\partial \phi_1}{\partial r}+z\frac{\partial \phi_3}{\partial z}\right)-\gamma_2\phi_1-\gamma_3\phi_3\right]+\frac{1}{r}\frac{\partial \psi}{\partial \theta},\\
u_\theta &= \frac{1}{r}\frac{\partial}{\partial \theta}\left[\phi_{01}+\phi_{03}+\gamma_1\left(r\frac{\partial \phi_1}{\partial r}+z\frac{\partial \phi_3}{\partial z}\right)-\gamma_2\phi_1-\gamma_3\phi_3\right]-\frac{\partial \psi}{\partial r}, \qquad (1)\\
u_z &= \frac{\partial}{\partial z}\left[k_1\left(\phi_{01}-\gamma_3\phi_1\right)+k_2\left(\phi_{03}-\gamma_2\phi_3\right)+\gamma_1\left(k_1 r\frac{\partial \phi_1}{\partial r}+k_2 z\frac{\partial \phi_3}{\partial z}\right)\right]
\end{aligned}
$$

where

$$\nabla_1^2\phi_{01}+\nu_1\frac{\partial^2\phi_{01}}{\partial z^2}=0,\quad \nabla_1^2\phi_{03}+\nu_2\frac{\partial^2\phi_{03}}{\partial z^2}=0,\quad \nabla_1^2\phi_1+\nu_2\frac{\partial^2\phi_1}{\partial z^2}=0,$$
$$\nabla_1^2\phi_3+\nu_1\frac{\partial^2\phi_3}{\partial z^2}=0,\quad \nabla_1^2\psi+\nu_3\frac{\partial^2\psi}{\partial z^2}=0,\quad \nabla_1^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \tag{2}$$

$$\gamma_1=\begin{cases}1 \text{ for } \nu_1=\nu_2\\ 0 \text{ for } \nu_1\neq\nu_2\end{cases},\quad \gamma_2=\begin{cases}\dfrac{2c_{11}\nu_2}{c_{11}\nu_2-c_{44}} \text{ for } \nu_1=\nu_2\\ \dfrac{2\nu_2}{\nu_1-\nu_2}\dfrac{c_{11}\nu_1-c_{44}}{c_{11}\nu_2-c_{44}} \text{ for } \nu_1\neq\nu_2\end{cases},$$
$$\gamma_3=\begin{cases}0 \quad \text{for } \nu_1=\nu_2\\ \dfrac{2\nu_2}{\nu_1-\nu_2} \text{ for } \nu_1\neq\nu_2\end{cases}, \tag{3}$$

$$k_1=\frac{c_{11}\nu_1-c_{44}}{c_{13}+c_{44}},\quad k_2=\frac{c_{11}\nu_2-c_{44}}{c_{13}+c_{44}},\quad \nu_3=\frac{c_{44}}{c_{66}}=\frac{2c_{44}}{c_{11}-c_{12}} \tag{4}$$

and c_{ij} is the elastic constant of transversely isotropic solids and is five in number [13], and ν_1 and ν_2 are the roots of

$$c_{11}c_{44}\nu^2+\left[c_{13}(c_{13}+2c_{44})-c_{11}c_{33}\right]\nu+c_{33}c_{44}=0. \tag{5}$$

If solution (1) is restricted to axi-symmetric problems, it becomes

$$u_r=\frac{\partial}{\partial r}\left[\phi_{01}+\phi_{03}+\gamma_1\left(r\frac{\partial\phi_1}{\partial r}+z\frac{\partial\phi_3}{\partial z}\right)-\gamma_2\phi_1-\gamma_3\phi_3\right],$$
$$u_z=\frac{\partial}{\partial z}\left[k_1(\phi_{01}-\gamma_3\phi_1)+k_2(\phi_{03}-\gamma_2\phi_3)+\gamma_1\left(k_1r\frac{\partial\phi_1}{\partial r}+k_2z\frac{\partial\phi_3}{\partial z}\right)\right] \tag{6}$$

and the operator ∇_1^2 in (2) becomes

$$\nabla_1^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}. \tag{7}$$

The generalized Hooke's law of the transversely isotropic solids in the axi-symmetric problems is

$$\sigma_{rr}=c_{11}\varepsilon_{rr}+c_{12}\varepsilon_{\theta\theta}+c_{13}\varepsilon_{zz},\quad \sigma_{\theta\theta}=c_{12}\varepsilon_{rr}+c_{11}\varepsilon_{\theta\theta}+c_{13}\varepsilon_{zz},$$
$$\sigma_{zz}=c_{13}\varepsilon_{rr}+c_{13}\varepsilon_{\theta\theta}+c_{33}\varepsilon_{zz},\quad \sigma_{zr}=2c_{44}\varepsilon_{zr} \tag{8}$$

where σ_{ij} and ε_{ij} are components of stress and strain, respectively. The strain components are expressed in the displacement components as

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\quad \varepsilon_{\theta\theta}=\frac{u_r}{r},\quad \varepsilon_{zz}=\frac{\partial u_z}{\partial z},\quad \varepsilon_{zr}=\frac{1}{2}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right). \tag{9}$$

3. Solution to a state of plane stress

We set the origin of the cylindrical coordinates on the extension of the middle plane of a thick annular plate. The desired solution to a state of plane stress is a solution that satisfies the following conditions:

$\sigma_{zz}=0$, $\sigma_{zr}=0$ for arbitary domains. (10)

We set the potential functions satisfying the first and third equations of (2) in the form

$$\phi_{01} = D_0 \ln r + D_0^* \left(\frac{r^2}{2} - \frac{z^2}{\nu_1} \right), \quad \phi_1 = F_0^* \left(\frac{r^2}{2} - \frac{z^2}{\nu_2} \right) \tag{11}$$

where D_0 to F_0^* are arbitrary constants. Expressions for the displacement components are obtained by substituting the potential functions (11) into (6). Furthermore, expressions for the stress components are obtained from (8) with the aid of (9) and the expressions for the displacement components. The expressions are

$$\begin{aligned}
&u_r = \frac{D_0}{r} + D_0^* r + F_0^* (2\gamma_1 - \gamma_2) r, \quad u_z = -\frac{2k_1}{\nu_1} D_0^* z + \frac{2k_1\gamma_3}{\nu_2} F_0^* z, \\
&\sigma_{rr} = -\frac{D_0}{r^2}(c_{11} - c_{12}) + D_0^* \left(c_{11} + c_{12} - \frac{2c_{13}}{\nu_1} k_1 \right) + F_0^* \left[(c_{11} + c_{12})(2\gamma_1 - \gamma_2) + 2c_{13} \frac{k_1\gamma_3}{\nu_2} \right], \\
&\sigma_{\theta\theta} = -\frac{D_0}{r^2}(c_{12} - c_{11}) + D_0^* \left(c_{12} + c_{11} - \frac{2c_{13}}{\nu_1} k_1 \right) + F_0^* \left[(c_{12} + c_{11})(2\gamma_1 - \gamma_2) + 2c_{13} \frac{k_1\gamma_3}{\nu_2} \right], \\
&\sigma_{zz} = 2D_0^* \left(c_{13} - c_{33} \frac{k_1}{\nu_1} \right) + 2F_0^* \left[c_{13}(2\gamma_1 - \gamma_2) + c_{33} \frac{k_1\gamma_3}{\nu_2} \right], \quad \sigma_{zr} = 0.
\end{aligned} \tag{12}$$

