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Global Weak Convergence of Successive Approximations for Nonlinear Ordinary Differential Equations in Banach Spaces

by Yasuhiko YABU* and Shigeo KATO**

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Abstract

The global existence of the solutions of the Cauchy problem

$$x' = f(t, x), \qquad x(0) = x_0 \in E$$

in a Banach space E are studied. We give a sufficient condition for the above equation to have a unique global solution, and prove that the successive approximations for the above equation converge weakly uniformly to the unique solution on any bounded interval of $[0, \infty)$.

1. Introduction

Let E be a Banach space with the dual space E^*. We denote by $\|\ \ \|$ the norm in E and E^*. In this paper we consider the Cauchy problem

$$\text{(CP)} \qquad x' = f(t, x), \qquad x(0) = x_0 \in E,$$

where f is an E-valued mapping defined on $[0, \infty) \times E$. By a solution u of (CP), we mean that u is strongly absolutely continuous on any bounded interval of $[0, \infty)$ and satisfies $u(0) = x_0$ and $u'(t) = f(t, u(t))$ for a. e. $t \in [0, \infty)$.

In the previous paper [5] we gave a weak local convergence theorem for the successive approximations for (CP). It is our object in this paper to give a weak global convergence theorem for the successive approximations for (CP) on $[0, \infty)$.

2. Main results

Let f be a mapping from $[0, \infty) \times E$ into E satisfying the following conditions:

(F_1) $f(\cdot, x)$ is locally strongly measurable in t for each $x \in E$, and for a. e. $t \in [0, \infty)$, $f(t, \cdot)$ is weakly continuous in x from E into E.

(F_2) There exists an $m \in L^1_{loc}[0, \infty)$ such that

$$\|f(t, x)\| \leqq m(t) \qquad \text{for a. e. } t \in [0, \infty) \text{ and all } x \in E.$$

We define the successive approximations for (CP) as follows:

$$\text{(2.1)} \qquad u_n(t) = x_0 + \int_0^t f\big(s, u_{n-1}(s)\big)\, ds \qquad (n \geqq 1),$$

where u_0 is an arbitrary continuous function from $[0, \infty)$ into E.

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* Department of General Education, Dōto College.
** Department of Common courses, Kitami Institute of Technology.

To show the weak convergence of $\{u_n\}$ we consider the following Kamke-type uniqueness function g satisfying the following conditions:

(G_1) $g=g(t,\tau)$ is a function from $(t_0, t_0+a]\times[0,2r]$ into $[0,\infty)$ which is measurable in t for each τ, and continuous nondecreasing in τ for each t.

(G_2) For each $\delta\in(0,a)$, $w\equiv 0$ is the only absolutely continuous function defined on $[t_0, t_0+\delta]$ which satisfies $w(t_0)=0$ and $w'(t)=g(t,w(t))$ for a. e. $t\in(t_0, t_0+\delta)$.

(G_3) There exists a function α defined on $(t_0, t_0+a]$ such that

$$g(t,\tau)\leqq\alpha(t) \qquad \text{for } (t,\tau)\in(t_0,t_0+a]\times[0,2r]$$

and $\alpha\in L^1(t_0+\varepsilon,\ t_0+a)$ for every $\varepsilon\in(0,a)$.

We proved in [5] the following local weak convergence theorem.

Theorem 1. Let E be a weakly sequentially complete Banach space. Suppose that f is an E-valued mapping defined on $[0,a]\times S(x_0,r)$ satisfying (F_1) and (F_2). Suppose furthermore that, for each $\phi\in E^*$, $\|\phi\|=1$, there exists a function g_ϕ satisfying (G_1)-(G_3) on $(0,a]\times[0,2r]$ such that

$$(2.2)\qquad |\langle f(t,x)-f(t,y),\phi\rangle|\leqq g_\phi(t,|\langle x-y,\phi\rangle|)$$

for each (t,x), $(t,y)\in(0,a]\times S(x_0,r)$. Then the successive approximations $\{u_n\}$ defined by (2. 1) converge weakly uniformly on some interval $[0,T]$ to a unique solution u of (CP). Here, we denote by $S(x_0,r)$ the closed ball of center x_0 with radius r and denote by $\langle x,\phi\rangle$ the value of ϕ at x.

