Dispersion of Harmonic Flexural Waves in Fiber Reinforced Rectangular Beam

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Abstract

The stress waves, propagating in a rectangular beam reinforced by equi-distantly spaced fibers in the longitudinal direction, are investigated as an eigenvalue problem of the dynamic finite prism method by means of finite Fourier integration transforms.

The discussions are focussed on the dispersion of a harmonic flexural wave and the comparison of the wave velocity curves with and without reinforcement is shown in figures.

Introduction

In analyzing a structural element to be fabricated from a fiber-reinforced composite, one of the principal difficulties lies in constructing a suitable mathematical model describing the mechanical behavior of the composite. The customary approach consists in replacing the composite by a homogeneous medium whose material constants are determined in terms of the material constants and the goemetry of each constituent of the composite.

The authors, going in another way, attempt to handle the problem with the aid of finite prism method and finite Fourier integration transforms to keep the discreteness of the matter. It is, however, assumed that fiber reinforcement are equi-distantly spaced and their inertia effect is to be neglected.

For many steady-state and transient dynamic problems of structural mechanics, it is helpful and sometimes required to know the dispersive characteristics of free harmonic flexural waves, i. e., the dependence of the phase and group velocity on the wave length.

Many reseachers have so far studied on the stress wave problem in a rectangular prism ; Rayleigh[1], Timoshenko[2], Volterra[3], Mindlin[4], Engström[5], Tanaka[6], and such. The present paper adds a data of the wave velocity curves concerning a reinforced rectanglar beam.

Nomenclature

c : phase velocity

$c_p = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$ (primary wave velocity)

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$c_s = \sqrt{\dfrac{\mu}{\rho}}$ (secondary wave velocity)

l : wave length

ρ : density of material

ρ_s : density of reinforcement

μ_s : Lame's constant of fiber reinforcement

ν, ν_s : Poisson's ratio

λ_2, λ_3 : width and depth of the element in the y and z direction, respectively

A_s : section area of reinforcement

E_s : elastic modulus of reinforcement

$a = 1+(-1)^j$

$b = 1-(-1)^k$

$\bar{D}_j = 1 - \dfrac{D_j}{6}$

$\bar{D}_k = 1 - \dfrac{D_k}{6}$

$D_j = 2\left(1-\cos\dfrac{j\pi}{n}\right)$

$D_k = 2\left(1-\cos\dfrac{k\pi}{r}\right)$

$\dot{f} = \dfrac{\partial f}{\partial x}$

$f' = \dfrac{\partial f}{\partial t}$

$c_r = \sqrt{\dfrac{\mu_s}{\rho}}$

$A_p =$ section area of prism element

Formulation of Mathematical Models

Firstly, let us assume the harmonic wave as

$$u = U \cos \frac{2\pi}{l}(x-ct) \tag{1}$$

$$v = V \sin \frac{2\pi}{l}(x-ct) \tag{2}$$

$$w = W \sin \frac{2\pi}{l}(x-ct) \tag{3}$$

in which the displacements take linear distribution over the section, namely

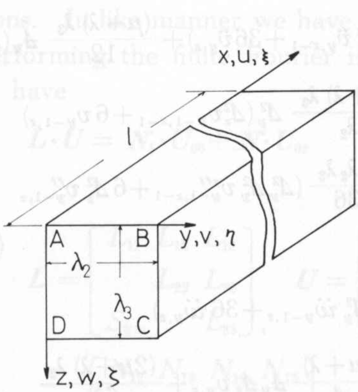

Fig. 1. A Prism Element.

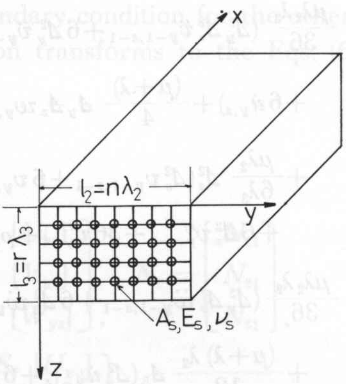

Fig. 2. Fiber-reinforced Beam.

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} U_A & U_B & U_C & U_D \\ V_A & V_B & V_C & V_D \\ W_A & W_B & W_C & W_D \end{bmatrix} \begin{bmatrix} (1-\eta)\cdot(1-\zeta) \\ \eta\cdot(1-\zeta) \\ \eta\cdot\zeta \\ (1-\eta)\cdot\zeta \end{bmatrix} \tag{4}$$

where U_A, V_A, W_A are the displacements at the vertex A in the x, y, z directions respectively. $\eta=y/\lambda_2$ and $\zeta=z/\lambda_3$.