Imposing the first condition of (10) on σ_{zz} in (12), we obtain the following relationship between the arbitrary constants :

$$D_0^* = -F_0^* \frac{\nu_1}{\nu_2} \left\{ \frac{c_{13}}{c_{13}\nu_1 - c_{33}k_1} [\nu_1\gamma_3 + \nu_2(2\gamma_1 - \gamma_2)] - \gamma_3 \right\}. \tag{13}$$

If we use the following relationship and notation :

$$\nu_1\gamma_3 + \nu_2(2\gamma_1 - \gamma_2) = -\frac{2c_{44}\nu_2}{c_{11}\nu_2 - c_{44}}, \quad \gamma_5 = \frac{c_{13}}{c_{13}\nu_1 - c_{33}k_1}, \tag{14}$$

we have

$$D_0^* = F_0^* \frac{\nu_1}{\nu_2} \left(\frac{2c_{44}\nu_2\gamma_5}{c_{11}\nu_2 - c_{44}} + \gamma_3 \right). \tag{15}$$

By substituting relationships (14) and (15) into (12), we obtain the solution to the state of plane stress as the result :

$$\begin{aligned}
&u_r = \frac{D_0}{r} + F_0(\nu_1\gamma_5 - 1) r, \quad u_z = -2F_0 k_1 \gamma_5 z, \\
&\sigma_{rr} = -\frac{D_0}{r^2}(c_{11} - c_{12}) - F_0 \{ c_{11} + c_{12} - \gamma_5 [\nu_1(c_{11} + c_{12}) - 2c_{13}k_1] \}, \\
&\sigma_{\theta\theta} = -\frac{D_0}{r^2}(c_{12} - c_{11}) - F_0 \{ c_{12} + c_{11} - \gamma_5 [\nu_1(c_{12} + c_{11}) - 2c_{13}k_1] \}, \\
&\sigma_{zz} = \sigma_{zr} = 0
\end{aligned} \tag{16}$$

where F_0 is a new arbitrary constant and is expressed in

$$\frac{2c_{44}}{c_{11}\nu_2 - c_{44}} F_0^* = F_0. \tag{17}$$

4. Solution to a state of generalized plane stress

We denote the thickness of a thick annular plate by h and set the origin of the cylindrical coordinates on the extension of the middle plane of the plate. The desired solution to a state of generalized plane stress is a solution that satisfies the following conditions :

$$\sigma_{zz}=0,\quad \sigma_{zr}=0 \quad \text{at} \quad z=\pm h/2. \tag{18}$$

We set the potential functions satisfying the second and fourth equations of (2) in the form

$$\begin{aligned}\phi_{03}&=A_0 z+A_0' z\ln r+A_0^{(3)}z\left(\frac{r^2}{2}-\frac{z^2}{3\nu_2}\right)+A_0^{(4)}z\left(r^2\ln r-r^2-\frac{2z^2}{3\nu_2}\ln r\right),\\ \phi_3&=C_0 z\ln r+C_0' z\left(\frac{r^2}{2}-\frac{z^2}{3\nu_1}\right)+C_0^* z\left(r^2\ln r-r^2-\frac{2z^2}{3\nu_1}\ln r\right)\end{aligned} \tag{19}$$

where A_0 to C_0^* are arbitrary constants. Expressions for the displacement components are obtained by substituting the potential functions (19) into (6). Furthermore, expressions for the stress components are obtained from (8) and (9). The expressions are

$$\begin{aligned}u_r&=rz\left[A_0^{(3)}+C_0'(\gamma_1-\gamma_3)\right]+A_0'\frac{z}{r}+A_0^{(4)}z\left(2r\ln r-r-\frac{2}{3\nu_2}\frac{z^2}{r}\right)+C_0(\gamma_1-\gamma_3)\frac{z}{r}\\&\quad+C_0^* z\left[(\gamma_1-\gamma_3)r(2r\ln r-1)-\frac{2(3\gamma_1-\gamma_3)}{3\nu_1}\frac{z^2}{r}\right],\end{aligned}$$

$$\begin{aligned}u_z&=A_0k_2+A_0^{(3)}k_2\left(\frac{r^2}{2}-\frac{z^2}{\nu_2}\right)+C_0'k_2\left(\frac{\gamma_1-\gamma_2}{2}r^2-\frac{3\gamma_1-\gamma_2}{\nu_1}z^2\right)+A_0'k_2\ln r\\&\quad+A_0^{(4)}k_2\left[r^2(\ln r-1)-\frac{2z^2}{\nu_2}\ln r\right]+C_0k_2(\gamma_1-\gamma_2)\ln r+C_0^*k_2\\&\quad\times\left[(\gamma_1-\gamma_2)r^2(\ln r-1)-\frac{2(3\gamma_1-\gamma_2)}{\nu_1}z^2\ln r\right],\end{aligned}$$

$$\begin{aligned}\sigma_{rr}&=A_0^{(3)}z\left(c_{11}+c_{12}-2c_{13}\frac{k_2}{\nu_2}\right)+C_0'z\left[(\gamma_1-\gamma_3)(c_{11}+c_{12})-2c_{13}\frac{k_2}{\nu_1}(3\gamma_1-\gamma_2)\right]\\&\quad-A_0'(c_{11}-c_{12})\frac{z}{r^2}+A_0^{(4)}z\left[2\left(c_{11}+c_{12}-2c_{13}\frac{k_2}{\nu_2}\right)\ln r+(c_{11}-c_{12})\left(1+\frac{2}{3\nu_2}\frac{z^2}{r^2}\right)\right]\\&\quad-C_0(\gamma_1-\gamma_3)(c_{11}-c_{12})\frac{z}{r^2}+C_0^*z\left\{2\left[(\gamma_1-\gamma_3)(c_{11}+c_{12})-2c_{13}\frac{k_2}{\nu_1}(3\gamma_1-\gamma_2)\right]\ln r\right.\\&\quad\left.+(c_{11}-c_{12})\left[\gamma_1-\gamma_3+\frac{2(3\gamma_1-\gamma_3)}{3\nu_1}\frac{z^2}{r^2}\right]\right\},\end{aligned} \tag{20}$$

$$\begin{aligned}\sigma_{\theta\theta}&=A_0^{(3)}z\left(c_{12}+c_{11}-2c_{13}\frac{k_2}{\nu_2}\right)+C_0'z\left[(\gamma_1-\gamma_3)(c_{12}+c_{11})-2c_{13}\frac{k_2}{\nu_1}(3\gamma_1-\gamma_2)\right]\\&\quad-A_0'(c_{12}-c_{11})\frac{z}{r^2}+A_0^{(4)}z\left[2\left(c_{12}+c_{11}-2c_{13}\frac{k_2}{\nu_2}\right)\ln r+(c_{12}-c_{11})\left(1+\frac{2}{3\nu_2}\frac{z^2}{r^2}\right)\right]\end{aligned}$$

$$-C_0(\gamma_1-\gamma_3)(c_{12}-c_{11})\frac{z}{r^2}+C_0^* z\left\{2\left[(\gamma_1-\gamma_3)(c_{12}+c_{11})-2c_{13}\frac{k_2}{\nu_1}(3\gamma_1-\gamma_2)\right]\ln r\right.$$
$$\left.+(c_{12}-c_{11})\left[\gamma_1-\gamma_3+\frac{2(3\gamma_1-\gamma_3)}{3\nu_1}\frac{z^2}{r^2}\right]\right\},$$

$$\sigma_{zz}=2A_0^{(3)}z\left(c_{13}-c_{33}\frac{k_2}{\nu_2}\right)+2C_0'z\left[c_{13}(\gamma_1-\gamma_3)-c_{33}k_2\frac{3\gamma_1-\gamma_2}{\nu_1}\right]+4A_0^{(4)}\left(c_{13}-c_{33}\frac{k_2}{\nu_2}\right)z\ln r$$
$$+4C_0^* z\left[c_{13}(\gamma_1-\gamma_3)-c_{33}\frac{k_2}{\nu_1}(3\gamma_1-\gamma_2)\right]\ln r,$$

$$\sigma_{zr}=c_{44}(1+k_2)\left\{A_0^{(3)}r+C_0'\left(\gamma_1-\frac{k_2\gamma_2+\gamma_3}{1+k_2}\right)r+\frac{A_0'}{r}+A_0^{(4)}\left[r(2\ln r-1)-\frac{2}{\nu_2}\frac{z^2}{r}\right]\right.$$
$$\left.+\frac{C_0}{r}\left(\gamma_1-\frac{k_2\gamma_2+\gamma_3}{1+k_2}\right)+C_0^*\left[r(2\ln r-1)\left(\gamma_1-\frac{k_2\gamma_2+\gamma_3}{1+k_2}\right)-\frac{2}{\nu_1}\frac{z^2}{r}\left(3\gamma_1-\frac{k_2\gamma_2+\gamma_3}{1+k_2}\right)\right]\right\}.$$