Remark. In the previous paper [5] we assumed that E is weakly complete, that is, E is complete in the topology $\sigma(E,E^*)$. We used in [5], however, only the weakly sequential completeness of E. We note that E is weakly complete if and only if E is finite dimensional.

Now, we can state the following main results of this paper.

Theorem 2. Let E be a weakly sequentially complete Banach space. Suppose that (F_1) and (F_2) are satisfied and suppose furthermore that for each $\phi\in E^*$, $\|\phi\|=1$ and for each $t_0\geqq 0$ there exist positive constants a, r and a function g_ϕ satisfying (G_1)-(G_3) on $(t_0,\ t_0+a]\times[0,2r]$ such that

$$(2.3)\qquad |\langle f(t,x)-f(t,y),\phi\rangle|\leqq g_\phi(t,|\langle x-y,\phi\rangle|)$$

for each (t,x), $(t,y)\in(t_0,t_0+a]\times E$ whenever $|\langle x-y,\phi\rangle|\leqq 2r$. Then the successive approximations $\{u_n\}$ defined by (2. 1) converge weakly uniformly on any bounded interval of $[0,\infty)$ to a unique solution u of (CP).

Corollary. In Theorem 2, if f is weakly continuous on $[0,\infty)\times E$, then the conclusion of Theorem 2 holds true. Moreover, u is weakly continuously diferentiable on $[0,\infty)$ and satisfies (CP) for every $t\geqq 0$ in the sense of weak derivative.

3. Proof of main results

Before proving Theorem 2 we prepare the following two lemmas.

Lemma 3. 1. Let g satisfy the conditions (G_1), (G_2) and (G_3) on $(t_0, t_0+a]\times$

$[0, 2r]$, and let w be an absolutely continuous function from $[t_0, t_0+a]$ into $[0, 2r]$. Suppose furthermore that $w(t_0)=0$ and

$$w'(t) \leqq g(t, w(t)) \quad \text{for a. e. } t \in (t_0, t_0+a].$$

Then $w \equiv 0$ on $[t_0, t_0+a]$.

For a proof see [4].

Lemma 3. 2. Suppose that f satisfies the conditions (F_1) and (F_2). Then for each locally strongly measurable function z from $[0, \infty)$ into E, $f(t, z(t))$ is locally strongly measurable and locally Bochner integrable on $[0, \infty)$.

Proof. We omitted the proof of this lemma in the previous paper [5]. We give the proof here, however, for completeness. Let $T>0$ be fixed. Then, by (F_1), there exists a sequence of finitely valued functions z_n on $[0, T]$ and a set $N \subset [0, T]$ of measure zero such that

$$\lim_{n\to\infty} z_n(t) = z(t) \text{ and } \operatorname{weak-lim}_{n\to\infty} f\big(t, z_n(t)\big) = f\big(t, z(t)\big)$$

for $t \in [0, T]-N$. Since $f(\cdot, z_n(\cdot))$ is strongly measurable on $[0, T]$ for each $n \geqq 1$ by (F_1), $f(\cdot, z(\cdot))$ is weakly measurable on $[0, T]$. Since $\Gamma_0 = \bigcup_{n\geqq 1} \{f(t, z_n(t))\,;\, t \in [0, T]\}$ is separable by (F_1), the closed linear hull Γ of Γ_0 is also separable. Since $\{f(t, z(t))\,;\, t\in[0, T]-N\}$ is contained in the weak closure of Γ_0 and since Γ is also weakly closed, $f(\cdot, z(\cdot))$ is almost separably valued on $[0, T]$. Consequently, by Pettis's theorem, $f(\cdot, z(\cdot))$ is strongly measurable on $[0, T]$. It thus follows that $f(\cdot, z(\cdot))$ is locally strongly measurable on $[0, \infty)$. Moreover, (F_2) implies that $f(\cdot, z(\cdot))$ is locally Bochner integrable on $[0, \infty)$.