Putting u, v, w into the equation of equilibrium

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} = \rho \frac{\partial^2}{\partial t} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{5}$$

and applying the Galerkin's method with the base functions same as the displacement distribution over the cross section, we can get necessary stiffness matrix and mass matrix between the vertex forces and the displacements. After so doing, the equiilbrium of forces of node can be expressed by the difference equations as follows[9] : (y, z: interger in the y, z direction)

$$\begin{aligned} &\frac{(2\mu+\lambda)\,\lambda_2\lambda_3}{36}\,(\varDelta_y^2\varDelta_z^2\ddot{u}_{y-1,z-1}+6\varDelta_y^2\ddot{u}_{y-1,z}+6\varDelta_z^2\ddot{u}_{y,z-1}+36\ddot{u}_{y,z}) \\ &+\frac{\mu\lambda_3}{6\lambda_2}\varDelta_y^2(\varDelta_z^2 u_{y-1,z-1}+6u_{y-1,z})+\frac{\mu\lambda_2}{6\lambda_3}\varDelta_z^2(\varDelta_y^2 u_{y-1,z-1}+6u_{y,z-1}) \\ &+\frac{(\mu+\lambda)\,\lambda_3}{12}\varDelta_y(\varDelta_z^2\dot{v}_{y,z-1}+6\dot{v}_{y,z})+\frac{(\mu+\lambda)\,\lambda_2}{12}\varDelta_z(\varDelta_y^2\dot{w}_{y-1,z}+6\dot{w}_{y,z}) \\ &+E_s A_s \ddot{u}_{y,z}=\frac{\rho\lambda_2\lambda_3}{36}(\varDelta_y^2\varDelta_z^2 u''_{y-1,z-1}+6\varDelta_y^2 u''_{y-1,z}+6\varDelta_z^2 u''_{y,z-1}+36u''_{y,z}) \\ &+\rho_s A_s u''_{y,z} \end{aligned} \tag{6}$$

$$\frac{\mu\lambda_2\lambda_3}{36}(\Delta_y^2\Delta_z^2\ddot{v}_{y-1,z-1}+6\Delta_y^2\ddot{v}_{y-1,z}+6\Delta_z^2\ddot{v}_{y,z-1}+36\ddot{v}_{y,z})+\frac{(\mu+\lambda)\lambda_3}{12}\Delta_y(\Delta_z^2\dot{u}_{y,z-1}$$
$$+6\dot{u}_{y,z})+\frac{(\mu+\lambda)}{4}\Delta_y\Delta_z w_{y,z}+\frac{(2\mu+\lambda)\lambda_3}{6\lambda_2}\Delta_y^2(\Delta_z^2 v_{y-1,z-1}+6v_{y-1,z})$$
$$+\frac{\mu\lambda_2}{6\lambda_3}\Delta_z^2(\Delta_z^2 v_{y-1,z-1}+6v_{y,z-1})=\frac{\rho\lambda_2\lambda_3}{36}(\Delta_y^2\Delta_y^2 v^{z\prime\prime}_{y-1,z-1}+6\Delta_y^2 v''_{y-1,z}$$
$$+6\Delta_z^2 v''_{y,z-1}+36v''_{y,z})+\rho_s A_s v''_{y,z} \tag{7}$$

$$\frac{\mu\lambda_2\lambda_3}{36}(\Delta_y^2\Delta_z^2\ddot{w}_{y-1,z-1}+6\Delta_z^2\ddot{w}_{y,z-1}+6\Delta_y^2\ddot{w}_{y-1,z}+36\ddot{w}_{y,z})$$
$$+\frac{(\mu+\lambda)\lambda_2}{12}\Delta_8(\Delta_y^2\dot{u}_{y-1z}+6\dot{u}_{y,z})+\frac{(\mu+\lambda)}{4}\Delta_y\Delta_z v_{y,z}+\frac{(2\mu+\lambda)\lambda_2}{6\lambda_3}$$
$$\Delta_z^2(\Delta_y^2 w_{y-1,z-1})+6w_{y,z-1})+\frac{\mu\lambda_3}{6\lambda_2}\Delta_y^2(\Delta_z^2 w_{y-1,z-1}+6w_{y-1,z})$$
$$=\frac{\rho\lambda_2\lambda_3}{36}(\Delta_y^2\Delta_z^2 w''_{y-1,z-1}+6\Delta_y^2 w''_{y-1,z}+6\Delta_z^2 w''_{y,z-1}+36w''_{y,z})+\rho_s A_s w''_{y,z} \tag{8}$$

where

$$\Delta^2 f(x-1)=f(x-1)-2f(x)+f(x-1) \text{ and}$$
$$\Delta f(x-1)=f(x+1)-f(x-1)\,.$$

