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Some Remarks on the Existence of Solutions of Nonlinear Ordinary Differential Equations in Banach Spaces

by Yasuhiko YABU* and Shigeo KATO**

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Abstract

The existence of the solutions of the Cauchy problem

$$x' = f(t, x), \qquad x(0) = x_0 \in E$$

in a Banach space E are studied. We give a sufficient condition for the above equation to have a unique solution, and prove that the successive approximations for the above equation converge weakly to the unique solution.

1. Introduction

Let E be a Banach space with the dual space E^*. The norms in E and E^* are denoted by $\| \quad \|$. We denote by $S(x, r)$ the closed ball of center x with radius r. In this paper we consider the Cauchy problem

$$\text{(CP)} \quad x' = f(t, x), \qquad x(0) = x_0 \in E,$$

where f is E-valued mapping defined on $[0, a] \times S(x_0, r)$. By a solution of (CP), we mean a strongly absolutely continuous function u defined on some interval $[0, T]$ $(0 < T \leqq a)$ satisfying $u(0) = x_0$ and $u'(t) = f(t, u(t))$ for $a.e.\ t \in [0, T]$.

It is our object in this paper to give the weak convergence theorem for the successive approximations for (CP) under similar conditions to S. Kato [3] in which he treats the strong convergence. The techniques employed in this paper are similar to those of [3].

Throughout this paper we assume that E_w, the space E with its weak topology, is complete.

2. Preliminaries

Let f be a mapping from $[0, a] \times S(x_0, r)$ into E satisfying the following conditions:

(F_1) $f(\cdot, x)$ is strongly measurable in t for each $x \in S(x_0, r)$, and for $a.e.\ t \in [0, a]$, $f(t, \cdot)$ is weakly continuous in x, ie., $f(t, \cdot)$ is continuous from $S(x_0, r)$ with relative topology of E_w into E_w.

(F_2) There exists an $m \in L^1(0, a)$ such that

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* Department of General Education, Dōto College.

** Department of Common Courses, Kitami Institute of Technology.

$$\|f(t,x)\| \leqq m(t) \text{ for } a.e.\ t \in [0,a] \text{ and all } x \in S(x_0, r).$$

We define the successive approximations for (CP) as follows:

$$u_n(t) = x_0 + \int_0^t f(s, u_{n-1}(t))ds \quad (n \geqq 1), \tag{1.1}$$

where u_0 is an arbitrary continuous function from $[0, a]$ into $S(x_0, r)$.

To show the weak convergence of $\{u_n\}$ we consider a Kamke-type uniqueness function g satisfying the following conditions:

(G_1) $g=g(t,\tau)$ is a function from $(0,a] \times [0, 2r]$ into R^+ which is measurable in t for each τ, and continuous nondecreasing in τ for each t.

(G_2) For each $\delta \in (0, a]$, $w \equiv 0$ is the only absolutely continuous function defined on $[0, \delta]$ which satisfies $w(0)=0$ and $w'(t) = g(t, w(t))$ for $a.e.\ t \in (0, \delta]$.

(G_3) There exists a function α defined on $(0, a]$ such that

$$g(t,\tau) \leqq \alpha(t) \quad \text{for } (t,\tau) \in (0,a] \times [0, 2r]$$

and $\alpha \in L^1(\varepsilon, a)$ for every $\varepsilon \in (0, a)$.

Before stating the main results we prepare the following two lemmas.

Lemma 2.1. Let g satisfy the conditions (G_1), (G_2) and (G_3), and let w be an absolutely continuous function from $[0, a]$ to $[0, 2r]$. Suppose furthermore that $w(0)=0$ and

$$w'(t) \leqq g(t, w(t)) \quad \text{for } a.e.\ t \in (0, a].$$

Then $w \equiv 0$ on $[0, a]$.

For a proof see Lemma 2.3 in [5].

Lemma 2.2. Suppose that f satisfies (F_1) and (F_2). Then for each strongly measurable function z from $[0, a]$ into $S(x_0, r)$, $f(\cdot, z(\cdot))$ is strongly measurable and Bochner integrable on $[0, a]$.

For a proof see Lemma 2.2 in [3].

3. Main results

Now, we can state the following main results.

