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On a generalization of ε-countable sup property in Riesz spaces

by Kiro ISOBE*

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1. Introduction and preliminaries

The purpose of this paper is to demonstrate a generalization of ε-countable sup property established by W. A. J. Luxemburg [1] in the theory of Riesz spaces. For a proper understanding of the kind of result we are interested in we shall begin by recalling the various formulae and concepts which are involved. For terminology and notation not explained below we refer to books [2] and [3].

In what follows R will always denote a Riesz space. The simple formulae are inserted without proofs.

(1) $\sup_{\lambda\in\Lambda} a_\lambda = -\inf_{\lambda\in\Lambda}(-a_\lambda)$,

$\inf_{\lambda\in\Lambda} a_\lambda = -\sup_{\lambda\in\Lambda}(-a_\lambda)$.

(2) $\alpha>0$ implies $\sup_{\lambda\in\Lambda} \alpha a_\lambda = \alpha \sup_{\lambda\in\Lambda} a_\lambda$ and $\inf_{\lambda\in\Lambda} \alpha a_\lambda = \alpha \inf_{\lambda\in\Lambda} a_\lambda$.

(3) $\sup_{\lambda\in\Lambda}(a_\lambda+a) = \sup_{\lambda\in\Lambda} a_\lambda + a$,

$\inf_{\lambda\in\Lambda}(a_\lambda+a) = \inf_{\lambda\in\Lambda} a_\lambda + a$.

(4) $\sup_{\lambda\in\Lambda,\gamma\in\Gamma}(a_\lambda+b_\gamma) = \sup_{\lambda\in\Lambda} a_\lambda + \sup_{\gamma\in\Gamma} b_\gamma$.

$\inf_{\lambda\in\Lambda,\gamma\in\Gamma}(a_\lambda+b_\gamma) = \inf_{\lambda\in\Lambda} a_\lambda + \inf_{\gamma\in\Gamma} b_\gamma$.

(5) $\sup_{\lambda\in\Lambda,\gamma\in\Gamma_\lambda} a_{\lambda\gamma} = \sup_{\lambda\in\Lambda}(\sup_{\gamma\in\Gamma_\lambda} a_{\lambda\gamma})$,

$\inf_{\lambda\in\Lambda,\gamma\in\Gamma_\lambda} a_{\lambda\gamma} = \inf_{\lambda\in\Lambda}(\inf_{\gamma\in\Gamma_\lambda} a_{\lambda\gamma})$.

(6) $\sup_{\lambda\in\Lambda}(\sup(a_\lambda, b)) = \sup(\sup_{\lambda\in\Lambda} a_\lambda, b)$,

$\inf_{\lambda\in\Lambda}(\inf(a_\lambda, b)) = \inf(\inf_{\lambda\in\Lambda} a_\lambda, b)$.

(7) $a+b=\sup(a, b)+\inf(a, b)$.

Proof. Since (1) and (3)

$$\sup(a, b)-(a+b)=\sup(-b, -a)=-\inf(a, b).$$

Thus $a+b=\sup(a, b)+\inf(a, b)$.

* Department of Liberal Arts, Kitami Institute of Technology.

(8) $\sup_{\lambda\in\Lambda}(\inf(a_\lambda, b)) = \inf(\sup_{\lambda\in\Lambda} a_\lambda, b)$,

$\inf_{\lambda\in\Lambda}(\sup(a_\lambda, b)) = \sup(\inf_{\lambda\in\Lambda} a_\lambda, b)$.

Proof. Putting $\sup_{\lambda\in\Lambda} a_\lambda = a$, we have

$$\inf(a_\lambda, b) \leq \inf(a, b) \text{ for all } \lambda\in\Lambda.$$

Since (7), if $\inf(a_\lambda, b) \leq c$ for all $\lambda\in\Lambda$, then

$$c \geq \inf(a_\lambda, b) = (a_\lambda + b) - \sup(a_\lambda, b)$$
$$\geq (a_\lambda + b) - \sup(a, b) \text{ for all } \lambda\in\Lambda.$$

Thus $c - b + \sup(a, b) \geq a_\lambda$ for all $\lambda\in\Lambda$.
Therefore

$$c - b + \sup(a, b) \geq a \text{ and } c \geq a + b - \sup(a, b) = \inf(a, b).$$

Hence we have

$$\sup_{\lambda\in\Lambda}(\inf(a_\lambda, b)) = \inf(\sup_{\lambda\in\Lambda} a_\lambda, b).$$

The latter formula is proved similarly.