Solution

Decomposing the stress wave into a symmetrical case and antisymmetrical case with respect to both middle planes of the depth and the width, we have only to treat the one fourth of the nodal lines. The surfaces $z=0$ and $y=0$, being free from stress, we may write the boundary conditions:

$$T_{\left(\substack{y,y+1\\0}\right)}(x)+T_{\left(\substack{y,y-1\\0}\right)}(x)=0 \qquad \text{(for } z=0), \tag{9}$$

$$Y_{\left(\substack{y,y+1\\0}\right)}(x)+Y_{\left(\substack{y,y-1\\0}\right)}(x)=0 \qquad \text{(for } z=0) \tag{10}$$

$$Z_{\left(\substack{y,y+1\\0}\right)}(x)+Z_{\left(\substack{y,y-1\\0}\right)}(x)=0 \qquad \text{(for } z=0), \tag{11}$$

$$T_{\left(\substack{0\\z,z+1}\right)}(x)+T_{\left(\substack{0\\z,z-1}\right)}(x)=0 \qquad \text{(for } y=0), \tag{12}$$

$$Y_{\left(\substack{0\\z,z+1}\right)}(x)+Y_{\left(\substack{0\\z,z-1}\right)}(x)=0 \qquad \text{(for } y=0), \tag{13}$$

$$X_{\left(\substack{0\\z,z+1}\right)}(x)+Z_{\left(\substack{0\\z,z-1}\right)}(x)=0 \qquad \text{(for } y=0), \tag{14}$$

$$T_{00}(x)=0 \qquad \text{(for } y=z=0), \tag{15}$$

$$Y_{00}(x)=0 \qquad \text{(for } y=z=0), \tag{16}$$

$$Z_{00}(x)=0 \qquad \text{(for } y=z=0)\,. \tag{17}$$

in which T, Y and Z are vertex forces of the prism element in the x, y and z

directions. In like manner we have the boundary condition for the other surfaces.

Performing the finite Fourier integration transforms to the Eqs. (6), (7) and (8), we have

$$\boldsymbol{L}\cdot\boldsymbol{U} = \boldsymbol{N}_1\cdot U_{00} + \boldsymbol{N}\cdot \boldsymbol{U}_{0s} \tag{18}$$

where

$$\boldsymbol{L} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ & L_{22} & L_{23} \\ S_{ym} & & L_{33} \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} \boldsymbol{S}_j & \boldsymbol{S}_k & [U_{yz}] \\ \boldsymbol{R}_j & \boldsymbol{S}_k & [V_{yz}] \\ \boldsymbol{S}_j & \boldsymbol{R}_k & [W_{yz}] \end{bmatrix}, \quad \boldsymbol{N}_1 = \begin{bmatrix} N_{11} \\ N_{21} \\ N_{31} \end{bmatrix},$$

$$\boldsymbol{N} = \begin{bmatrix} N_{12} & N_{13} & N_{14} & N_{15} \\ N_{22} & N_{23} & N_{24} & N_{25} \\ N_{32} & N_{33} & N_{34} & N_{35} \end{bmatrix}, \quad \boldsymbol{U}_{0s} = \begin{bmatrix} \boldsymbol{S}_j\,[U_{y0}] \\ \boldsymbol{S}_k\,[U_{0z}] \\ \boldsymbol{R}_j\,[V_{y0}] \\ \boldsymbol{R}_k\,[W_{0z}] \end{bmatrix},$$

$$\boldsymbol{S}_j\,[f(y)] = \sum_{y=1}^{n-1} f(y)\sin\frac{j\pi y}{n},$$

$$\boldsymbol{R}_j\,[f(y)] = \sum_{y=1}^{n-1} f(y)\cos\frac{j\pi y}{n} + \frac{1}{2}f(n)\,(-1)^j + \frac{1}{2}f(0)\,.$$

