Some notes on order properties of certain directed ordered vector spaces

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1. Introduction

Let X be a directed ordered vector space, namely, X is an ordered vector space which is generated by its positive cone X^+. Corresponding to every element $a\in X$, we define a set P_a in X^+ such that

$$P_a=(x:\ 0\leq x\leq b \text{ for every } a\leq b\in X^+).$$

It is clear that

1) $a\geq 0$ implies $P_a=(x:\ 0\leq x\leq a)$,

2) $a\leq b$ implies $P_a\subset P_b$.

We do not assume that X is a Riesz space, namely, for every pair of elements a and b in X, it is not guaranteed that the supremum $\sup(a, b)$ exists. In the below, we substitute the following property (*) for the existence of $\sup(a, b)$:

(*) $P_a=(0)$ implies $a\leq 0$.

It is obvious that a Riesz space satisfies the property (*), but a directed ordered vector space with the property (*) need not be a Riesz space as, for example, in the case of the directed ordered vector space of all real differentiable functions on an interval (α, β).

Even if $\inf(a_\lambda:\ \lambda\in\Lambda)$ does not exist in X for a system of elements $a_\lambda\in X^+$ $(\lambda\in\Lambda)$ and if $\inf(a_\lambda:\ \lambda\in\Lambda)$ exists and $\neq 0$, then it goes without saying that we can find a lower bound $a>0$ of $(a_\lambda:\ \lambda\in\Lambda)$.

It is clear in X as a Riesz space that

3) if $\inf(a, b)=0$ and $\alpha>0$, then $\inf(\alpha a, b)=0$,

4) if $\inf(a, b)=0$ and $\inf(a, c)=0$, then $\inf(a, b+c)=0$,

5) if $\sup(a_\lambda:\ \lambda\in\Lambda)=a$, then $\sup(a_\lambda+b:\ \lambda\in\Lambda)=a+b$ for any $b\in X$.

2. Archimedean directed ordered vector space

Let X be a directed ordered vector space with the property (*).

(a) X is called Archimedean whenever $\inf \dfrac{1}{n}a=0$ for all $a\in X^+$.

(b) A directed ordered linear subspace $I\subset X$ is called an ideal whenever $a\in I$ implies $P_a\subset I$.

Proposition. A directed ordered linear subspace $I\subset X$ is an ideal if and only if $a, b\in I$ and $a\leq x\leq b$ imply $x\in I$.

Proof. Let I be an ideal. If $a, b\in I$ and $a\leq x\leq b$, then $0\leq x-a\in P_{b-a}$.

Hence $x \in I$.

Conversely, suppose that $a, b \in I$ and $a \leq x \leq b$ imply $x \in I$. If $u \in I$, then we can find an element v, which is $u \leq v \in X^+$, in I. Hence, $u \in I$ implies $P_u \subset P_v \subset I$ and the proof is finished.

There exists the smallest ideal including a non-empty set $A \subset X^+$, namely, such ideal consists of all elements x expressed $\alpha_1 a_1 + \cdots + \alpha_m a_m \leq x \leq \beta_1 b_1 + \cdots + \beta_n b_n$ for some finite number of elements $a_1, \cdots, a_m, b_1, \cdots, b_n$ and some real numbers $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n$.

(c) An ideal $I \subset X$ is called order dense in an ideal $J \subset X$ whenever every element $u \in J^+$ is expressed as $u = \sup(X: u \geq x \in I^+)$.

(d) An ideal $I \subset X$ is called quasi order dense in an ideal $J \subset X$ whenever for every $0 < u \in J$ there exists an element $0 < v \in I$ such that $0 < v \leq u$.

Now, we assume that X is Archimedean and an ideal $I \subset X$ is quasi order dense in an ideal $J \subset X$. If I is not order dense in J, then there exstis some element $0 < u \in J$ which is not expressed as $u = \sup(x: u \geq x \in I^+)$ whether the supremum $\sup(X: u \geq x \in I^+)$ exists or not. Consequently, we can find an element $v \in X$ which is an upper bound of $(x: u \geq x \in I^+)$ and $P_{u-v} \supsetneqq (0)$. Taking up an element $0 < p \in P_{u-v}$, we have $p \in J$ and there exists an element $q \in I$ which $p \geq q > 0$ by our assumption.

Since $0 < q \leq p \leq u$, we have $v \geq q$ and

$$u - v = (u - q) - (v - q), \qquad u - q \geq 0, \qquad v - q \geq 0.$$

Since $u - q \geq p \geq q > 0$, we have $u \geq 2q$, $v \geq 2q$ and

$$u - v = (u - 2q) - (v - 2q), \qquad u - 2q \geq 0, \qquad v - 2q \geq 0.$$

Since $u - 2q \geq p \geq q > 0$, we have $u \geq 3q$ and $v \geq 3q$ and so forth.

Therefore we conclude $u \geq nq$, $n = 1, 2, \cdots$. It contradicts our assumption that X is Archimedean.

Conversely, we assume that every ideal $I \subset X$ which is quasi order dense in an ideal $J \subset X$ is order dense in J. If X is not Archimedean, then there exists some element $0 < u \in X$ such that

$$\left(x:\ 0 < x \leq \frac{1}{n} u, \qquad n = 1, 2, \cdots\right) = A \neq \phi.$$

Clearly, we have $0 < x \leq a$ for some $a \in A$ implies $x \in A$ and $f, g \in A$ implies $\lambda f + \mu g \in A$ for any $\lambda, \mu > 0$. Furthermore, we put

$$(x: u \geq x > 0 \text{ and } \inf(a, x) = 0 \text{ for all } a \in A) = B$$

if it is not empty. Clearly, $x_1, \cdots, x_n \in B$ implies $\frac{1}{n}(x_1 + \cdots + x_n) \in B$.

For every $0 < v \leq u$, there exists an element x such that $v \geq x \in A \cup B$.

Let I be the smallest ideal including $A \cup B$ and let J be the smallest ideal including (u). Clearly, I is quai order dense in J, because $0 < f \in J$ implies $0 < \frac{1}{\alpha} f \leq u$ for some $\alpha > 0$. Taking up an element $u \geq g \in I^+$, we have $g \leq x + \alpha y$

for some $x \in A$, some $y \in B$ and some $\alpha > 0$. We shall now show that $g + x_0 \leq u$ for any $x_0 \in A$. We take an element $x_0 \in A$ and fixed it. Putting $x_1 = x + x_0$, we have $g \leq x + \alpha y \leq x_1 + \alpha y$. Since $g + x_1 - u = x_1 - (u - g)$, $x_1 > 0$ and $u - g \geq 0$, it is obvious that $p \in P_{g+x_1-u}$ implies $p \leq x_1$. Since $x_1 \in A$, $u - x_1 > x_1$ and $g + x_1 - u = g - (u - x_1) \leq g - x_1 \leq \alpha y$, it is obvious that $p \in P_{g+x_1-u}$ implies $p \leq \alpha y$. Since $\inf(x_1, \alpha y) = 0$, we conclude $P_{g+x_1-u} = (0)$. Consequently, $g + x_0 \leq g + x_1 \leq u$. Hence, $u = \sup(x : u \geq x \in I^+) = \sup(x + x_0 : u \geq x \in I^+) = u + x_0$. This result contradicts that $x_0 > 0$. So we have the following:

Theorem. A directed ordered vector space X is Archimedean if and only if every ideal $I \subset X$ which is quasi order dense in an ideal $J \subset X$ is order dense in J.

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References

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