Proof of Theorem 2. Let $\{u_n\}$ be the sequence of successive approximations for (CP) defined by

$$(3.1) \qquad u_n(t) = x_0 + \int_0^t f\big(s, u_{n-1}(s)\big)\,ds \qquad (t \geqq 0,\ n \geqq 1)\,,$$

where u_0 is an arbitrary continuous function from $[0, \infty)$ into E and the integral is a Bochner sense.

It follows from (3. 1) and (F_2) that

$$\|u_n(t) - x_0\| \leqq \int_0^t \big\|f\big(s, u_{n-1}(s)\big)\big\|\,ds \leqq \int_0^t m(\tau)\,d\tau$$

for $t \geqq 0$ and $n \geqq 1$. And we have

$$(3.2) \qquad \|u_n(t) - u_n(s)\| \leqq \left|\int_s^t \big\|f\big(\tau, u_{n-1}(\tau)\big)\big\|\,d\tau\right| \leqq \left|\int_s^t m(\tau)\,d\tau\right| \leqq |M(t)-M(s)|$$

for $s, t \geqq 0$ and $n \geqq 1$, where $M(t) = \int_0^t m(\tau)\,d\tau$ for $t \geqq 0$.

For any $T>0$, letting $s=0$ in (3. 2) we have

$$\|u_n(t)\| \leqq \|u_n(0)\| + M(t) \leqq \|x_0\| + M(T) \qquad \text{for } t \in [0, T]\,,$$

and hence $\{u_n\}$ is equicontinuous and uniformly bounded on $[0, T]$. Let $A = \{t \geqq 0\,;\, \{u_n\}$ converges weakly uniformly on $[0, t]\}$ and let $t_0 = \sup A$. Then, by Theorem 1, $t_0 > 0$. We have only to show that $t_0 < +\infty$ leads to a contradiction.

Since $\{u_n\}$ is equicontinuous on $[0, T]$ for each $T \geqq t_0$, given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\|u_n(t) - u_n(t_0)\| < \varepsilon/3 \qquad \text{whenever } |t - t_0| \leqq \delta \text{ and } n \geqq 1.$$

By the definition of t_0, $\lim_{n\to\infty} \langle u_n(t_0 - \delta), \phi\rangle$ exists for each $\phi \in E^*$, $\|\phi\| = 1$, therefore, there exists an n_0 such that

$$|\langle u_n(t_0-\delta) - u_m(t_0-\delta), \phi\rangle| < \varepsilon/3 \qquad \text{for all } m,\ n \geqq n_0$$

and so that

$$|\langle u_n(t_0) - u_m(t_0), \phi\rangle| \leqq \|u_n(t_0) - u_n(t_0-\delta)\| + |\langle u_n(t_0-\delta) - u_m(t_0-\delta), \phi\rangle| + \|u_m(t_0-\delta) - u_m(t_0)\| < \varepsilon$$

for all m, $n \geqq n_0$. Consequently, $\lim_{n\to\infty} \langle u_n(t_0), \phi\rangle$ exists for all $\phi \in E^*$ and, so that, there exists $z_0 \in E$ such that $\{u_n(t_0)\}$ converges weakly to z_0 by the weakly sequential completeness of E.

Corresponding to t_0 and $\phi \in E^*$, $\|\phi\| = 1$ there exist positive constants a, r and a function g_ϕ satisfying the conditions (G_1)-(G_3) on $(t_0, t_0+a] \times [0, 2r]$ such that

$$|\langle f(t,x) - f(t,y), \phi\rangle| \leqq g_\phi(t, |\langle x-y, \phi\rangle|)$$

for a.e. $t \in (t_0, t_0+a]$ and x, $y \in E$ whenever $|\langle x-y, \phi\rangle| \leqq 2r$. By the equicontinuity of $\{u_n\}$ we can find an η such that $0 < \eta < \min\{a, r\}$, $\int_{t_0}^{t_0+\eta} m(t)\,dt \leqq r$ and $\|u_n(t) - u_n(t_0)\| \leqq r/2$ for all $t \in [t_0, t_0+\eta]$ and $n \geqq 1$. From now on, we will denote $I = [t_0, t_0+\eta]$. Since $\{u_n(t_0)\}$ converges weakly to z_0, there exists an n_0 such that $|\langle u_n(t_0) - z_0, \phi\rangle| \leqq r/2$ for all $n \geqq n_0$. It therefore follows that

