On a certain sufficient condition of the countable sup property in Riesz spaces

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1. Introduction

Let L be a Riesz space. In [1], W. A. J. Luxemburg considered the following properties (i) and (ii) in L.

(i) A Riesz space is said to have the *countable sup property* whenever every non-empty subset D of L having a supremum contains an at most countable subset having the same supremum as D.

(ii) Every non-empty subset D of L which is bounded above contains an at most countable subset having the same set of upper bounds as D.

We have the following theorem in [1].

Theorem. *If L is Archimedean**), then L has the countable sup property if and only if L has the property* (ii).

It is obvious that (ii) implies (i) even if L is not Archimedean.

It is not necessary that L is Archimedean though L holds (i) and (ii) simultaneously, (for such example, we have the lexicograghic plane $\boldsymbol{R}^2$). The purpose of this note is to show a certain example which has the property (i), but not (ii).

2. Example

Let X be the linear space of all real functions on the real line $\boldsymbol{R}$ such that $f(\lambda)\neq 0$ holds only at most countably many in $\boldsymbol{R}$. We introduce a partial ordering in X by defining that $f\leqq g$ means that $f(\lambda)\leq g(\lambda)$ for all $\lambda\in\boldsymbol{R}$. Evidently, X is a Riesz space with respect to this partial ordering. Let L be the linear space of the product space $\boldsymbol{R}\times X$. We introduce a partial ordering in L by defining that $(\alpha, f)\leqq(\beta, g)$ means either $\alpha<\beta$ or $\alpha=\beta, f\leq g$. Evidently L is a non-Archimedean Riesz space with respect to this partial ordering.

(1) L has the property (i).

Proof. We shall assume that D is a non-empty subset of L having a supremum $\sup D=(\alpha, f)$. We shall write $x=(\alpha_x, f_x)$ for every $x\in D$.

From the definition of the partial ordering in L, it follows that $\sup\limits_{x\in D}\alpha_x\leqq\alpha$.

If $\sup\limits_{x\in D}\alpha_x<\alpha$, then we have $\sup\limits_{x\in D}\alpha_x<\beta<\alpha$ and $(\alpha_x, f_x)\leqq(\beta, g)<(\alpha, f)$ for all $x\in D$ and for any $g\in X$. Contradiction.

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** L is called Archimedean whenever $\inf (1/n)\ u=0$ for every $0<u\in L$.

Hence, $\sup_{x \in D} \alpha_x = \alpha$. If $\alpha_x < \alpha$ for all $x \in D$, then we have $(\alpha_x, f_x) \leqq (\alpha, h) < (\alpha, f)$ for all $x \in D$ and for $f > h \in X$. Contradiction.

Hence, $D_1 = (x: x \in D, \alpha_x = \alpha) \neq \phi$. Evidently, it follows that sup $D_1 = \sup D = (\alpha, f)$. From the definition of the partial ordering in L, it follows that $\sup_{x \in D} f_x(\lambda) \leqq f(\lambda)$ for all $\lambda \in \boldsymbol{R}$.

If $\sup_{x \in D_1} f_x(\lambda_0) < f(\lambda_0)$ for some $\lambda_0 \in R$, then we have a function $k \in X$ such that

$$k(\lambda) = \begin{cases} f(\lambda) & (\lambda \neq \lambda_0) \\ \sup_{x \in D_1} f_x(\lambda_0) & (\lambda = \lambda_0) \end{cases}$$

For such function k, it follows that $(\alpha_x, f_x) \leqq (\alpha, k) < (\alpha, f)$ for all $x \in D_1$. Contradiction. Hence, $\sup_{x \in D_1} f_x(\lambda) = f(\lambda)$ for all $\lambda \in R$.

We set the following:

$$M_f = (\lambda: f(\lambda) \neq 0), \quad M_x = (\lambda: f_x(\lambda) \neq 0) \ (x \in D_1) \text{ and}$$
$$M = \bigcup_{x \in D_1} M_x .$$

a) The case of $M_f = \phi$ and $M = \phi$.

In this case, D_1 is composed of only one element $(\alpha, f) = (\alpha, 0)$ and $\sup D_1 = \sup D = (\alpha, f) = (\alpha, 0)$.

b) The case of $M_f = \phi$ and $M \neq \phi$.