(9) $\sup_{\lambda\in\Lambda,\gamma\in\Gamma}(\sup(a_\lambda, b_\gamma)) = \sup(\sup_{\lambda\in\Lambda} a_\lambda, \sup_{\gamma\in\Gamma} b_\gamma)$,

$\inf_{\lambda\in\Lambda,\gamma\in\Gamma}(\inf(a_\lambda, b_\gamma)) = \inf(\inf_{\lambda\in\Lambda} a_\lambda, \inf_{\gamma\in\Gamma} b_\gamma)$.

(10) $\sup_{\lambda\in\Lambda,\gamma\in\Gamma}(\inf(a_\lambda, b_\gamma)) = \inf(\sup_{\lambda\in\Lambda} a_\lambda, \sup_{\gamma\in\Gamma} b_\gamma)$,

$\inf_{\lambda\in\Lambda,\gamma\in\Gamma}(\sup(a_\lambda, b_\gamma) = \sup(\inf_{\lambda\in\Lambda} a_\lambda, \inf_{\gamma\in\Gamma} b_\gamma)$.

Proof. Putting $a = \sup_{\lambda\in\Lambda} a_\lambda$ and $b = \sup_{\gamma\in\Gamma} b_\gamma$, we have

$$\inf(a_\lambda, b_\gamma) \leq \inf(a, b) \text{ for all } \lambda\in\Lambda, \gamma\in\Gamma.$$

Let c be an upper bound of $\inf(a_\lambda, b_\gamma)$ $(\lambda\in\Lambda, \gamma\in\Gamma)$. From (8), it follows that $\inf(a, b_\gamma) \leq c$ for all $\gamma\in\Gamma$. Thus $\inf(a, b) \leq c$.
Therefore $\sup_{\lambda\in\Lambda,\gamma\in\Gamma}(a_\lambda, b_\gamma) = \inf(a, b)$. The latter formula is proved similarly.

(11) $a = a^+ - a^-$.

Proof. Since (1) and (7) it follows that

$$a = a + 0 = \sup(a, 0) + \inf(a, 0) = \sup(a, 0) - \sup(-a, 0)$$
$$= a^+ - a^-.$$

(12) $|a| = \sup(a, -a)$.

Proof. $|a| = a^+ + a^- = \sup(a, 0) + \sup(-a, 0) = \sup(a, -a, 0)$.
Since $\sup(a, -a) \geq a$ and $\sup(a, -a) \geq -a$, it follows that $2\sup(a, -a) \geq 0$ and $\sup(a, -a) \geq 0$. Therefore $|a| = \sup(a, -a, 0) = \sup(a, -a)$.

Lemma 1, *If* X *is a linear subspace of* R, *then the following conditison are mutually equivalent.*

(i) $X \ni a$ *implies* $a^+ \in X$.

(ii) $X \ni a$ *implies* $a^- \in X$.

(iii) $X \ni a, b$ *implies* $\sup(a, b) \in X$.

(iv) $X \ni a, b$ *implies* $\inf(a, b) \in X$.

(v) $X \ni a$ *implies* $|a| \in X$.

Proof. (i)⇒(ii). If $a \in X$, then $-a \in X$. Hence $(-a)^+ = a^- \in X$.

(ii)⇒(iii). If $a, b \in X$, then $a-b \in X$. Hence $(a-b)^- = \sup(b-a, 0) \in X$, we have $\sup(b-a, 0)+a = \sup(b, a) \in X$.

(iii)⇒(iv). If $a, b \in X$, then $-a, -b \in X$ and $\sup(-a, -b) = -\inf(a, b) \in X$ by (1). Thus $\inf(a, b) \in X$.

(iv)⇒(v). If $a \in X$, then $-a \in X$ and $-\inf(a, -a) = \sup(-a, a) = |a| \in X$ by (12).

(v)⇒(i). We assume that $X \ni a$ implies $|a| \in X$. $|a| = \sup(a, -a) \in X$ by (12) and $\sup(a, -a)+a = \sup(2a, 0) = 2\sup(a, 0) = 2a^+ \in X$ by (2), (3) and (12). Hence, $a \in X$ implies $a^+ \in X$.