In the first place, if we impose the first condition of (18) on σ_{zz} in (20), we obtain

$$\begin{aligned}&\frac{A_0^{(3)}}{\nu_2}(c_{13}\nu_2-c_{33}k_2)+\frac{C_0'}{\nu_1}\left[c_{13}\nu_1(\gamma_1-\gamma_3)-c_{33}k_2(3\gamma_1-\gamma_2)\right]=0,\\&\frac{A_0^{(4)}}{\nu_2}(c_{13}\nu_2-c_{33}k_2)+\frac{C_0^*}{\nu_1}\left[c_{13}\nu_1(\gamma_1-\gamma_3)-c_{33}k_2(3\gamma_1-\gamma_2)\right]=0.\end{aligned}\tag{21}$$

From the above two equations, we obtain the following relationships :

$$\begin{aligned}A_0^{(3)}&=C_0'\frac{\nu_2}{\nu_1}\left\{\frac{c_{13}}{c_{13}\nu_2-c_{33}k_2}\left[\nu_1\gamma_3+\nu_2(2\gamma_1-\gamma_2)\right]-(3\gamma_1-\gamma_2)\right\},\\A_0^{(4)}&=C_0^*\frac{\nu_2}{\nu_1}\left\{\frac{c_{13}}{c_{13}\nu_2-c_{33}k_2}\left[\nu_1\gamma_3+\nu_2(2\gamma_1-\gamma_2)\right]-(3\gamma_1-\gamma_2)\right\}.\end{aligned}\tag{22}$$

In the second place, if we impose the second condition of (18) on σ_{zr} in (20) and use the relationships (14), (22) and

$$\gamma_2-\gamma_3=\frac{2c_{11}\nu_2}{c_{11}\nu_2-c_{44}},\quad k_1k_2=1,\quad \nu_1\nu_2=\frac{c_{33}}{c_{11}},\tag{23}$$

we obtain

$$\begin{aligned}&A_0^{(3)}+C_0'\left(\gamma_1-\frac{k_2\gamma_2+\gamma_3}{1+k_2}\right)=0,\quad A_0^{(4)}+C_0^*\left(\gamma_1-\frac{k_2\gamma_2+\gamma_3}{1+k_2}\right)=0,\\&A_0'+C_0\left(\gamma_1-\frac{k_2\gamma_2+\gamma_3}{1+k_2}\right)-\frac{C_0^*h^2}{\nu_1}\frac{c_{44}\nu_2}{c_{11}\nu_2-c_{44}}\left[\frac{c_{11}}{c_{44}(1+k_2)}-\frac{c_{13}}{c_{13}\nu_2-c_{33}k_2}\right]=0.\end{aligned}\tag{24}$$

From the third equation of (24), we obtain

$$A_0'=-C_0\left(\gamma_1-\frac{k_2\gamma_2+\gamma_3}{1+k_2}\right)+\frac{C_0^*h^2}{\nu_1}\frac{c_{44}\nu_2}{c_{11}\nu_2-c_{44}}\left[\frac{c_{11}}{c_{44}(1+k_2)}-\frac{c_{13}}{c_{13}\nu_2-c_{33}k_2}\right].\tag{25}$$

Thus, relationships (22) and (25) among the arbitrary constants satisfied the conditions of generalized plane stress, i.e., (18). If we substitute relationships (14) and (22) to (25) into (20) and use the following notation :

$$\gamma_6 = \frac{c_{13}}{c_{13}\nu_2 - c_{33}k_2}, \quad \nu_4 = \frac{c_{11}}{c_{44}}, \tag{26}$$

we obtain the solution to the state of generalized plane stress as the result :

$$\begin{aligned}
u_r &= -z\left\{\frac{A_0^{(2)}}{r} - C_0^{(1)} r(\nu_2\gamma_6 - 1) - C_0^{(2)}\left[r(2\ln r - 1)(\nu_2\gamma_6 - 1) - \frac{2z^2}{3r}(\gamma_6 - \nu_4)\right]\right\},\\
u_z &= A_0^{(1)} + A_0^{(2)}\ln r + \frac{k_2}{2}C_0^{(1)}\left\{r^2(\nu_2\gamma_6 - 1 + \nu_1\nu_4) + \frac{h^2}{2k_2}\left[\gamma_6(1+k_2) - \nu_4\right] - 2\gamma_6 z^2\right\}\\
&\quad + k_2 C_0^{(2)}\left\langle r^2(\ln r - 1)(\nu_2\gamma_6 - 1 + \nu_1\nu_4) + \left\{\frac{h^2}{2k_2}\left[\gamma_6(1+k_2) - \nu_4\right] - 2\gamma_6 z^2\right\}\ln r\right\rangle,\\
\sigma_{rr} &= z\left\langle \frac{A_0^{(2)}}{r^2}(c_{11} - c_{12}) + C_0^{(1)}\left\{\gamma_6\left[\nu_2(c_{11} + c_{12}) - 2c_{13}k_2\right] - (c_{11} + c_{12})\right\}\right.\\
&\quad \left. + C_0^{(2)}\left\{2\left[\gamma_6\left\{\nu_2(c_{11} + c_{12}) - 2c_{13}k_2\right\} - (c_{11} + c_{12})\right]\ln r + (c_{11} - c_{12})\left[\nu_2\gamma_6 - 1 + \frac{2z^2}{3r^2}(\gamma_6 - \nu_4)\right]\right\}\right\rangle,\\
\sigma_{\theta\theta} &= z\left\langle \frac{A_0^{(2)}}{r^2}(c_{12} - c_{11}) + C_0^{(1)}\left\{\gamma_6\left[\nu_2(c_{12} + c_{11}) - 2c_{13}k_2\right] - (c_{12} + c_{11})\right\}\right.\\
&\quad \left. + C_0^{(2)}\left\{2\left[\gamma_6\left\{\nu_2(c_{12} + c_{11}) - 2c_{13}k_2\right\} - (c_{12} + c_{11})\right]\ln r + (c_{12} - c_{11})\left[\nu_2\gamma_6 - 1 + \frac{2z^2}{3r^2}(\gamma_6 - \nu_4)\right]\right\}\right\rangle,\\
\sigma_{zz} &= 0,\\
\sigma_{zr} &= \frac{c_{44}C_0^{(2)}}{2}\left[\gamma_6(1+k_2) - \nu_4\right]\frac{h^2 - 4z^2}{r}
\end{aligned} \tag{27}$$

where $A_0^{(1)}, A_0^{(2)}, C_0^{(1)}$ and $C_0^{(2)}$ are new arbitrary constants and are expressed in

$$\begin{aligned}
&A_0 k_2 - \frac{C_0' h^2}{2\nu_1}\frac{c_{44}\nu_2}{c_{11}\nu_2 - c_{44}}\left[\nu_4 - \gamma_6(1+k_2)\right] = A_0^{(1)},\\
&\frac{2c_{44}\nu_2}{c_{11}\nu_2 - c_{44}}\left[\frac{C_0\nu_4 k_2}{1+k_2} + \frac{C_0^* h^2}{2\nu_1}\left(\frac{\nu_4}{1+k_2} - \gamma_6\right)\right] = -A_0^{(2)},\\
&-\frac{2c_{44}\nu_2}{\nu_1(c_{11}\nu_2 - c_{44})}C_0' = C_0^{(1)}, \quad -\frac{2c_{44}\nu_2}{\nu_1(c_{11}\nu_2 - c_{44})}C_0^* = C_0^{(2)}.
\end{aligned} \tag{28}$$