In this case, it follows that $f = 0$ and $M_{x_0} = (\lambda_1, \lambda_2, \cdots)$ for some $x_0 \in D_1$. For every λ_i $(i = 1, 2, \cdots)$, there exist some sequences $x_{in} \in D_1$ $(n = 1, 2, \cdots)$ such that $\sup_n f_{x_{in}}(\lambda_i) = f(\lambda_i)$.

Let D_0 be the totality of all x_{in} $(n = 1, 2, \cdots, i = 1, 2, \cdots)$ and x_0. Evidently, D_0 is at most countable and $D_0 \subset D_1 \subset D$. Since $f_x(\lambda) \leqq f(\lambda) = 0$ for all $x \in D_0$ and for all $\lambda \in \boldsymbol{R}$, if $\lambda \overline{\in} M_{x_0}$, then $f_{x_0}(\lambda) = 0$ and $\sup_{x \in D_0} f_x(\lambda) = f(\lambda)$. If $\lambda \in M_{x_0}$, then $\sup_{x \in D_0} f_x(\lambda) = f(\lambda)$.

Hence, $\sup D_0 = \sup D = (\alpha, f)$.

c) The case of $M_f \neq \phi$.

In this case, we can set that $M_f = (\mu_1, \mu_2, \cdots)$. For every μ_i $(i = 1, 2, \cdots)$, we can find some sequences $y_{in} \in D_1$ $(n = 1, 2, \cdots)$ such that $\sup_n f_{y_{in}}(\mu_i) = f(\mu_i)$. Setting $M_1 = \bigcup_{i,n} M_{y_{in}}$, it follows that M_1 is at most countable and we set $M_1 = (\nu_1, \nu_2, \cdots)$. For every ν_i $(i = 1, 2, \cdots)$, we can find some sequences $z_{in} \in D_1$ $(n = 1, 2, \cdots)$ such that $\sup_n f_{z_{in}}(\nu_i) = f(\nu_i)$. Let D_0 be the totality of all y_{in} and z_{in} $(n = 1, 2, \cdots, i = 1, 2, \cdots)$. Evidently, D_0 is at most countable and $D_0 \subset D_1 \subset D$. Since $M_f \subset M_1$ and $\sup_{x \in D_1} f_x(\lambda) = f(\lambda)$ for all $\lambda \in \boldsymbol{R}$, if $\lambda \overline{\in} M_1$, then $f_{y_{in}}(\lambda) = f(\lambda) = 0$ and if $\lambda \in M_1$, then $\lambda = \nu_i$ for some i and $\sup_n f_{z_{in}}(\lambda) = f(\lambda)$. Hence, $\sup D_0 = \sup D = (\alpha, f)$.

(2) L does not hold the property (ii).

Proof. In order to prove, we shall show that there exists some non-empty subset D of L which is bounded above such that it does not contain any at most countable subset having the same set of upper bounds as D. For every $\lambda \in \boldsymbol{R}$, we set a function $f_\lambda \in X$ such that

$$f_\lambda(\mu) = \begin{cases} 1 & (\mu = \lambda) \\ 0 & (\mu \neq \lambda) \end{cases}$$

Let D be the totality of all $(0, f_\lambda)$ $(\lambda \in \boldsymbol{R})$. Evidently, D is a non-empty upper bounded set of L. For any non-empty at most countable subset D_0 of D, we set $M = (\lambda : (0, f_\lambda) \in D_0)$. Evidently, M is at most countable. We set the following function $f \in X$ such that

$$f(\mu) = \begin{cases} 1 & (\mu \in M) \\ 0 & (\mu \notin M) \end{cases}.$$

Evidently, $(0, f_\lambda) \leqq (0, f)$ for all $(0, f_\lambda) \in D_0$. Since $M^c \neq \phi$, if $\lambda \notin M$, then f_λ is not smaller than f and it follows that L does not hold the property (ii).

Reference

[1] Luxemburg, W. A. J.: On some order properties of Riesz spaces and their relations. Archiv Der Math. Fasc. 5, Vol. XIX, 488–493, (1968).