(a) A linear subspace I of R is called an *ideal* whenever $a \in I$ and $|b| \leq |a|$ implies $b \in I$. The smallest ideal containing one element a is called a *principal ideal* and we denote it by I_a.

(b) An indeal B of R is called a *bnad* whenever $0 \leq a_\lambda \in B$ $(\lambda \in \Lambda)$ and $a = \sup_{\lambda \in \Lambda} a_\lambda$ implies $a \in B$.

Lemma 2. *Let* I *be an ideal of* R. *There exists the smallest band* B_I *containing* I *and*

$$B_I = \Big\{a-b:\ a = \sup_{\lambda \in \Lambda} a_\lambda \text{ for some } 0 \leq a_\lambda \in I\ (\lambda \in \Lambda) \text{ and }$$
$$b = \sup_{\gamma \in \Gamma} b_\gamma \text{ for some } 0 \leq b_\gamma \in I\ (\gamma \in \Gamma)\Big\}.$$

Proof. Evidently B_I is a linear subspace of R by (2) and (4). Before we prove that B_I is an ideal, we shall prove the following (i) and (ii).

(i) $a \in B_I$ implies $|a| \in B_I$.

(ii) $0 \leq a \in B_I$ implies $a = \sup_{\lambda \in \Lambda} a_\lambda$ for some $0 \leq a_\lambda \in I$ $(\lambda \in \Lambda)$.

Proof of (i). We put $a = b-c$, $b = \sup_{\lambda \in \Lambda} b_\lambda$ for some $0 \leq b_\lambda \in I$ $(\lambda \in \Lambda)$ and $c = \sup_{\gamma \in \Gamma} c_\gamma$ for some $0 \leq c_\gamma \in I$ $(\gamma \in \Gamma)$. Since $a^+ \leq b$, we have

$$a^+ = \inf(a^+, b) = \sup_{\lambda \in \Lambda}(\inf(b_\lambda, a^+) \text{ by } (8).$$

Since I is an ideal of R, it follows that $\inf(b_\lambda, a^+) \in I$ $(\lambda \in \Lambda)$.

Consequently, $a^+ \in B_I$ and $|a| \in B_I$ by Lemma 1.

Proof of (ii). We put $0 \leq a = b-c$, $b = \sup_{\lambda \in \Lambda} b_\lambda$ for some $0 \leq b_\lambda \in I$ $(\lambda \in \Lambda)$ and $c = \sup_{\gamma \in \Gamma} c_\gamma$ for some $0 \leq c_\gamma \in I$ $(\gamma \in \Gamma)$. Since $a \leq b$, we have

$$a = \inf(a, b) = \inf(a, \sup_{\lambda \in \Lambda} b_\lambda) = \sup_{\lambda \in \Lambda}(\inf(a, b_\lambda)) \text{ by } (8).$$

Since I is an ideal, it follows that $0 \leq \inf(a, b_\lambda) \in I$ $(\lambda \in \Lambda)$.

Putting $a_\lambda=\inf(a, b_\lambda)\in \mathrm{I}$ $(\lambda\in\Lambda)$, we have that (ii) is formed.

If $a\in \mathrm{B_I}$ and $|b|\leq|a|$, then $|a|\in \mathrm{B_I}$ by (i). Since (ii) we have

$$|a|=\sup_{\lambda\in\Lambda} a_\lambda \text{ for some } 0\leq a_\lambda\in \mathrm{I}\ (\lambda\in\Lambda).$$

Consequently

$$|b|=\inf(|a|, |b|)=\inf(\sup_{\lambda\in\Lambda} a_\lambda, |b|)$$

$$=\sup_{\lambda\in\Lambda}(\inf(a_\lambda, |b|)) \text{ by (8)}.$$

Since I is an ideal, it follows that

$$0\leq\inf(a_\lambda, |b|)\in \mathrm{I}\ (\lambda\in\Lambda) \text{ and } |b|\in \mathrm{B_I}.$$

Consequently $b^+\in \mathrm{B_I}$, $b^-\in \mathrm{B_I}$ and $b=b^+-b^-\in \mathrm{B_I}$. Thus $\mathrm{B_I}$ is an ideal.

