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A certain example of non-Archimedean Riesz space

by Kiro ISOBE

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1. Introduction

Let L be a Riesz space.

(a) L is called *Archimedean* whenever $\inf \frac{1}{n} u=0$ for all $0 \leqq u \in L$.

(b) A linear subspace A of L is called an *ideal* whenever $f \in A$ and $|g| \leqq |f|$ implies $g \in A$.

(c) An ideal A of L is called a *band* whenever $f_\tau \in A$ $(\tau \in \{\tau\})$ and $f_\tau \uparrow f$ implies $f \in A$.

(d) An ideal $A \subset L$ is called *order dense* in an ideal $B \subset L$ whenever the smallest band $\{A\}$ containing A satisfies $B \subset \{A\}$.

(e) If A is an ideal in L and if $\{f_\tau\}$ is a system of elements of A, then $\{f_\tau\}$ is called an *order basis* of A whenever the ideal generated by $\{f_\tau\}$ is order dense in A.

W. A. J. Luxemburg proved the following in [1].

Theorem 1. *If L is Archimedean, then the following conditions are mutually equivalent.*

(i) *Every ideal contained in some principal ideal has a countable order basis.*

(ii) *Any non-empty subset D of L which is bounded above has an at most countable subset having the same set of upper bounds as D.*

Now, considering the above theorem in the non-Archimedean case, we have already an example which holds (i) but not (ii) in [2].

The purpose of this paper is to show an example of non-Archimedean Riesz space which holds (ii) but not (i).

2. Example

Let X be the linear space of all real functions on the real line $\boldsymbol{R}$ such that $f(x) \neq 0$ holds only at most finitely many in R. We introduce a partial ordering in X by defining that $f \leq g$ means that $f(\lambda) \leq g(\lambda)$ for all $\lambda \in \boldsymbol{R}$. Evidently, X is a Riesz space with respect to this partial ordering.

Proposition 2. *X satisfies the condition* (i).

Proof. Let I be an ideal in X contained in some ideal I_a generated by an element a. For every $x \in X$ we put
$M_x = \{\lambda :\ x(\lambda) \neq 0,\ \lambda \in \boldsymbol{R}\}$.

Department of Liberal Arts, Kitami Institute of Technology.

For every $x \in X$, there exists some positive number α such that $|x| \leq \alpha |a|$. Since $M_x = M_{|x|} \subset M_{\alpha|a|} = M_{|a|} = M_a$, we have $M_x \subset M_a$ for all $x \in I$. Hence $\bigcup_{x \in I} M_x \subset M_a$. Consequently $\bigcup_{x \in I} M_x$ is an at most finite set. So, we can put $\bigcup_{x \in I} M_x = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$.

Putting

$$f_i(\lambda) = \begin{cases} 1 & (\lambda = \lambda_i) \\ 0 & (\lambda \neq \lambda_i) \end{cases}$$

we get the finite sequence $\{f_i\}_{i=1}^{k}$. To every λ_i $(i=1, 2, \cdots, k)$, there exists some $x \in I$ such that $\lambda_i \in M_x$. So, we have $f_i \leq \mu |x|$ for some positive number μ and we have $f_i \in I$ $(i=1, 2, \cdots, k)$.

For every $x \in I$, putting

$$M_x = \{\lambda_{x1}, \cdots, \lambda_{xn_x}\} \subset \{\lambda_1, \cdots, \lambda_k\},$$

we have $x = x(\lambda_{x1}) f_{x1} + \cdots + x(\lambda_{xn_x}) f_{xn_x}$. Therefore I is generated by $\{f_i\}_{i=1}^{k}$ and I has a finite order basis. The proof is complete.

Of course X is Archimedean. Using *Theorem* 1. and *Proposition* 2. we obtain the following result.

Proposition 3. *X satisfies the condition* (ii).

Let L be the linear space of the product space $\boldsymbol{R} \times X$. We introduce a partial ordering in L by defining that $(\alpha, x) \leq (\beta, y)$ means either $\alpha < \beta$ or $\alpha = \beta$, $f \leq g$. Evidently L is the non-Archimedean Riesz space with respect to this partial ordering.

Proposition 4. *L satisfies the condition* (ii).

Proof. We assume that D is a non-empty upper bounded subset of L, i.e., there exists some element (α, f) such that $(\alpha, f) \geq (\lambda, x)$ for all $(\lambda, x) \in D$. From the definition of ordering in L, we have $\alpha \geq \lambda$ for all $(\lambda, x) \in D$. We put $\beta = \sup_{(\lambda, x) \in D} \lambda \leq \alpha$.

Case I. We suppose that $\beta > \lambda$ for all $(\lambda, x) \in D$. In this case we can find the sequence $\{(\lambda_n, x_n)\}_{n=1}^{\infty} \subset D$ such that $\lambda_n \uparrow \beta$.

Let $(\gamma, g) \geq (\lambda_n, x_n)$ for all n. We have $\gamma \geq \lambda_n$ for all n.

Consequently $\gamma \geq \beta > \lambda$ for all $(\lambda, x) \in D$. Therefore (γ, g) is the upper bound of D.

If there exists some element $(\beta, x) \in D$, then to every $\mu \in \boldsymbol{R}$ we put $\alpha_\mu = \sup_{(\beta, x) \in D} x(\mu)$.

Case II. Suppose that there exists some element $(\beta, x) \in D$ and $\alpha_{\mu_0} = \infty$ for some μ_0. In this case, we can find some sequence $\{(\beta, x_n)\}_{n=1}^{\infty} \subset D$ such that $x_n(\mu_0) \uparrow \infty$. Let $(\gamma, g) \geq (\beta, x_n)$ for all n. We have $\gamma \geq \beta$. If $\gamma = \beta$, then $g(\mu_0) \geq x_n(\mu_0)$ for all n. Hence $g(\mu_0) = \infty$. It contradicts the statement that $g(\mu_0)$ is finite.

