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Some Notes on Order Properties in the Non-Archimedean Riesz Spaces

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Preliminary. Let L be a Riesz space.

(a) L is called *Archimedean* whenever $\inf \frac{1}{n} u = 0$ for every $0 < u \in L$.

(b) A linear subspace I of L is called an *ideal* whenever $|f| \leqq |g|$ and $g \in I$ imply $f \in I$.

For every non-empty set X of L, we have the ideal I generated by X, namely, I is the least ideal including X and we will designate it by I_X. An ideal which is generated by only one element of L is called a *principal ideal.*

(c) An ideal I of L is called *order dense* in an ideal J of L whenever every element $f \in J^+$ is expressed as $f = \sup(x : f \geqq x \in I^+)$.

(d) Let I be an ideal of L and $I \supset A \neq \phi$. A is called an *order basis* whenever I_A is order dense in I.

If $A (\subset I)$ is an at most countable set and an order basis in I, then A is called a *countable order basis* in I.

(e) An ideal I in L is called *quasi order dense* in an ideal J in L whenever for every $0 < u \in J$ there exists an element $0 < v \in I$ such that $0 < v \leqq u$.

(f) A non-empty subset A of an ideal I in L is called a *quasi order basis* of I whenever the ideal I_A is quasi order dense in I. If $A (\subset I)$ is an at most countable set and a quasi order basis of I, then A is called a *countable quasi order basis* of I.

It is clear that:

Proposition 1. *If an ideal I in L is order dense in an ideal J in L, then I is quasi order dense in J. If a non-empty subset A of an ideal I in L is a countable order basis of I, then A is a countable quasi order basis of I.*

(g) L is said to have the *countable sup property* whenever every non-empty subset D of L having a supremum contains an at most countable subset having the same supremum as D.

(h) L is said to have *0-countable inf property* whenever every non-empty subset D of L having $\inf D = 0$ contains an at most countable subset having the infimum 0.

It is obvious that:

Proposition 2. (g) is equivalent to (h).

(i) L is said to have the *countable upper bounds property* whenever every non-empty subset D of L which is bounded above contains an at most countable subset having the same set of upper bounds as D.

It is clear that:

Proposition 3. (i) *implies* (g).

W. A. J. Luxembrug has brought many interesting relations on order properties in Archimedean Riesz spaces in [1], by way of example, he has shown the following:

Theorem 4. *If the Riesz space L is Archimedean, then the following properties* (i), (ii) *and* (iii) *are equivalent mutually.*

(i) *An ideal which is included by a principal ideal has a countable order basis.*

(ii) *L has the countable sup property.*

(iii) *L has the countable upper bounds property.*

Now, considering the above theorem in case of non-Archimdedan Riesz space, we have the negative results.

1. The relations between a countable order basis and the countable sup property in non-Archimdedan Riesz spaces.

Let L be a Riesz space and that we do not assume that L is Archimedean. We shall show that it does not necessarily follow that (i) implies (ii).

1-1. ω-well ordered sets. Let S be a infinite set whose cardinal number is $\bar{S}>\aleph_0$. By virtue of the well order theorem, we can fix the well order on S. This well ordered set is denoted by S again. We put $\min S=0$ and an interval $[0, s]=(x\in S:\ 0\leqq x\leqq s)$ for every $s\in S$. We put $A=(s\in S:\ \overline{\overline{[0, s]}}>\aleph_0)$ and $\min A=a$ if $A\neq\phi$. We set the subset X in S as follows:

$$X=\begin{cases} S & (A=\phi) \\ [0, a]\backslash(a) & (A\neq\phi) \end{cases}$$

Such a well ordered set X is called a *ω-well ordered set* and it has the following properties:

Proposition 5. *If the set X is a ω-well ordered set, then $\overline{\overline{X}}>\aleph_0$ and every at most countable subset Y of X has its upper bound.*

Proof. Since the construction of X, we have $\overline{\overline{X}}>\aleph_0$ and $\overline{\overline{[0, x]}}\leqq\aleph_0$ for all $x\in X$. If we deny our conclusion for every at most countable subset Y of X, then for every $x\in X$, there exists some $y\in Y$ such that $x\leqq y$ and this fact yields $[0, x]\subset[0, y]$ and $X=\bigcup_{x\in X}[0, x]=\bigcup_{y\in Y}[0, y]$. Since $\overline{\overline{Y}}\leqq\aleph_0$ and $\overline{\overline{[0, y]}}\leqq\aleph_0$ for all $y\in Y$, we have $\overline{\overline{X}}\leqq\aleph_0$. Contradiction.