Finally we shall show that $\mathrm{B_I}$ is a band. If $a=\sup_{\lambda\in\Lambda} a_\lambda$ for some $0\leq a_\lambda\in \mathrm{B_I}$ $(\lambda\in\Lambda)$, then we have $a_\lambda=\sup_{\gamma\in\Gamma_\lambda} b_{\lambda\gamma}$ for some $0\leq b_{\lambda\gamma}\in \mathrm{I}$ $(\lambda\in\Lambda, \gamma\in\Gamma_\lambda)$. Since (5), it follows that

$$a=\sup_{\lambda\in\Lambda}(\sup_{\gamma\in\Gamma_\lambda} b_{\lambda\gamma})=\sup_{\lambda\in\Lambda,\gamma\in\Gamma_\lambda} b_{\lambda\gamma}\in \mathrm{B_I}.$$

Thus $\mathrm{B_I}$ is a band. It is clear that $\mathrm{B_I}$ is the smallest band containing I and the proof is completed.

(c) An ideal I of R is called *order dense* in an ideal J of R whenever the smallest band $\mathrm{B_I}$ containing I satisfies that $\mathrm{J}\subset \mathrm{B_I}$.

(d) An ideal I of R is called *quasi order dense* in an ideal J of R whenever for every $0<a\in \mathrm{J}$ there exists an element $0<b\in \mathrm{I}$ such that $0<b\leq a$.

(e) If I is an ideal in R and if a_λ $(\lambda\in\Lambda)$ is a system of elements of I, then a_λ $(\lambda\in\Lambda)$ is called a *(quasi) order basis* whenever the ideal generated by a_λ $(\lambda\in\Lambda)$ is (quasi) order dense in I.

2. The main theorem

(f) A Riesz space R is said to have *(a, b)-countable sup property* whenever for every system $0\leq a_\lambda$ $(\lambda\in\Lambda)$ with $\sup_{\lambda\in\Lambda} a_\lambda=a$ and $0<b<a$ with $\mathrm{I}_a=\mathrm{I}_{a-b}$, there exists a_{λ_n} $(\lambda_n\in\Lambda, n=1, 2, \cdots)$ such that

$$\sup_n(\inf(a_{\lambda_n}, b)=b.$$

Theorem 3. *Let* R *be a Riesz space. If every ideal of* R *contained in some principal ideal has a countable quasi order basis, then* R *has the* (a, b)-*countable sup property for all* $0<b<a$ *with* $\mathrm{I}_a=\mathrm{I}_{a-b}$.

Proof. Suppose that $a, b\in \mathrm{R}$, $0<b<a$, $\mathrm{I}_a=\mathrm{I}_{a-b}$ and

$$a=\sup_{\lambda\in\Lambda} a_\lambda \text{ for some } 0\leq a_\lambda\in \mathrm{R}\ (\lambda\in\Lambda).$$

We put $c_\lambda=(a_\lambda-b)^+$ $(\lambda\in\Lambda)$ and let I be the ideal generated by c_λ $(\lambda\in\Lambda)$.

$\sup_{\lambda\in\Lambda} c_\lambda = a-b$ by (3) and (6).

So, I is order dense in I_{a-b} $(=I_a)$. From this hypothesis, I has a countable quasi order basis, i. e., there exist countably many elements $x_n \in I$ $(n=1, 2, \cdots)$ such that the ideal generated by them is quasi order dense in I. Since we can say that

$$|x_k| \leq \alpha_{k1} c_{\lambda_{k1}} + \cdots + \alpha_{kn_k} c_{\lambda_{kn_k}}$$

for some positive integer n_k and some positive numbers $\alpha_{k1}, \cdots, \alpha_{kn_k}$ and some $\lambda_{k1}, \cdots, \lambda_{kn_k} \in \Lambda$, then we can find countably many elements c_{λ_n} $(\lambda \in \Lambda, n=1, 2, \cdots)$ such that the ideal J generated by them is quasi order dense in I. So, J is quasi order dense in I_a $(=I_{a-b})$.

$$c_{\lambda_n} = (a_{\lambda_n} - b)^+ n = 1, 2, \cdots.$$

clearly $\inf(a_{\lambda_n}, b) \leq b$ for all $n=1, 2, \cdots$. Assuming that

$$\inf(a_{\lambda_n}, b) \leq c \text{ for all } n=1, 2, \cdots,$$

then $b-c \leq b - \inf(a_{\lambda_n}, b) = b + \sup(-a_{\lambda_n}, -b) = \sup(b-a_{\lambda_n}, 0) = (a_{\lambda_n} - b)^-$ for all $n = 1, 2, \cdots$.