in which

$$L_{11} = \left(1-\frac{c^2}{c_p^2}\right)\left((\lambda_2\lambda_3\bar{D}_j\bar{D}_k + A_s\frac{\rho_s}{\rho}\right)\left(\frac{2\pi}{l}\right)^2 + \frac{c_s^2}{c_p^2}\left\{\frac{\lambda_3}{\lambda_2}D_j\bar{D}_k + \frac{\lambda_2}{\lambda_3}D_k\bar{D}_j\right\}$$

$$+2\,(1+\nu_s)\,\frac{c_r^2}{c_p^2}\,A_s\left(\frac{2\pi}{l}\right)^2,$$

$$L_{22} = \left(1-\frac{c^2}{c_s^2}\right)\left(\lambda_2\lambda_3\bar{D}_j\bar{D}_k + A_s\frac{\rho_s}{\rho}\right)\left(\frac{2\pi}{l}\right)^2 + \frac{c_p^2}{c_s^2}\,\frac{\lambda_2}{\lambda_3}\,D_j\bar{D}_k + \frac{\lambda_2}{\lambda_3}\,\bar{D}_jD_k\,,$$

$$L_{12} = \left(1-\frac{c_s^2}{c_p^2}\right)\lambda_3\sin\frac{j\pi}{n}\,\bar{D}_k\left(\frac{2\pi}{l}\right),$$

$$L_{23} = \left(\frac{c_p^2}{c_s^2}-1\right)\sin\frac{j\pi}{n}\,\sin\frac{k\pi}{r}\,,$$

$$N_{11} = -\left\{\left(1-\frac{c^2}{c_p^2}\right)\frac{\lambda_2\lambda_3}{36}\left(\frac{2\pi}{l}\right)^2 - \frac{c_s^2}{c_p^2}\,\frac{1}{6}\left(\frac{\lambda_3}{\lambda_2}+\frac{\lambda_2}{\lambda_3}\right)\right\}\sin\frac{j\pi}{n}\,\sin\frac{k\pi}{r}\,ab\,,$$

$$N_{21} = \frac{\lambda_3}{6}\left\{\left(\frac{c_p^2}{c_s^2}-1\right)\frac{1}{2}\left(1+\cos\frac{k\pi}{r}\right)+2-\frac{c_p^2}{c_s^2}\right\}\sin\frac{k\pi}{r}\,ab\left(\frac{2\pi}{l}\right)$$

$$=\frac{\lambda_3}{6}\,b\left(\frac{2\pi}{l}\right)N_{25}\,,$$

$$N_{12} = -\left\{\left(1-\frac{c^2}{c_p^2}\right)\frac{\lambda_2\lambda_3}{6}\left(1-\frac{D_j}{6}\right)\left(\frac{2\pi}{l}\right)^2 + \frac{1}{6}\,\frac{c_s^2}{c_p^2}\left(\frac{\lambda_3}{\lambda_2}+\frac{\lambda_2}{\lambda_3}\right)\right.$$

$$\left. D_j - \frac{c_s^2}{c_p^2}\,\frac{\lambda_2}{\lambda_3}\right\}\sin\frac{k\pi}{r}\,b$$

$$N_{14} = \left(\frac{c_s^2}{c_p^2}-1\right)\frac{\lambda_3}{6}\sin\frac{j\pi}{n}\sin\frac{k\pi}{r}\,b\left(\frac{2\pi}{l}\right),$$

$$N_{22} = \left(1-\frac{c_p^2}{c_s^2}\right)\frac{\lambda_3}{6}\sin\frac{j\pi}{n}\sin\frac{k\pi}{r}\,b\left(\frac{2\pi}{l}\right),$$

$$N_{23} = \left\{\left(\frac{c_p^2}{c_s^2}-1\right)\frac{1}{2}\left(1-\cos\frac{j\pi}{n}\right)+2-\frac{c_p^2}{c_s^2}\right\}\lambda_3\bar{D}_k\,a\left(\frac{2\pi}{l}\right),$$

$$N_{24} = -\left\{\left(1-\frac{c^2}{c_s^2}\right)\frac{\lambda_2\lambda_3}{6}\bar{D}_j\left(\frac{2\pi}{l}\right)^2+\frac{c_p^2}{c_s^2}\frac{\lambda_3}{6\lambda_2}D_j+\frac{\lambda_2}{\lambda_3}\bar{D}_j\right\}\sin\frac{k\pi}{r}\,b$$

Replacement of λ_2, j, n and a by λ_3, k, r and b in L_{12}, L_{22}, N_{21}, N_{12}, N_{14}, N_{23}, N_{24} and N_{25} yields L_{13}, L_{33}, N_{31}, N_{13}, N_{15}, N_{32}, N_{35} and N_{34} in which $N_{15}=N_{33}$.