Theorem. Suppose that (F_1) and (F_2) are satisfied. Suppose furthermore that, for each $\phi \in E^*$, $\|\phi\| = 1$, there exists a function g_ϕ satisfying (G_1)-(G_3) such that

$$|\langle f(t,x) - f(t,y), \phi\rangle| \leqq g_\phi(t, |\langle x-y, \phi\rangle|) \tag{3.1}$$

for each (t, x), $(t, y) \in (0, a] \times S(x_0, r)$. Then the successive approximations $\{u_n\}$ defined by (1.1) converge weakly uniformly on some interval $[0, T]$ to a unique solution u of (CP). Here, we denote by $\langle x, \phi\rangle$ the value of ϕ at x.

Corollary. In Theorem, if f is weakly continuous on $[0, a] \times S(x_0, r)$, then the conclusion of Theorem holds true. Moreover, u is weakly continuously differentiable on $[0, T]$ and satisfies (CP) for every $t \in [0, T]$ in the sense of weak derivative.

4. Proof of main results

Proof of Theorem. Let $T\in(0, a]$ be such that $\int_0^T m(t)\,dt<r$ and set $I=[0, T]$. We remark first that, by the assumptions (F_1), (F_2) and Lemma 2.2, $f(\cdot, z(\cdot))$ is strongly measurable and Bochner integrable on I for each strongly measurable function $z: I\to S(x_0, r)$.

Let $\{u_n\}$ be the sequence of successive approximations for (CP) defined by

$$u_n(t)=x_0+\int_0^t f(s, u_{n-1}(s))\,ds \qquad (t\in I,\ n\geqq 1), \tag{4.1}$$

where u_0 is an arbitrary continuous function from $[0, a]$ into $S(x_0, r)$, and the integral is a Bochner sense.

It follows from (4.1) and (F_2) that

$$\|u_n(t)-x_0\|\leqq\int_0^t\|f(s, u_{n-1}(s))\|ds\leqq\int_0^t m(\tau)d\tau\leqq r$$

for each $t\in I$ and $n\geqq 1$. This implies that $u_n(t)\in S(x_0, r)$ for each $t\in I$ and $n\geqq 1$. On the other hand, we have

$$\|u_n(t)-u_n(s)\|\leqq\left|\int_s^t\|f(\tau, u_{n-1}(\tau))\|d\tau\right|\leqq\left|\int_s^t m(\tau)d\tau\right|\leqq|M(t)-M(s)| \tag{4.2}$$

for each $s, t\in I$ and $n\geqq 1$, where $M(t)=\int_0^t m(\tau)\,d\tau$ for $t\in I$.

Letting $s=0$ in (4.2) we have

$$\|u_n(t)\|\leqq\|u_n(0)\|+M(t)\leqq\|x_0\|+M(T)\leqq\|x_0\|+r,$$

and hence $\{u_n\}$ is equicontinuous and uniformly bounded on I. For each $\phi\in E^*$, $\|\phi\|=1$, we define the functions $w_{m,n}(\cdot\,;\phi)$ and $w_n(\cdot\,;\phi)$ by

$$w_{m,n}(t\,;\phi)=|\langle u_m(t)-u_n(t), \phi\rangle| \qquad (t\in I,\ m\geqq n\geqq 1)$$

and

$$w_n(t\,;\phi)=\sup\{w_{m,n}(t\,;\phi)\,;\ m\geqq n\} \qquad (t\in I,\ n\geqq 1).$$

Then, by (4.2), we have

$$\begin{aligned}|w_{m,n}(t\,;\phi)-w_{m,n}(s\,;\phi)|&=|\,|\langle u_m(t)-u_n(t), \phi\rangle|-|\langle u_m(s)-u_n(s), \phi\rangle|\,|\\&\leqq|\langle u_m(t)-u_m(s), \phi\rangle-\langle u_n(t)-u_n(s), \phi\rangle|\\&\leqq\|u_m(t)-u_m(s)\|+\|u_n(t)-u_n(s)\|\leqq 2|M(t)-M(s)|\end{aligned}$$

and hence

$$\begin{aligned}|w_n(t\,;\phi)-w_n(s\,;\phi)|&\leqq\sup\{|w_{m,n}(t\,;\phi)-w_{m,n}(s\,;\phi)|\,;\ m\geqq n\}\\&\leqq 2|M(t)-M(s)|\end{aligned}$$

for all $s, t\in I$ and $n\geqq 1$. Thus we have the following.