Assuming that $(b-c)^+ > 0$, then $(b-c)^+ \in I_a$. Since J is quasi order dense in I_a, there is an element $x \in J$ such that $0 < x \leq (b-c)^+$.

Such an element x is experssed

$$0 < x \leq \alpha_1 c_{\lambda_1} + \cdots + \alpha_{n_0} c_{\lambda n_0}$$

for some positive integer n_0 and some positive numbers $\alpha_1, \cdots, \alpha_{n_0}$.

On the other hand, $0 < x \leq (b-c)^+ \leq (a_{\lambda_n} - b)^-$ for all $n=1, 2, \cdots$.

$$\text{So, } \inf(x, c_{\lambda_k}) = 0 \ (k=1, 2, \cdots, n_0).$$

Consequently

$$x = \inf(x, \alpha_1 c_{\lambda_1} + \cdots + \alpha_{n_0} c_{\lambda n_0}) = 0.$$

Contradiction.

Thus, $(b-c)^+ = 0$ and $b \leq c$. Therefore $\sup_n (\inf(a_{\lambda_n}, b)) = b$ and the proof is comleted.

(g) A Riesz space R is said to have the *ε-countable sup property* whenever for every directed system $\{a_\lambda\}$ with $0 \leq a_\lambda \uparrow a$ and $0 < \varepsilon \leq 1$ there exists a sequence $\{a_{\lambda n}\} \subset \{a_\lambda\}$ such that $a_{\lambda n} \uparrow$ and

$$\sup_n (\inf(a_{\lambda n}, \varepsilon a)) = \varepsilon a.$$

Concerning the ε-countable sup property, W. A. J. Luxemburg proved the following theorem in [1].

Theorem 4. *Let* R *be a Riesz space. If every ideal of* R *contained in some principal ideal has a countable quasi order basis, then* R *has the ε-countable sup property for all* $0 < \varepsilon < 1$.

The above theorem is implied by Theorem 3., because of $0<\varepsilon a<a$ and $I_{(1-\varepsilon)a}=I_a$.

(h) Riesz space R is called *Archimedean* whenever $\inf_n (1/n)a=0$ for all $0\leq a\in R$.

(i) If a Riesz space has the 1-countable sup property, R is said to have the *countable sup property.*

If the Riesz space is Archimedean, then the countable sup property can be expressed in terms of the ε-countable sup property as follows.

Lemma 5. *If* R *is Archimedean, then* R *has the countable sup property if and only if* R *has the* ε*-countable sup property for all* $0<\varepsilon<1$.

For proof of Lemma 5., we refer the reader to Lemma 5. in [1].

If a countable subset A satisfies the following property: for every $0<u\in R$ there exists an element $0<a\in A$ such that $0<a\leq u$, then A is a countable quasi order basis in R and furthermore every ideal in R has a countable quasi order basis.

We denote the class of all continuous real valued functions on [0, 1] by the symbol C [0,1]. For $f, g\in C[0,1]$ we define $f\leq g$ whenever $f(x)\leq g(x)$ for all $0\leq x\leq 1$. So, C [0, 1] is Archimedean Riesz space. Let $A\subset C[0,1]$ be the totality of all functions as follows

$$a(x)=\begin{cases} 0 & (0\leq x<r_1) \\ \dfrac{r_2}{r_2-r_1}(x-r_1) & (r_1\leq x<r_2) \\ \dfrac{r_2}{r_2-r_3}(x-r_3) & (r_2\leq x<r_3) \\ 0 & (r_3\leq x\leq 1) \end{cases}$$

for all rational numbers $0<r_1<r_2<r_3<1$.

Clearly A is a countable subset of C [0, 1] and furthermore for every $0<f\in C[0,1]$ there exists some $a\in A$ such that $0<a\leq f$. Hence we have the following corollary by Theorem 4. and Lemma 5.

Corollary 6. C [0, 1] *holds the countable sup property.*

References

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