And satisfying the boundary conditions (9)$\sim$(17), we can obtain the eigenvalue concerning phase velocity by the iteration method. Inversion theorem leads to the solution which are for the finite Fourier integration transforms:

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \frac{4}{nr}\sum_{j=0}^{n}\sum_{k=0}^{r}\boldsymbol{U}\cdot[I]\begin{bmatrix} \sin\dfrac{j\pi y}{n}\cdot\sin\dfrac{k\pi z}{r} \\ \cos\dfrac{j\pi y}{n}\ \sin\dfrac{k\pi z}{r} \\ \sin\dfrac{j\pi y}{n}\ \cos\dfrac{k\pi z}{r} \end{bmatrix} \tag{19}$$

in which $[I]$ is a unit matrix, n and r are the numbers of the nodes in the y and z directions, respectively.

Numerical Examples

In order to illustrate the numberical calculation of presenting formulas, rectangular beam with square cross section of the side B $(=8\lambda_2 =8\lambda_3)$ as shown in Fig. 3, is considered. The following values are taken in the calculation. The computation was carried out by FACOM 230-75 Hokkaido university.

$$B = H = 15\,\text{cm},\ E = 2.1\times10^5\,\text{kg/cm}^2,\ \rho=\gamma/g,\ \gamma=2.4\,\text{t/m}^3,\ \gamma_s=7.85\,\text{t/m}^3$$
$$\nu = 0.18,\ \nu_s=0.3,\ V_p=c_p/c_s=1.601,\ c_p=3.051\,\text{km/sec},$$
$$c_s = 1.906\,\text{km/sec},\ E_s=2.1\times10^6\,\text{kg/cm}^2,\ \alpha=B/l,$$
$$V_E = c/c_s,\ EPS=1\times10^{-4}.$$

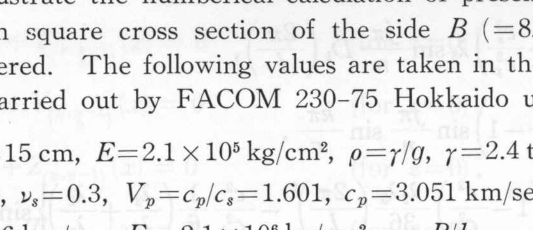

Fig. 3. Fiber-reinforced Square Beam.

Dispersion curves of harmonic flexural waves with and without reinforcement are shown in Fig. 4. And axial displacement mode of the waves are shown in Fig. 5 for $\alpha=0.5$ and Fig. 6 for $\alpha=2.5$.