5. Application of the present solutions to the axi-symmetric bending of a transversely isotropic, thick annular plate

In this article, we will consider an application of the two solutions (16) and (27) to the axi-symmetric bending of a transversely isotropic, thick annular plate. Since the stress component σ_{zz} in the solutions is identically zero, the solutions cannot satisfy some of the loading conditions at the top face of the plate. However, they can be used for a homogeneous solution satisfying the boundary conditions at the edges. Therefore, a particular solution satisfying the loading conditions is needed and will be deduced from a part of the generalized Elliott solution. We consider a plate with thickness h, inner radius a, outer radius b and width $d=b-a$, subjected to a uniformly distributed annular load over the top face, as shown in Fig. 1.

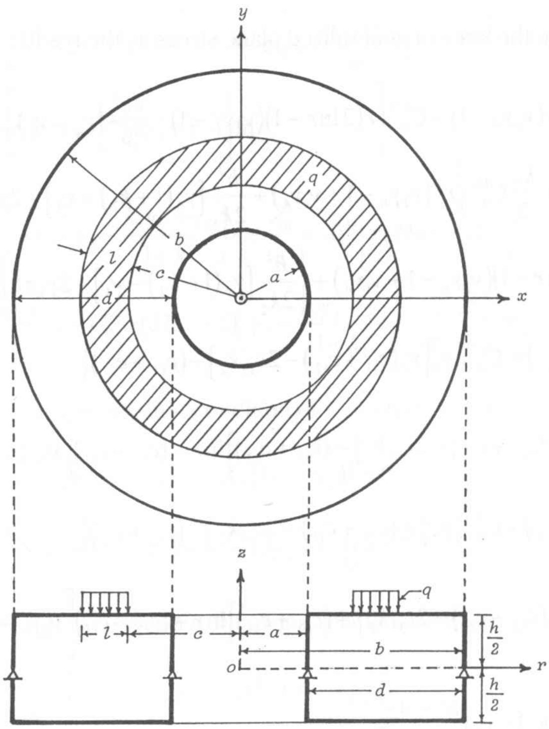

Fig. 1 Coordinate system of thick annular plate

5.1 Particular solution

If we solve the second and fourth equations of (2) by the method of separation of variables, we obtain

$$\begin{aligned}\phi_{03} &= \sum_{s=1}^{\infty}\left[J_0(\alpha_s r)\left(A_s^{(1)}\cosh\frac{\alpha_s z}{\sqrt{\nu_2}}+L_s^{(1)}\sinh\frac{\alpha_s z}{\sqrt{\nu_2}}\right)+Y_0(\alpha_s r)\left(A_s^{(2)}\cosh\frac{\alpha_s z}{\sqrt{\nu_2}}+L_s^{(2)}\sinh\frac{\alpha_s z}{\sqrt{\nu_2}}\right)\right],\\ \phi_3 &= \sum_{s=1}^{\infty}\left[J_0(\alpha_s r)\left(C_s^{(1)}\cosh\frac{\alpha_s z}{\sqrt{\nu_1}}+M_s^{(1)}\sinh\frac{\alpha_s z}{\sqrt{\nu_1}}\right)+Y_0(\alpha_s r)\left(C_s^{(2)}\cosh\frac{\alpha_s z}{\sqrt{\nu_1}}+M_s^{(2)}\sinh\frac{\alpha_s z}{\sqrt{\nu_1}}\right)\right]\end{aligned} \tag{29}$$

where $A_s^{(1)}$ to $M_s^{(2)}$ are arbitrary constants to be determined from the loading conditions and $\alpha_s=\lambda_s/a$. Expressions for the displacement components are obtained by substituting the potential functions (29) into (6). Furthermore, expressions for the stress components are obtained from (8) and (9). For example, the expressions for the stress components σ_{zz} and σ_{zr} are

$$\sigma_{zz} = -\sum_{s=1}^{\infty} \alpha_s^2 \left[J_0(\alpha_s r) f_7(z) + Y_0(\alpha_s r) g_7(z) \right],$$
$$\sigma_{zr} = -\frac{c_{44}(1+k_2)}{\sqrt{\nu_2}} \sum_{s=1}^{\infty} \alpha_s^2 \left[J_1(\alpha_s r) f_8(z) + Y_1(\alpha_s r) g_8(z) \right] \tag{30}$$

where

$$f_7(z) = A_s^{(1)}\left(c_{13} - c_{33}\frac{k_2}{\nu_2}\right)\cosh\frac{\alpha_s z}{\sqrt{\nu_2}} + L_s^{(1)}\left(c_{13} - c_{33}\frac{k_2}{\nu_2}\right)\sinh\frac{\alpha_s z}{\sqrt{\nu_2}}$$
$$- C_s^{(1)}\left\{\left[c_{13}\gamma_3 + c_{33}\frac{k_2}{\nu_1}(2\gamma_1 - \gamma_2)\right]\cosh\frac{\alpha_s z}{\sqrt{\nu_1}} - \gamma_1\left(c_{13} - c_{33}\frac{k_2}{\nu_1}\right)\frac{\alpha_s z}{\sqrt{\nu_1}}\sinh\frac{\alpha_s z}{\sqrt{\nu_1}}\right\}$$
$$- M_s^{(1)}\left\{\left[c_{13}\gamma_3 + c_{33}\frac{k_2}{\nu_1}(2\gamma_1 - \gamma_2)\right]\sinh\frac{\alpha_s z}{\sqrt{\nu_1}} - \gamma_1\left(c_{13} - c_{33}\frac{k_2}{\nu_1}\right)\frac{\alpha_s z}{\sqrt{\nu_1}}\cosh\frac{\alpha_s z}{\sqrt{\nu_1}}\right\}, \tag{31}$$

$$f_8(z) = A_s^{(1)}\sinh\frac{\alpha_s z}{\sqrt{\nu_2}} + L_s^{(1)}\cosh\frac{\alpha_s z}{\sqrt{\nu_2}} + C_s^{(1)}\sqrt{\frac{\nu_2}{\nu_1}}\left[\left(\gamma_1 - \frac{k_2\gamma_2 + \gamma_3}{1+k_2}\right)\sinh\frac{\alpha_s z}{\sqrt{\nu_1}} + \gamma_1\frac{\alpha_s z}{\sqrt{\nu_1}}\cosh\frac{\alpha_s z}{\sqrt{\nu_1}}\right]$$
$$+ M_s^{(1)}\sqrt{\frac{\nu_2}{\nu_1}}\left[\left(\gamma_1 - \frac{k_2\gamma_2 + \gamma_3}{1+k_2}\right)\cosh\frac{\alpha_s z}{\sqrt{\nu_1}} + \gamma_1\frac{\alpha_s z}{\sqrt{\nu_1}}\sinh\frac{\alpha_s z}{\sqrt{\nu_1}}\right]$$

and $g_7(z)$ and $g_8(z)$ are obtained when superscript (1) of the arbitrary constants in $f_7(z)$ and $f_8(z)$ is altered into superscript (2).