$$|\langle u_n(t) - z_0, \phi\rangle| \leqq \|u_n(t) - u_n(t_0)\| + |\langle u_n(t_0) - z_0, \phi\rangle| \leqq r$$

for all $t \in I$ and $n \geqq n_0$, and so that

$$(3.3) \qquad |\langle u_m(t) - u_n(t), \phi\rangle| \leqq |\langle u_m(t) - z_0, \phi\rangle| + |\langle u_n(t) - z_0, \phi\rangle| \leqq 2r$$

for all $t \in I$ and m, $n \geqq n_0$.

For each $\phi \in E^*$, $\|\phi\| = 1$, we define the functions $w_{m,n}(\cdot\,;\phi)$ and $w_n(\cdot\,;\phi)$ by

$$w_{m,n}(t\,;\phi) = |\langle u_m(t) - u_n(t), \phi\rangle| \qquad (t \in I,\ m \geqq n \geqq 1)$$

and

$$w_n(t\,;\phi) = \sup\{w_{m,n}(t\,;\phi)\,;\ m \geqq n\} \qquad (t \in I,\ n \geqq 1).$$

Then by (3.2), we have

$$|w_{m,n}(t\,;\phi) - w_{m,n}(s\,;\phi)| = \big|\,|\langle u_m(t) - u_n(t), \phi\rangle| - |\langle u_m(s) - u_n(s), \phi\rangle|\,\big|$$
$$\leqq |\langle u_m(t) - u_m(s), \phi\rangle - \langle u_n(t) - u_n(s), \phi\rangle|$$
$$\leqq \|u_m(t) - u_m(s)\| + \|u_n(t) - u_n(s)\| \leqq 2|M(t) - M(s)|$$

and hence

$$|w_n(t\,;\,\phi)-w_n(s\,;\,\phi)|\leqq\sup\left\{|w_{m,n}(t\,;\,\phi)-w_{m,n}(s\,;\,\phi)|\,;\,m\geqq n\right\}$$
$$\leqq 2|M(t)-M(s)|$$

for all $s,\ t\in I$ and $n\geqq 1$. Thus we have the following.

(3. 4) $\quad w_{n+1}(t\,;\,\phi)\leqq w_n(t\,;\,\phi) \qquad (t\in I,\ n\geqq 1)$.

(3. 5) $\quad w_n(\cdot\,;\,\phi)$ is absolutely continuous on I for each $n\geqq 1$, and $\{w_n(\cdot\,;\,\phi)\}$ is equicontinuous and uniformly bounded on I.

(3. 6) $\quad \lim_{n\to\infty} w_n(t_0\,;\,\phi)=0$.

From (3. 4) there exists a function $w(\cdot\,;\,\phi)$ defined on I such that

$$w_n(t\,;\,\phi)\to w(t\,;\,\phi) \text{ pointwise on } I.$$

On the other hand, (3. 5) implies that the convergence $w_n(t\,;\,\phi)\to w(t\,;\,\phi)$ is uniformly on I. It follows that $w(\cdot\,;\,\phi)$ is absolutely continuous on I. Let t and Δt be such that $t,\ t+\Delta t\in I$. Then we have by (3. 1), (3. 3) and (G_1)

$$|w_{m+1,n+1}(t+\Delta t\,;\,\phi)-w_{m+1,n+1}(t\,;\,\phi)|$$
$$=|\langle u_{m+1}(t+\Delta t)-u_{m+1}(t),\phi\rangle-\langle u_{n+1}(t+\Delta t)-u_{n+1}(t),\phi\rangle|$$
$$=|\left\langle\int_t^{t+\Delta t} f\big(s,u_m(s)\big)\,ds,\phi\right\rangle-\left\langle\int_t^{t+\Delta t} f\big(s,u_n(s)\big)\,ds,\phi\right\rangle|$$
$$=\left|\int_t^{t+\Delta t}\left\langle f\big(s,u_m(s)\big)-f\big(s,u_n(s)\big),\phi\right\rangle ds\right|$$
$$\leqq\int_t^{t+\Delta t} g_\phi\big(s,w_{m,n}(s\,;\,\phi)\big)\,ds\leqq\int_t^{t+\Delta t} g_\phi\big(s,w_n(s\,;\,\phi)\big)\,ds.$$