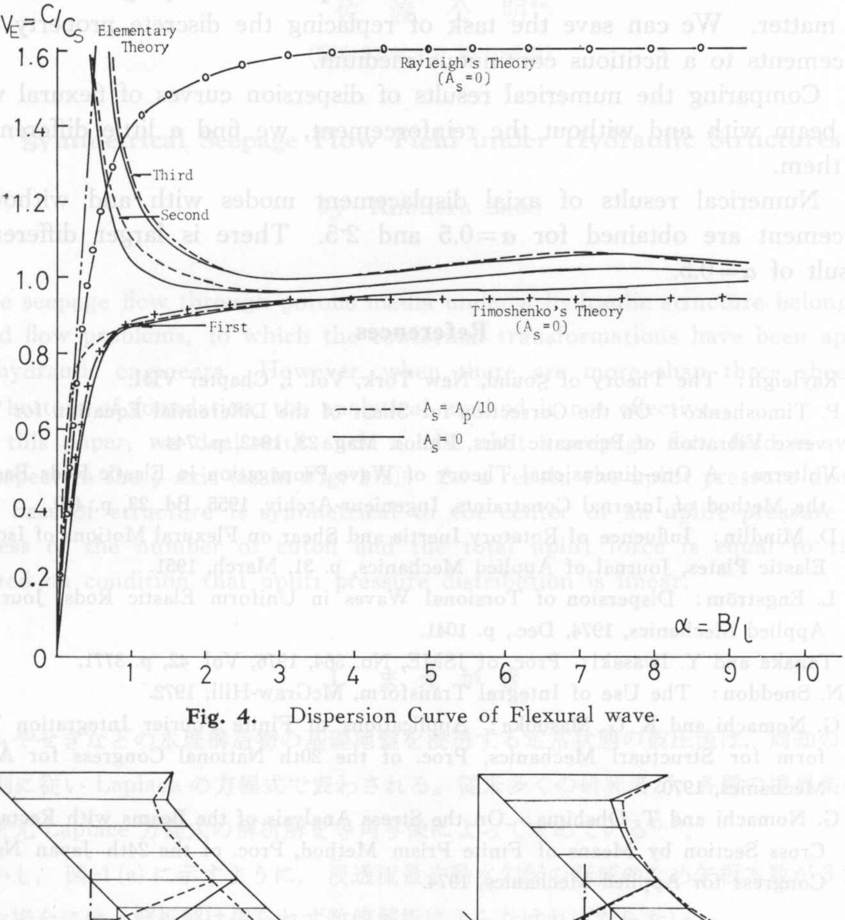

Fig. 4. Dispersion Curve of Flexural wave.

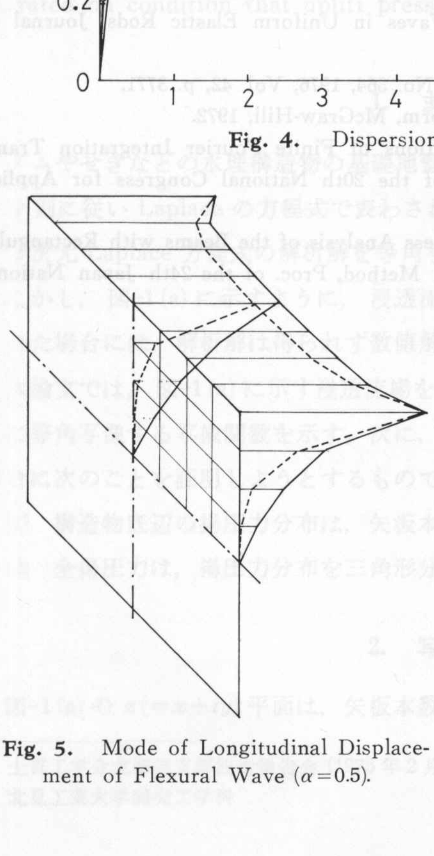

Fig. 5. Mode of Longitudinal Displacement of Flexural Wave ($\alpha=0.5$).

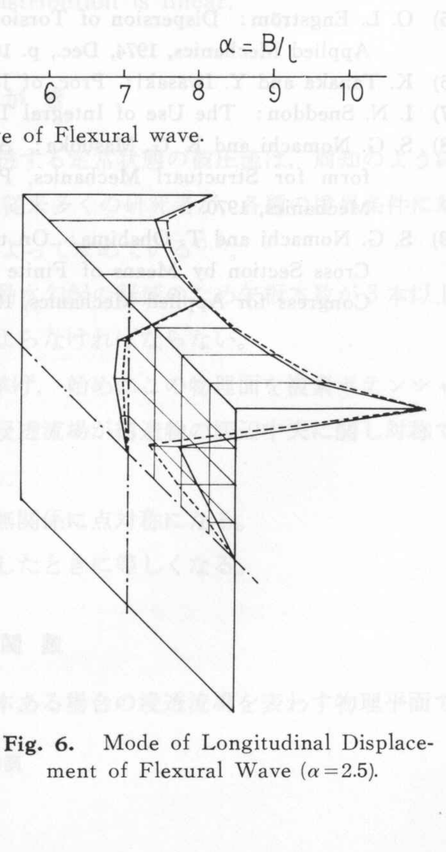

Fig. 6. Mode of Longitudinal Displacement of Flexural Wave ($\alpha=2.5$).

Final Remarks

1) Making use of finite prism method and finite Fourier integration transforms, we can handle the fiber reinforced beam problem, keeping the discreteness of the matter. We can save the task of replacing the discrete property of the reinforcements to a fictitious continuous medium.

2) Comparing the numerical results of dispersion curves of flexural wavers in the beam with and without the reinforcement, we find a little difference between them.

3) Numerical results of axial displacement modes with and without the reinforcement are obtained for $\alpha=0.5$ and $2\cdot5$. There is larger difference in the result of $\alpha=0.5$.

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