We consider the plate whose top face is subjected to the uniformly distributed annular load and whose bottom face is free from surface tractions. The loading conditions for that case become

$$\sigma_{zr} = 0, \quad \sigma_{zz} = -p(r) \quad \text{at} \quad z = h/2,$$
$$\sigma_{zr} = 0, \quad \sigma_{zz} = 0 \quad \text{at} \quad z = -h/2 \tag{32}$$

where

$$p(r) = \begin{cases} q & \text{for} \quad c < r < c + l \\ 0 & \text{for} \quad a \le r < c \quad \text{or} \quad c + l < r \le b. \end{cases} \tag{33}$$

In the first place, imposing the first and third conditions of (32) on σ_{zr} in (30), we obtain

$$L_s^{(1)} = -M_s^{(1)}\frac{\cosh\zeta_1}{\cosh\zeta_2}\sqrt{\frac{\nu_2}{\nu_1}}\left(\gamma_1 - \frac{k_2\gamma_2 + \gamma_3}{1+k_2} + \gamma_1\zeta_1\tanh\zeta_1\right),$$
$$A_s^{(1)} = -C_s^{(1)}\frac{\sinh\zeta_1}{\sinh\zeta_2}\sqrt{\frac{\nu_2}{\nu_1}}\left(\gamma_1 - \frac{k_2\gamma_2 + \gamma_3}{1+k_2} + \gamma_1\zeta_1\coth\zeta_1\right),$$
$$L_s^{(2)} = -M_s^{(2)}\frac{\cosh\zeta_1}{\cosh\zeta_2}\sqrt{\frac{\nu_2}{\nu_1}}\left(\gamma_1 - \frac{k_2\gamma_2 + \gamma_3}{1+k_2} + \gamma_1\zeta_1\tanh\zeta_1\right),$$
$$A_s^{(2)} = -C_s^{(2)}\frac{\sinh\zeta_1}{\sinh\zeta_2}\sqrt{\frac{\nu_2}{\nu_1}}\left(\gamma_1 - \frac{k_2\gamma_2 + \gamma_3}{1+k_2} + \gamma_1\zeta_1\coth\zeta_1\right) \tag{34}$$

where

$$\zeta_1 = \frac{\alpha_s h}{2\sqrt{\nu_1}}, \quad \zeta_2 = \frac{\alpha_s h}{2\sqrt{\nu_2}}. \tag{35}$$

By substituting relationships (34) into σ_{zz} in (30), we obtain

$$\sigma_{zz}=\sum_{s=1}^{\infty}\alpha_s^2\left\{\left[C_s^{(1)}J_0(\alpha_s r)+C_s^{(2)}Y_0(\alpha_s r)\right]h_1(z)+\left[M_s^{(1)}J_0(\alpha_s r)+M_s^{(2)}Y_0(\alpha_s r)\right]h_2(z)\right\}, \tag{36}$$

where

$$\begin{aligned} h_1(z)&=\frac{\sinh\zeta_1}{\sinh\zeta_2}\sqrt{\frac{\nu_2}{\nu_1}}\left(\gamma_1-\frac{k_2\gamma_2+\gamma_3}{1+k_2}+\gamma_1\zeta_1\coth\zeta_1\right)\left(c_{13}-c_{33}\frac{k_2}{\nu_2}\right)\cosh\frac{\alpha_s z}{\sqrt{\nu_2}}\\ &+\left[c_{13}\gamma_3+c_{33}\frac{k_2}{\nu_1}(2\gamma_1-\gamma_2)\right]\cosh\frac{\alpha_s z}{\sqrt{\nu_1}}-\gamma_1\left(c_{13}-c_{33}\frac{k_2}{\nu_1}\right)\frac{\alpha_s z}{\sqrt{\nu_1}}\sinh\frac{\alpha_s z}{\sqrt{\nu_1}},\\ h_2(z)&=\frac{\cosh\zeta_1}{\cosh\zeta_2}\sqrt{\frac{\nu_2}{\nu_1}}\left(\gamma_1-\frac{k_2\gamma_2+\gamma_3}{1+k_2}+\gamma_1\zeta_1\tanh\zeta_1\right)\left(c_{13}-c_{33}\frac{k_2}{\nu_2}\right)\sinh\frac{\alpha_s z}{\sqrt{\nu_2}}\\ &+\left[c_{13}\gamma_3+c_{33}\frac{k_2}{\nu_1}(2\gamma_1-\gamma_2)\right]\sinh\frac{\alpha_s z}{\sqrt{\nu_1}}-\gamma_1\left(c_{13}-c_{33}\frac{k_2}{\nu_1}\right)\frac{\alpha_s z}{\sqrt{\nu_1}}\cosh\frac{\alpha_s z}{\sqrt{\nu_1}}. \end{aligned} \tag{37}$$

At this point, if we set

$$C_s^{(2)}=-\varepsilon_s C_s^{(1)},\quad M_s^{(2)}=-\varepsilon_s M_s^{(1)} \tag{38}$$

where

$$\varepsilon_s=\frac{J_0(\alpha_s a)}{Y_0(\alpha_s a)}=\frac{J_0(\lambda_s)}{Y_0(\lambda_s)} \tag{39}$$

and introduce cylinder functions defined as

$$C_0(\alpha_s r)=J_0(\alpha_s r)-\varepsilon_s Y_0(\alpha_s r),\quad C_1(\alpha_s r)=J_1(\alpha_s r)-\varepsilon_s Y_1(\alpha_s r), \tag{40}$$

we obtain

$$\sigma_{zz}=\sum_{s=1}^{\infty}\alpha_s^2 C_0(\alpha_s r)\left[C_s^{(1)}h_1(z)+M_s^{(1)}h_2(z)\right]. \tag{41}$$

If λ_s is taken by the positive root of the following transcendental equation :

$$J_0(\kappa\lambda_s)Y_0(\lambda_s)-J_0(\lambda_s)Y_0(\kappa\lambda_s)=0,\quad \kappa=b/a, \tag{42}$$

the cylinder function $C_0\,(\alpha_s r)$ holds

$$C_0(\alpha_s a)=C_0(\lambda_s)=0,\quad C_0(\alpha_s b)=C_0(\kappa\lambda_s)=0. \tag{43}$$

Expanding $p(r)$ in (33) into Bessel series under (43), we obtain

$$p(r)=\sum_{s=1}^{\infty}t_s C_0(\alpha_s r) \tag{44}$$

where

$$t_s=\frac{\int_a^b p(r)rC_0(\alpha_s r)dr}{\int_a^b rC_0^2(\alpha_s r)dr}=\frac{2q}{\lambda_s\left[\kappa^2C_1^2(\kappa\lambda_s)-C_1^2(\lambda_s)\right]}\left(\frac{c}{a}\right)\left\{\left(1+\frac{l}{c}\right)C_1\left[\frac{\lambda_s}{a}(c+l)\right]-C_1\left(\lambda_s\frac{c}{a}\right)\right\}. \tag{45}$$