1-2. A Riesz space which is composed of all real functions on a ω-well ordered set. Let X be a ω-well ordered set and let L be the totality of all real functions on X. To every $f\in L$, we put $x_f=\min(x\in X: f(x)\neq 0)$. Taking a positive cone P in L such that $P=(f\in L: f=0$ or $f(x_f)>0)$. Putting $P=L^+$, L becomes a linearly ordred Riesz space, namely, $f\leqq g$ means that $g-f\in L^+$. For such Riesz space L, we have the following property:

Proposition 5. *Every ideal in L is a principal ideal and L does not hold the countable sup property.*

Proof. Let I be an ideal in L. In case of $I=(0)$, it is obvious that I is a principal ideal. In case of $I\neq(0)$, we put $\min_{0\neq f\in I}(x_f)=x_0$. Therefore we can find some function $f_0\neq 0$ in I such that $x_{f_0}=x_0$ and we have $|f_0|\,(x_0)>0$. Since $x_f\geqq x_0$ for all $0\neq f\in I$, if $x_f>x_0$, then $|f|\leqq|f_0|$ and if $x_f=x_0$, then $|f|\leqq\lambda\,|f_0|$ for some positive real number λ. Consequently, the ideal I coinsides with the principal ideal I_{f_0} by (b). Next, we shall prove that L does not hold the countable sup property. We put $D=(f\in L:\ 0<f)$. It is obvious that $\inf D=0$. For every non-empty at most countable subset D_0 of D, we put $Y=(x_f\in X:\ f\in D_0)$. Since X is ω-well ordered set and Y is an at most countable subset of X, we can find an upper bound x_0 of Y by the *proposition* 5. Therefore, we take the following function f_0 on X

$$f_0(x)=\begin{cases}1 & (x=x_0)\\ 0 & (x\neq x_0)\end{cases}.$$

It is obvious that $0<f_0\in L$ and $f_0\leqq f$ for all $f\in D_0$. Thus, L does not hold the 0-countable inf property, namely, L does not hold the countable sup property by the *proposition* 2.

1-3. A Riesz space which holds (ii) but not (i). We have the example of Riesz space which has the property (ii) but not (iii) in [2]. This example is in the following:

Let X be the linear space of all real functions on the real line $\boldsymbol{R}$ such that $f(\xi)\neq 0$ holds only at most countably many in $\boldsymbol{R}$. We introduce a partially ordering in X by defining that $f\leqq g$ means that $f(\xi)\leqq g(\xi)$ for all $\xi\in\boldsymbol{R}$. Let L be a linear space of the product space $\boldsymbol{R}\times X$. We introduce a partially ordering in L by defining that $(\alpha,f)\leqq(\beta,g)$ means either $\alpha<\beta$ or $\alpha=\beta$, $f\leqq g$. We shall show that this Riesz space L as above-mentioned example does not hold (i) below. we put a system of real functions $f_\lambda(\lambda\in\boldsymbol{R})$ such that

$$f_\lambda(\xi)=\begin{cases}1 & (\xi=\lambda)\\ 0 & (\xi\neq\lambda)\end{cases}\qquad(\lambda\in\boldsymbol{R}).$$

Let $A\,(\subset L)$ be the totality of all $(0,f_\lambda)$ $(\lambda\in\boldsymbol{R})$. It is evident that $0\leqq(0,f_\lambda)<(1,0)$ for all $\lambda\in\boldsymbol{R}$. Hence $I_A\subset I_{(1,0)}$ (which is the ideal generated by $(1,0)$), namely, the ideal I_A is included by the principal idenal $I_{(1,0)}$. If $I_A\ni(\alpha,\varphi)$, then $|(\alpha,\varphi)|\leqq\mu_1(0,f_{\lambda_1})+\cdots+\mu_n(0,f_{\lambda_n})$ for some positive real numbers $\mu_1,\cdots,\mu_n$ and some real numbers $\lambda_1,\cdots,\lambda_n$. Thus, $\alpha=0$ and $M_\varphi=(\xi\in R:\ \varphi(\xi)\neq 0)\subset(\lambda_1,\cdots,\lambda_n)$. We take every at most countable subset B of I_A. If $B\ni(\alpha,\varphi)$, then $\alpha=0$ and $\varphi(\xi)\neq 0$ holds only finitely many in $\boldsymbol{R}$. therefore, $M=\bigcup_{(0,\varphi)\in B}(\xi\in R:\ \varphi(\xi)\neq 0)$ is at most countable. Hence, we can take a real number $\lambda\overline{\in}M$ and $0<(0,f_\lambda)\in I_A$. If we can find an element $(\alpha,\varphi)\,(=(0,\varphi))\in I_B$ such that $(0,f_\lambda)\geqq(\alpha,\varphi)\,(=(0,\varphi))>0$, then $(\alpha,\varphi)\leqq 0$. Contradiction.

Hence, we conclude that I_B is not quasi order dence in I_A.

Consequently, by *proposition* 1, the ideal I_A which is included by the principal ideal $I_{(1,0)}$ does not have any countable order basis.

2. The relations between a countable order basis and the countable upper bounds property in non-Archimedean Riesz spaces

Already, we have shown the example of the Riesz space which holds (i) but not (ii) in 1–2. This Riesz space as this example holds (i) but not (iii), because (iii) implies (ii). (cf. [1]). There is not saying that (iii) implies (i) or not in the non-Archimedean Riesz spaces.

References

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