北見工業大学研究報告 Memoirs of the Kitami Institute of Technology
第13巻 第1号 (昭和56年) Vol. 13 No. 1 (1981)

Remarks on some order properties in non-Archimedean Riesz spaces*)

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(April 1, 1981)

1. Introduction and preliminaries

We shall first begin by recalling the various concepts. In what follows L will always denote a Riesz space.

(a) L is called Archimedean whenever $\inf(1/n)u=0$ for all $0\leq u\in L$.

(b) A linear subspace A of L is called an ideal whenever $f\in A$ and $|g|\leq|f|$ implies $g\in A$. An ideal generated by only one element is called a principal ideal.

(c) An ideal A of L is called a band whenever $f_\tau\in A(\tau\in\{\tau\})$ and $f_\tau\uparrow f$ implies $f\in A$.

(d) An ideal $A\subset L$ is called order dense in an ideal $B\subset L$ whenever the smallest band $\{A\}$ containing A satisfies $B\subset\{A\}$.

(e) If A is an ideal in L and $\{f_\tau\}$ is a system of elements of A, then $\{f_\tau\}$ is called an order basis of A whenever the ideal generated by $\{f_\tau\}$ is order dense in A.

We have the following theorem in [1] (see Theorem 6).

Theorem A. If L is Archimedean, then the following conditions are mutually equivalent.

(i) Every ideal contained in some principal ideal has a countable order basis.

(ii) Every ideal contained in some ideal with a countable order basis has also a countable order basis.

The purpose of this note is to show that the above theorem holds in non-Archimedean case.

2. The main result

Lemma. If A and B are two ideals of L, then $\{A\cap B\}=\{A\}\cap\{B\}$.

Proof. $\{A\cap B\}\subset\{A\}\cap\{B\}$. Follows immediately from the definition (d).

Next we shall show that $\{A\cap B\}\supset\{A\}\cap\{B\}$. Take an element $u\in\{A\}\cap\{B\}$, then $u^+\in\{A\}\cap\{B\}$ and we can find two systems $\{u_\lambda\}\subset A$ and $\{v_\gamma\}\subset B$ such that $0\leq u_\lambda\uparrow u^+$ and $0\leq v_\gamma\uparrow u^+$ respectively (see [2] Lemma 2). Therefore $A\cap B\ni \inf_{\lambda,\gamma}(u_\lambda, v_\gamma)\uparrow u^+$. Thus $u^+\in\{A\cap B\}$. Since $-u\in\{A\}\cap\{B\}$ we have $(-u)^+=u^-\in\{A\cap B\}$. Thus $u=u^+-u^-\in\{A\cap B\}$ and the proof is finished.

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*) Presented at a meeting in Sapporo on January 17, 1981 held to discuss ordered vector spaces.

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Theorem B. The conditions (i) and (ii) are mutually equivalent.

Proof. (ii)⇒(i). Follows immediately from the condition (ii).

(i)⇒(ii). Let $A_0 \subset A$ be ideals in L, where A has the countable order basis $\{u_n : n=1, 2, \cdots\}$; we may assume that $0 \leq u_n \uparrow$. Let U be the ideal generated by $\{u_n : n=1, 2, \cdots\}$. U is order dense in A, i. e., $A \subset \{U\}$, in fact $\{A\}=\{U\}$. For $n=1, 2, \cdots$, let U_n be the ideal generated by u_n. By (i) $A_0 \cap U_n = A_n$ has a countable order basis $\{v_{n,m} : m=1, 2, \cdots\}$; we may assume that $0 \leq v_{n,m} \uparrow$ in m. Let V_n be the ideal generated by $\{v_{n,m} : m=1, 2, \cdots\}$. V_n is order dense in A_n, i. e., $A_n \subset \{V_n\}$, in fact $\{A_n\}=\{V_n\}$. We shall show that $\{v_{n,m} : m, n=1, 2, \cdots\}$ is an order basis of A_0. For this purpose, let V be the ideal generated by $\{v_{n,m} : m, n=1, 2, \cdots\}$. Then we have $V_n \subset V$ for every n, and $A_n = A_0 \cap U_n \subset \{V_n\} = \{A_n\} \subset \{V\}$. $(*)$

Since $u_n \uparrow$, $\bigcup_{n=1}^{\infty} U_n$ is the ideal of L and $U \subset \bigcup_{n=1}^{\infty} U_n$. So we have

$$A_0 \subset A \subset \{U\} \subset \{\bigcup_{n=1}^{\infty} U_n\}.$$

By $(*)$ and Lemma we have

$$A_0 \subset \{\bigcup_{n=1}^{\infty} U_n\} \cap \{A_0\} = \left\{(\bigcup_{n=1}^{\infty} U_n) \cap A_0\right\} = \left\{\bigcup_{n=1}^{\infty} (U_n \cap A_0)\right\} = \{\bigcup_{n=1}^{\infty} A_n\} \subset \{V\} .$$

This completes the proof.

References

[1] W. A. J. Luxemburg: On some order properties of Riesz spaces and their relations. Archiv Der Math.; Vol. XIX, 488-492 (1968).

[2] K. Isobe: On a generalization of ε-countable sup property in Riesz spaces. Mem. Kitami Inst. Tech.; Vol. 11, No. 2, 181-186 (1980).