In the second place, imposing the second and fourth conditions of (32) on σ_{zz} in (41) with the aid of (44), we obtain

$$C_s^{(1)} = \frac{t_s}{2\alpha_s^2 i_1 \sinh\zeta_1}, \quad M_s^{(1)} = \frac{t_s}{2\alpha_s^2 j_1 \cosh\zeta_1} \tag{46}$$

where

$$\begin{aligned} i_1 &= \gamma_1\zeta_1\left(c_{13} - c_{33}\frac{k_2}{\nu_1}\right) - \sqrt{\frac{\nu_2}{\nu_1}}\left(c_{13} - c_{33}\frac{k_2}{\nu_2}\right)\left[\gamma_1(1+\zeta_1\coth\zeta_1) - \frac{k_2\gamma_2+\gamma_3}{1+k_2}\right]\coth\zeta_2 \\ &\quad - \left[c_{13}\gamma_3 + c_{33}\frac{k_2}{\nu_1}(2\gamma_1 - \gamma_2)\right]\coth\zeta_1, \\ j_1 &= \gamma_1\zeta_1\left(c_{13} - c_{33}\frac{k_2}{\nu_1}\right) - \sqrt{\frac{\nu_2}{\nu_1}}\left(c_{13} - c_{33}\frac{k_2}{\nu_2}\right)\left[\gamma_1(1+\zeta_1\tanh\zeta_1) - \frac{k_2\gamma_2+\gamma_3}{1+k_2}\right]\tanh\zeta_2 \\ &\quad - \left[c_{13}\gamma_3 + c_{33}\frac{k_2}{\nu_1}(2\gamma_1 - \gamma_2)\right]\tanh\zeta_1. \end{aligned} \tag{47}$$

By substituting (38) and (46) into (34), the arbitrary constants are determined in

$$A_s^{(1)} = -\frac{i_2 t_s}{2\alpha_s^2\sinh\zeta_2}, \quad L_s^{(1)} = -\frac{j_2 t_s}{2\alpha_s^2\cosh\zeta_2}, \quad A_s^{(2)} = \frac{\varepsilon_s i_2 t_s}{2\alpha_s^2\sinh\zeta_2}, \quad L_s^{(2)} = \frac{\varepsilon_s j_2 t_s}{2\alpha_s^2\cosh\zeta_2} \tag{48}$$

where

$$\begin{aligned} i_2 &= \frac{1}{i_1}\sqrt{\frac{\nu_2}{\nu_1}}\left[\gamma_1(1+\zeta_1\coth\zeta_1) - \frac{k_2\gamma_2+\gamma_3}{1+k_2}\right], \\ j_2 &= \frac{1}{j_1}\sqrt{\frac{\nu_2}{\nu_1}}\left[\gamma_1(1+\zeta_1\tanh\zeta_1) - \frac{k_2\gamma_2+\gamma_3}{1+k_2}\right]. \end{aligned} \tag{49}$$

Thus, all the arbitrary constants included in the particular solution were exactly determined by (46), (48) and (38). Therefore, the particular solution can be expressed in the closed form. For example, the expression for the stress component σ_{rr} is

$$\begin{aligned} \sigma_{rr} = -\frac{1}{2}\sum_{s=1}^{\infty} t_s \Bigg\langle &\left[(c_{11}-c_{12})\frac{C_1(\alpha_s r)}{\alpha_s r} - \left(c_{11} - c_{13}\frac{k_2}{\nu_2}\right)C_0(\alpha_s r)\right]\left(i_2\frac{\cosh n_2 z}{\sinh\zeta_2} + j_2\frac{\sinh n_2 z}{\cosh\zeta_2}\right) \\ &+ \left\{\gamma_3(c_{11}-c_{12})\frac{C_1(\alpha_s r)}{\alpha_s r} - \left[c_{11}\gamma_3 + c_{13}\frac{k_2}{\nu_1}(2\gamma_1-\gamma_2)\right]C_0(\alpha_s r)\right\}\left(\frac{\cosh n_1 z}{i_1\sinh\zeta_1} + \frac{\sinh n_1 z}{j_1\cosh\zeta_1}\right) \\ &- \gamma_1\left[(c_{11}-c_{12})\frac{C_1(\alpha_s r)}{\alpha_s r} - \left(c_{11} - c_{13}\frac{k_2}{\nu_1}\right)C_0(\alpha_s r)\right] n_1 z\left(\frac{\sinh n_1 z}{i_1\sinh\zeta_1} + \frac{\cosh n_1 z}{j_1\cosh\zeta_1}\right)\Bigg\rangle \end{aligned} \tag{50}$$

where $n_1 = \alpha_s/\sqrt{\nu_1}$ and $n_2 = \alpha_s/\sqrt{\nu_2}$.

Since boundary conditions at the edges cannot be prescribed by the stress component, stress resultants and stress couples are needed. They are defined as

$$T_{rr} = \int_{-h/2}^{h/2}\sigma_{rr}dz, \quad Q_r = \int_{-h/2}^{h/2}\sigma_{rz}dz, \quad M_r = \int_{-h/2}^{h/2}z\sigma_{rr}dz, \quad M_\theta = \int_{-h/2}^{h/2}z\sigma_{\theta\theta}dz. \tag{51}$$

By substituting solutions (16), (27) and (50) into (51), we obtain the expressions for the stress resultants and the stress couples.

5.2 System of linear algebraic equations

We consider the two circular edges of the plate simply supported. The boundary conditions for that case become

$$T_{rr}=0,\quad (u_z)_{z=0}=0,\quad M_r=0 \quad \text{at } r=a \text{ and } r=b. \tag{52}$$

From the first condition of (52), we obtain a system of linear algebraic equations with D_0 and F_0 in the form

$$\begin{aligned}
&\frac{D_0}{a^2}(c_{11}-c_{12})+F_0\{c_{11}+c_{12}-\gamma_5[\nu_1(c_{11}+c_{12})-2c_{13}k_1]\}=-\sqrt{\nu_1}\frac{a}{h}(c_{11}-c_{12})\sum_{s=1}^{\infty}\frac{t_s}{\lambda_s^2}C_1(\lambda_s)m_1,\\
&\frac{D_0}{b^2}(c_{11}-c_{12})+F_0\{c_{11}+c_{12}-\gamma_5[\nu_1(c_{11}+c_{12})-2c_{13}k_1]\}=-\frac{\sqrt{\nu_1}}{\kappa}\frac{a}{h}(c_{11}-c_{12})\sum_{s=1}^{\infty}\frac{t_s}{\lambda_s^2}C_1(\kappa\lambda_s)m_1
\end{aligned} \tag{53}$$

where

$$m_1=i_2\sqrt{\frac{\nu_2}{\nu_1}}+\frac{1}{i_1}[\gamma_3-\gamma_1(\xi_1\coth\xi_1-1)]. \tag{54}$$