(4. 3) $w_{n+1}(t;\phi)\leqq w_n(t;\phi) \quad (t\in I,\ n\geqq 1)$.

(4. 4) $w_n(\cdot;\phi)$ is absolutely continuous on I for each $n\geqq 1$, and $\{w_n(\cdot;\phi)\}$ is equicontinuous and uniformly bounded on I.

(4. 5) $w_n(0;\phi)=0 \quad (n\geqq 1)$.

From (4. 3), there exists a function $w(\cdot;\phi)$ from I to R^+ such that $w_n(\cdot;\phi)\to w(\cdot;\phi)$ pointwise on I. On the other hand, (4. 4) implies that the convergence $w_n(\cdot;\phi)\to w(\cdot;\phi)$ is uniform on I. It thus follows that $w(\cdot;\phi)$ is absolutely continuous on I.

Let t and $\Delta t>0$ be such that $t,\ t+\Delta t\in I$. Then we have by (3. 1) and (G_1)

$$\begin{aligned}
&|w_{m+1,n+1}(t+\Delta t;\phi)-w_{m+1,n+1}(t;\phi)|\\
&=|\langle u_{m+1}(t+\Delta t)-u_{m+1}(t),\phi\rangle-\langle u_{n+1}(t+\Delta t)-u_{n+1}(t),\phi\rangle|\\
&=|\langle\int_t^{t+\Delta t} f(s,u_m(s))ds,\phi\rangle-\langle\int_t^{t+\Delta t} f(s,u_n(s))ds,\phi\rangle|\\
&=|\int_t^{t+\Delta t}\langle f(s,u_m(s))-f(s,u_n(s)),\phi\rangle\, ds|\\
&\leqq\int_t^{t+\Delta t} g_\phi(s,w_{m,n}(s;\phi))\,ds\leqq\int_t^{t+\Delta t} g_\phi(s,w_n(s;\phi))\,ds\,.
\end{aligned}$$

Since $w_n(\cdot;\phi)\to w(\cdot;\phi)$ uniformly on I, we have by letting $n\to\infty$

$$|w(t+\Delta t;\phi)-w(t;\phi)|\leqq\int_t^{t+t} g_\phi(s,w(s;\phi))\,ds.$$

Thus we have

$$|w'(t;\phi)|\leqq g_\phi(t,w(t;\phi)) \qquad \text{for } a.e.\ t\in(0,T]\,.$$

Since $w(0;\phi)=0$, we deduce now that $w(t;\phi)\equiv 0$ on I by Lemma 2. 1, and this implies that $\{u_n\}$ is weakly Cauchy uniformly on I. By virtue of the weak completeness of E, $\{u_n\}$ converges weakly uniformly on I to a certain weakly continuous function u.

We next show that u is a unique solution of (CP) on I. Let $\phi\in E^*$, then

$$\langle u_{n+1}(t),\phi\rangle=\langle x_0,\phi\rangle+\int_0^t\langle f(s,u_n(s)),\phi\rangle\, ds$$

for $t\in I$ and $n\geqq 1$. Since $\{u_n\}$ converges weakly uniformly on I to u, we have by letting $n\to\infty$

$$\langle u(t),\phi\rangle=\langle x_0,\phi\rangle+\int_0^t\langle f(s,u(s)),\phi\rangle\, ds$$

for $t\in I$. Here, we have used the dominated convergence theorem. Since $\phi\in E^*$ is arbitrary, u is a solution of (CP) on I.

To show the uniqueness, let v be another solution of (CP) on I and let $w(t)=|\langle u(t)-v(t),\phi\rangle|$ for $t\in I$, where $\phi\in E^*$, $\|\phi\|=1$. Then w is absolutely continuous on I and

$$w'(t) \leqq |\langle f(t, u(t)) - f(t, v(t)), \phi\rangle|$$
$$\leqq g_\phi(t, |\langle u(t) - v(t), \phi\rangle|) = g_\phi(t, w(t))$$

for $a.e.\, t \in (0, T]$. Since $w(0)=0$, it then follows from Lemma 2.1 that $w \equiv 0$ on I. This completes the proof of Theorem.

Proof of Corollary. The weak continuity of f on $[0, a] \times S(x_0, r)$ satisfies the conditions (F_1) and (F_2). Now, the desired result follows from the weak continuity of $f(t, u(t))$ on I.

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