Therefore we have $\gamma > \beta$. Consequently $(\gamma, g) \geq (\lambda, x)$ for all $(\lambda, x) \in D$.

If $\alpha_\mu < \infty$ for all $\mu \in \boldsymbol{R}$, then we put $A = \{\mu : \mu \in \boldsymbol{R},\ \alpha_\mu > 0\}$.

Case III. Suppose that there exists some element $(\beta, x) \in D$ and $\alpha_\mu < \infty$ for all $\mu \in \boldsymbol{R}$ and $\bar{A} \geq \aleph_0$. Taking a countable set $\{\mu_1, \mu_2, \cdots, \mu_n, \cdots\} \subset A$, we can find a sequence $\{(\beta, x_n)\}_{n=1}^{\infty} \subset D$ such that $x_n(\mu_n) > 0$. If $(\gamma, g) \geq (\beta, x_n)$ for all n, then $\gamma \geq \beta$.

If $\gamma = \beta$, then $g \geq x_n$ for all n. So $g(\mu_n) \geq x(\mu_n)$ for all n.

This fact is contradicts the statement that M_g is a finite set. Hence $\gamma > \beta$ and $(\gamma, g) \geq (\beta, x)$ for all $(\beta, x) \in D$.

Case IV. Suppose that there exists some element $(\beta, x) \in D$ and $\alpha_\mu < \infty$ for all $\mu \in \boldsymbol{R}$ and A is a finite set. If $A = \emptyset$, then $0 \geq x$ for all $(\beta, x) \in D$. If $A \neq \emptyset$, then we put

$$h(\mu) = \begin{cases} \alpha_\mu & (\mu \in A) \\ 0 & (\mu \notin A) \end{cases}.$$

So we have $X \ni h \geq x$ for all $(\beta, x) \subset D$. According to *Proposition* 3. we can find some countable set $\{x_n\}_{n=1}^{\infty}$ such that $(\beta, x_n) \in D$ for all n and has the same set of upper bounds as $B = \{x : (\beta, x) \in D\}$.

If $(\gamma, g) \geq (\beta, x_n)$ for all n, then $\gamma \geq \beta$. If $\gamma > \beta$, then $(\gamma, g) \geq (\lambda, x)$ for all $(\lambda, x) \in D$. If $\gamma = \beta$, then $g \geq x_n$ for all n.

So $g \geq x$ for all $x \in B$. Therefore we conclude $(\gamma, g) \geq (\lambda, x)$ for all $(\lambda, x) \in D$ and the proof is complete.

Proposition 5. *L does not satisdfy the condition* (i).

Proof. To every $\lambda \in \boldsymbol{R}$, we put

$$f_\lambda(\mu) = \begin{cases} 1 & (\mu = \lambda) \\ 0 & (\mu \neq \lambda) \end{cases}.$$

Let I be the ideal generated by $\{(0, f_\lambda)\}_{\lambda \in R}$. Clearly $I \subset I_{(1,0)}$. Taking every countable set

$$\{(0, x_1), (0, x_2), \cdots, (0, x_n), \cdots\} \subset I,$$

we have

$$|(0, x_n)| \leq \alpha_{n1}(0, f_{\lambda_{n1}}) + \cdots + \alpha_{nk_n}(0, f_{\lambda_{nk_n}})$$

for some positive number $\alpha_{n1}, \cdots, \alpha_{nk_n}$ and $f_{\lambda_{n1}}, \cdots, f_{\lambda_{nk_n}}$. So, corresponding to $(0, x_n)$, we have a finite set $F_n = \{f_{\lambda_{n1}}, \cdots, f_{\lambda_{nk_n}}\}$ and the set $F = \bigcup_{n=1}^{\infty} F_n$ is an at most countable set. So there exists $f_{\lambda_0} \in F$. Let J be the ideal generated by $\{(0, x_n)\}_{n=1}^{\infty}$. We have $(0, f_{\lambda_0}) \perp J$ and so $(0, f_{\lambda_0}) \perp \{J\}$.

Therefore there is no countable subset which is a countable order basis of I and the proof is complete.

3. Additional result

Theorem 6. *If a Riesz space has a linear ordering, then the condition* (ii) *implies the condition* (i).

Proof. We assume that a Riesz space L has a linear ordering.

Let $I \subset L$ be an ideal included a principal ideal I_a which is generated by an element $a \in L$. Putting $D = \{x : x \in I, 0 \leq x \leq |a|\}$, there exists an at most countable set $\{x_n\}_{n=1}^{\infty} \subset D$ which has the same set of upper bounds as D. We may assume that $x_n \uparrow$. For any $x \in D$, if $x \geq x_n$ for all n, then $x_n \uparrow x$ and if there exists some x_{n_0} such that $x \leq x_{n_0}$, then $\inf(x, x_n) \uparrow x$. Let J be the ideal generated by $\{x_n\}_{n=1}^{\infty}$. We have $D \subset \{J\}$. For each $x \in I$ we can find $\alpha > 0$ such that $|x| \leq \alpha |a|$. So $\frac{1}{\alpha}|x| \in D \subset \{J\}$. Thus $I \subset \{J\}$. Consequently I has the countable order basis $\{x_n\}_{n=1}^{\infty}$. The proof is complete.

References

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[2] K. Isobe: Some notes on order properties in the non-Archimedean Riesz spaces. Mem. Kitami Inst. Tech.; Vol. 11, No. 1, 151–154 (1979).