From the second and third conditions of (52), we obtain a system of linear algebraic equations with $\overline{A}_0^{(1)}, A_0^{(2)}, \overline{C}_0^{(1)}$ and $C_0^{(2)}$ in the form

$$\begin{aligned}
&\overline{A}_0^{(1)}+\frac{k^2}{2}\overline{C}_0^{(1)}\left\{a^2(\nu_2\gamma_6-1+\nu_1\nu_4)+\frac{h^2}{2k_2}[\gamma_6(1+k_2)-\nu_4]\right\}-k_2C_0^{(2)}a^2(\nu_2\gamma_6-1+\nu_1\nu_4)=0,\\
&\overline{A}_0^{(1)}+A_0^{(2)}\ln\kappa+\frac{k^2}{2}\overline{C}_0^{(1)}\left\{b^2(\nu_2\gamma_6-1+\nu_1\nu_4)+\frac{h^2}{2k_2}[\gamma_6(1+k_2)-\nu_4]\right\}\\
&\qquad+k_2C_0^{(2)}\left\{b^2(\ln\kappa-1)(\nu_2\gamma_6-1+\nu_1\nu_4)+\frac{h^2}{2k_2}[\gamma_6(1+k_2)-\nu_4]\ln\kappa\right\}=0,\\
&\frac{A_0^{(2)}}{a^2}(c_{11}-c_{12})+\overline{C}_0^{(1)}\{\gamma_6[\nu_2(c_{11}+c_{12})-2c_{13}k_2]-(c_{11}+c_{12})\}+C_0^{(2)}(c_{11}-c_{12})\\
&\qquad\times\left[\nu_2\gamma_6-1+\frac{h^2}{10a^2}(\gamma_6-\nu_4)\right]=\frac{12a^2}{h^3}\nu_1(c_{11}-c_{12})\sum_{s=1}^{\infty}\frac{t_s}{\lambda_s^3}C_1(\lambda_s)m_2,\\
&\frac{A_0^{(2)}}{b^2}(c_{11}-c_{12})+\overline{C}_0^{(1)}\{\gamma_6[\nu_2(c_{11}+c_{12})-2c_{13}k_2]-(c_{11}+c_{12})\}\\
&\qquad+C_0^{(2)}\left\langle 2\{\gamma_6[\nu_2(c_{11}+c_{12})-2c_{13}\nu_2]-(c_{11}+c_{12})\}\ln\kappa+(c_{11}-c_{12})\left[\nu_2\gamma_6-1+\frac{h^2}{10b^2}(\gamma_6-\nu_4)\right]\right\rangle\\
&\qquad=\frac{12a^2}{h^3}\frac{\nu_1}{\kappa}(c_{11}-c_{12})\sum_{s=1}^{\infty}\frac{t_s}{\lambda_s^3}C_1(\kappa\lambda_s)m_2
\end{aligned} \tag{55}$$

where

$$m_2=j_2\frac{\nu_2}{\nu_1}(\xi_2-\tanh\xi_2)+\frac{1}{j_1}\{\gamma_3(\xi_1-\tanh\xi_1)-\gamma_1[(\xi_1^2+2)\tanh\xi_1-2\xi_1]\} \tag{56}$$

and $\overline{A}_0^{(1)}$ and $\overline{C}_0^{(1)}$ are new arbitrary constants and are expressed in

$$A_0^{(1)} + A_0^{(2)} \ln a = \overline{A}_0^{(1)}, \quad C_0^{(1)} + 2C_0^{(2)} \ln a = \overline{C}_0^{(1)}. \tag{57}$$

The systems of (53) and (55) can be numerically or exactly solved. In order to facilitate numerical calculations, it is convenient to replace the arbitrary constants previously used with the following ones :

$$\begin{aligned} &\frac{c_{44} D_0}{qa^2} = \overline{D}_0, \quad \frac{c_{44} F_0}{q} = \overline{F}_0, \quad \frac{c_{44} \overline{A}_0^{(1)}}{qa} = A_0^*, \quad \frac{c_{44} A_0^{(2)}}{qa} = \overline{A}_0^{(2)}, \\ &\frac{c_{44} a \overline{C}_0^{(1)}}{q} = \overline{C}_0^*, \quad \frac{c_{44} a C_0^{(2)}}{q} = \overline{C}_0^{(2)}, \quad \frac{t_s}{q} = \bar{t}_s. \end{aligned} \tag{58}$$

Once the arbitrary constants are determined, the values of the displacements, the stresses, the stress resultants and the stress couples can be completely determined.

5. 3 Numerical results

Numerical calculations were made for transversely isotropic and isotropic, thick annular plates with $b/a=6.0$, $c/a=3.0$, $l/a=1.0$ and various thickness-width ratios $e=h/d$. Graphite epoxy and E glass epoxy, as examples of transversely isotropic materials, and an isotropic material with Poisson's ratio $\nu=0.3$ were treated. The values of the elastic constants of these materials, as determined by Chen and Cheng [14,15], are given in Table 1. The roots of the transcendental equation (42) were calculated by the Regula-Falsi method. Numerical results were obtained by taking the first 70 terms for in the series. For the graphite epoxy, the variations of internal forces and stresses with the r-direction or the z-direction are shown in Figs. 2-8, respectively. Figure 2 shows the variations of shearing force Q_r and bending moments M_r and M_θ with the r-direction for $e=h/d=1/4$. The variation of Q_r is rapid for $3.0<r/a<4.0$ in the loaded area. Figure 3 shows the variation of σ_{rr} with the r-direction for $e=1/4$. The small values remain at $r/a=1.0$ and $r/a=6.0$, because the boundary conditions were prescribed by T_{rr} and M_r instead of σ_{rr} on the basis of Saint-Venant's principle. Figure 4 shows the variation of $\sigma_{\theta\theta}$ with the r-direction for $e=1/4$. The large values are yielded at $r/a=1.0$ due to stress concentration. Figures 5 and 6 show the variations of σ_{rr} and $\sigma_{\theta\theta}$ with the z-direction at $r/a=3.5$, respectively. The variations for $e=1/4$ differ widely from linear variations. Figure 7 shows the variation of σ_{zr} with the z-direction at $r/a=3.0$. The variation for $e=1/6$ nearly follows the parabolic law, but that for $e=1/4$ hardly follows the parabolic one. Figure 8 shows the remaining stresses σ_{rr} at $r=a$ and $r=b=6a$ for $e=1/4$. The value at $r/a=1.0$ is somewhat large due to stress concentration. Figures 3 and 8 demonstrate that the valid domain in the present method is $1.3a \leq r \leq 0.99b$ in case of $b/a=6.0$. Comparisons of the values of σ_{rr} at $r/a=3.5$, $\sigma_{\theta\theta}$ and σ_{zr} at $r/a=1.0$ among the graphite epoxy, the E glass epoxy and the isotropic material for $e=1/4$ are given in Table 2. The value of σ_{rr} at $z=0.5h$ for the graphite epoxy is 6.2% less than that in the isotropic material.

Table 1. Elastic constants c_{ij} (in units of 10Gpa)

c_{ij}	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}
Graphite-epoxy	0.82	0.26	0.32	8.68	0.41
E glass-epoxy	1.51	0.61	0.52	4.68	0.47
Isotropy	3.5	1.5	1.5	3.5	1.0

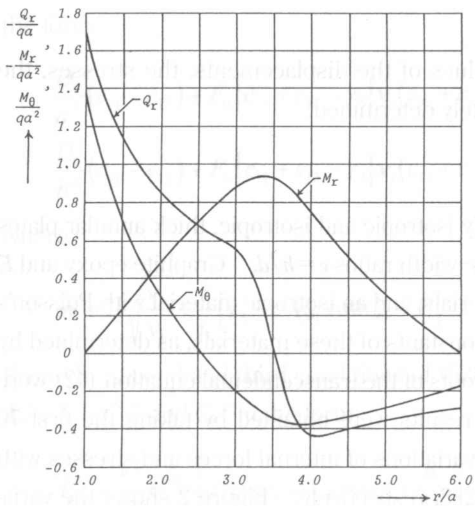

Fig. 2 Variations of internal forces in graphite epoxy ($e=h/d=1/4$)

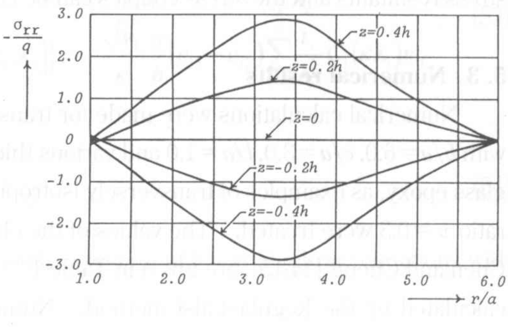

Fig. 3 Variations of σ_{rr} in graphite epoxy ($e=h/d=1/4$)