Since $w_n(t\,;\,\phi)\to w(t\,;\,\phi)$ uniformly on I, we have by letting $n\to\infty$

$$|w(t+\Delta t\,;\,\phi)-w(t\,;\,\phi)|\leqq\int_t^{t+\Delta t} g_\phi\big(s,w(s\,;\,\phi)\big)\,ds.$$

Thus we have

$$|w'(t\,;\,\phi)|\leqq g_\phi(s,w(s\,;\,\phi)) \qquad \text{for a. e. } t\in I.$$

Since $w(t_0\,;\,\phi)=0$, we deduce now that $w(t\,;\,\phi)\equiv 0$ on I by Lemma 3. 1, and this implies that $\{u_n\}$ is weakly Cauchy uniformly on I. By virtue of the weakly sequential completeness of E, $\{u_n\}$ converges weakly uniformly on I, which contradicts the definition of t_0.

We next show that u is a unique solution of (CP) on $[0,\infty)$. Let $\phi\in E^*$, then

$$\langle u_{n+1}(t),\phi\rangle=\langle x_0,\phi\rangle+\int_0^t\left\langle f\big(s,u_n(s)\big),\phi\right\rangle ds$$

for $t\geqq 0$ and $n\geqq 1$. Since, for any $T>0$, $\{u_n\}$ converges weakly uniformly on $[0,T]$ to u, we have by letting $n\to\infty$

$$\langle u(t),\phi\rangle=\langle x_0,\phi\rangle+\int_0^t\left\langle f\big(s,u(s)\big),\phi\right\rangle ds \qquad \text{for } t\in[0,T].$$

Here, we have used the dominated convergence theorem. Since $\phi\in E^*$ is arbitrary, u is a solution of (CP) on $[0, T]$. Since $T>0$ is arbitrary, it follows that u is a solution of (CP) on $[0, \infty)$.

To show the uniqueness, let v be another solution of (CP). For each $\phi\in E^*$, $||\phi||=1$, let $w(t;\phi)=|\langle u(t)-v(t),\phi\rangle|$. Then $w(t;\phi)$ is absolutely continuous on any bounded interval of $[0,\infty)$. By the assumption of Theorem 2, there exist positive constants a, r and a function g_ϕ defined on $(0,a]\times[0,2r]$ satisfying (G_1)-(G_3) such that

$$|\langle f(t,x)-f(t,y),\phi\rangle|\leqq g_\phi(t,|\langle x-y,\phi\rangle|)$$

for a. e. $t\in(0,a]$ and x, $y\in E$ whenever $|\langle x-y,\phi\rangle|\leqq 2r$. Since $w(0;\phi)=0$, there exists a $t_0>0$ such that

$$w(t;\phi)\leqq 2r \qquad t\in[0,t_0].$$

Let $b=\min\{a,t_0\}$, then we have

$$w'(t;\phi)\leqq|\langle f\big((t,u(t)\big)-f\big(t,v(t)\big),\phi\rangle|$$

$$\leqq g_\phi\big(t,|\langle u(t)-v(t),\phi\rangle|\big)=g_\phi\big(t,w(t;\phi)\big)$$

for a. e. $t\in(0,b]$. It follows from Lemma 3.1 that $w(t;\phi)\equiv0$ on $[0,b]$. Let $B=\{t\geqq0;\ w(\tau;\phi)=0 \text{ for } \tau\in[0,t]\}$ and $t_1=\sup B$. Then $t_1>0$ as mentioned as above. If $t_1<+\infty$, by the same method as the above mentioned proof, there exists an $\eta>0$ such that

$$w(t;\phi)\equiv0 \qquad \text{on } [t_1,t_1+\eta].$$

This contradicts the definition of t_1, and this completes the proof of Theorem 2.

Proof of Corollary. The weak continuity of f on $[0,\infty)\times E$ satisfies the conditions (F_1) and (F_2). Now, the desired result follows from the weak continuity of $f(t,u(t))$ on $[0,\infty)$.

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