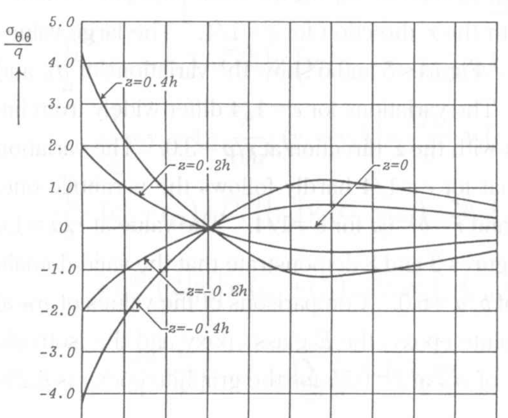

Fig. 4 Variations of $\sigma_{\theta\theta}$ in graphite epoxy ($e=h/d=1/4$)

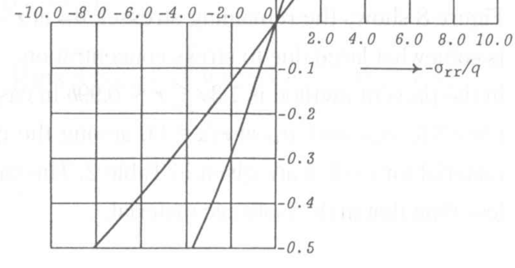

Fig. 5 Variations of σ_{rr} in graphite epoxy ($r/a=3.5, e=h/d$)

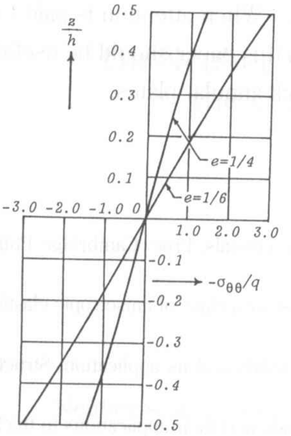

Fig. 6 Variations of $\sigma_{\theta\theta}$ in graphite epoxy ($r/a=3.5$, $e=h/d$)

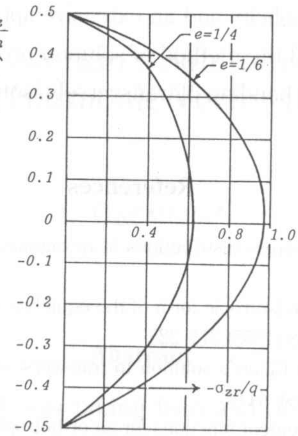

Fig. 7 Variations of σ_{zr} in graphite epoxy ($r/a=3.0$, $e=h/d$)

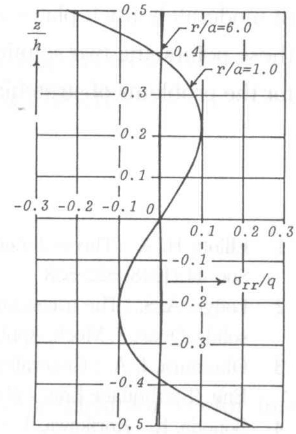

Fig. 8 Remaining stress σ_{rr} at $r=a$ and $r=b$ in graphite epoxy ($b=6a$, $e=h/d=1/4$)

Table 2. Comparison of stress values at $r=3.5a$ or $r=a$ for $e=1/4$

	σ_{rr}/q		$\sigma_{\theta\theta}/q$		σ_{zr}/q	
	$z=0.5h$	$z=0.2h$	$z=0.5h$	$z=0.2h$	$z=0$	$z=0.2h$
Graphite-epoxy	−3.76	−1.34	5.50	2.01	−2.05	−1.72
E glass-epoxy	−3.95	−1.32	5.42	1.91	−1.98	−1.67
Isotropy	−4.01	−1.29	5.36	1.92	−2.02	−1.70

6. Conclusions

From the generalized Elliott solution, axi-symmetric solutions to a state of plane stress and to a state of generalized plane stress in transversely isotropic, thick annular plates were deduced and were applied to axi-symmetric bending. The two solutions are exact elasticity solutions satisfying all the equations of equilibrium and compatibility, with no assumptions. Since the solutions are explicitly presented by components of displacement and stress, they may be simply applied to boundary-value problems. It has been confirmed that the solutions are coincident with the solutions in Love's theory when the elastic constants of transversely isotropic solids are replaced with those of isotropic solids. One method of solution for the problem of bending was proposed. The method is constituted by a homogeneous solution consisting of the two solutions and by a particular solution deduced from a part of the generalized Elliott solution. The particular solution makes three-dimensional supplements for the distribution of transverse shearing stress and for the influence of the transverse normal strain. In the present method, the loading conditions at the top and bottom faces of a thick plate are rigorously satisfied, whereas the boundary conditions at the edges are approximately satisfied by Saint-Venant's principle. The most important points of the theory of moderately thick plates are simplicity and an extensive applicability. When attention is paid to these points, the two solutions and the method of solution presented in this paper should be useful for the problems of stretching and bending of transversely isotropic, thick annular plates.

References

1 Elliott, H. A. : Three-dimensional stress distributions in hexagonal aeolotropic crystals. Proc. Cambridge Phil. Soc. 44 (1948) 522-533

2 Lodge, A. S. : The transformation to isotropic form of the equilibrium equations for a class of anisotropic elastic solids. Quart. J. Mech. Appl. Math. 8 (1955) 211-225

3 Okumura, I. A. : Generalization of Elliott's solution to transversely isotropic solids and its application. Struct. Eng./Earthquake Eng. 4 (1987) 401st-411s

4 Sonoda, K. ; Horikawa, T. : Displacement functions for an orthotropic elastic body and their applications to thick plate problems. Theor. Appl. Mech. 29 (1981) 117-126

5 Fan, J. ; Ye, J. : A series solution of the exact equation for thick orthotropic plates. Int. J. Solids Struct. 26 (1990) 773-778

6 Reissner, E. : A twelfth order theory of transverse bending of transversely isotropic plates. Z. angew. Math. Mech. 63 (1983) 285-289

7 Voyiadjis, G. Z. ; Baluch, M. H. : Refined theory for thick composite plates. Proc. ASCE, J. Eng. Mech. 114 (1988) 671-687

8 Wang, F.-Y. : Two-dimensional theories deduced from three-dimensional theory for a transversely isotropic body-I. Int. J. Solids Struct. 26 (1990) 455-470

9 Clark, R. A. ; Reissner, E. : A tenth-order theory of stretching of transversely isotropic sheets. J. Appl. Math. Phys. (ZAMP) 35 (1984) 883-889

10 Wang, F.-Y. : Two-dimensional theories deduced from three-dimensional theory for a transversely isotropic body-II. Int. J. Solids Struct. 28 (1991) 161-177

11 Okumura, I. A. : Solutions to stretching and bending of transversely isotropic, circular thick plates and their application. Struct. Eng./Earthquake Eng. 5 (1988) 293s-302s

12 Love, A .E. H. : A treatise on the mathematical theory of elasticity. 4th ed., New York : Dover pub. 1944

13 Saada, A. S. : Elasticity : theory and applications. New York : Pergamon Press Inc. 1974

14 Chen, C. H. ; Cheng, S. : Mechanical properties of anisotropic fiber-reinforced composites. Trans. ASME, J. Appl. Mech. 37 (1970) 186-189

15 Chen, C. H. ; Cheng, S. : Mechanical properties of fiber reinforced composites. J. Comp. Mater. 1 (1967) 30-41