A Certain Consideration on Derivatives and Rolle's Theorem

by Kiro ISOBE*

(Received September 29, 1975)

Let $f(x)$ be differentiable, $\alpha<x<\beta$. We put

$$r(x, y)=\begin{cases} \dfrac{f(x)-f(y)}{x-y}, & x\neq y, \\ f'(x), & x=y. \end{cases} \tag{1}$$

We have the following:

(i) $r(x, y)=r(y, x)$,

(ii) $r(x, z)(x-z)=r(x, y)(x-y)+r(y, z)(y-z)$,

(iii) $\lim_{x\to y} r(x, y)=r(y, y)$,

(iv) r is continuous at a point (x, y) of $x\neq y$.

Newly, having no relation to (1), let $r(x, y)$ be a real function of 2-variables, $\alpha<x, y<\beta$. In [1], if $r(x, y)$ satisfies the previous (i), (ii), (iii), (iv), then we obtain Rolle's theorem and mean value theorem, namely,

(v) if $r(a, b)=0$, $a<b$, then there exists $c(a<c<b)$ such that $r(c, c)=0$,

(vi) for $a, b(a<b)$, there exists $c(a<c<b)$ such that $r(a, b)=r(c, c)$.

The purpose of this note is to point out that if $r(x, y)$ satisfies (i), (ii), (iii), then we have (iv), (v), (vi) and Cauchy's theorem, namely,

(vii) if a function $s(x, y)$ satisfies (i), (ii), (iii) and $s(x, y)\neq 0$, $(a\leqq x, y\leqq b)$ for $a<b$, then there exists $c(a<c<b)$ such that $\dfrac{r(a, b)}{s(a, b)}=\dfrac{r(c, c)}{s(c, c)}$.

1. Continuity of $r(x, y)$

We assume that $r(x, y)$ holds (i), (ii), (iii). To a point (a, b) of $a\neq b$, we put $\rho=\dfrac{a-b}{3}$. To a point (x, y) which $|x-a|<\rho$ and $|y-b|<\rho$, we have $|x-y|>\rho$ and since (ii)

$$\begin{aligned} r(x, y)(x-y) &= r(x, a)(x-a)+r(a, y)(a-y) \\ &= r(x, a)(x-a)+r(a, b)(a-b)+r(b, y)(b-y), \end{aligned}$$

$$r(x, y)=r(x, a)\frac{x-a}{x-y}+r(a, b)\frac{a-b}{x-y}+r(b, y)\frac{b-y}{x-y}. \tag{2}$$

Since (i) and (iii)

$$\lim_{x\to a} r(x, a)=r(a, a) \quad \text{and} \quad \lim_{y\to b} r(b, y)=\lim_{y\to b} r(y, b)=r(b, b).$$

Consequently there exists $\delta(0<\delta<\rho)$ and $M(>0)$ such that if $|x-a|<\delta$

---

* Department of Liberal Arts, Kitami Institute of Technology.

and $|y-b|<\delta$, then $|r(x,a)|\leqq M$ and $|r(y,b)|\leqq M$. Thus, since (2), if $|x-a|<\delta$ and $|y-b|<\delta$, then

$$|r(x,y)|\leqq|r(x,a)|\left|\frac{x-a}{x-y}\right|+|r(a,b)|\left|\frac{a-b}{x-y}\right| +|r(b,y)|\left|\frac{b-y}{x-y}\right|<2M+3|r(a,b)|.$$

Consequently we have the following:

(A) $r(x,y)$ is bounded in a neighborhood of a point (a,b) of $a\neq b$.

In the next place, to a point (a,b) of $a\neq b$,

$$r(a,b)(a-b)=r(a,x)(a-x)+r(x,y)(x-y)+r(y,b)(y-b),$$

$$r(a,b)=r(a,x)\frac{a-x}{a-b}+r(x,y)\frac{x-y}{a-b}+r(y,b)\frac{y-b}{a-b},$$

$$r(x,y)-r(a,b)=-r(a,x)\frac{a-x}{a-b}+r(x,y)\left(1-\frac{x-y}{a-b}\right)-r(y,b)\frac{y-b}{a-b}. \tag{3}$$

Since (A), there exists $\sigma(>0)$ and $\mathrm{A}(>0)$ such that if $|x-a|<\sigma$ and $|y-b|<\sigma$, then $|r(x,y)|\leqq \mathrm{A}$. Consequently, since (3), if $|x-a|<\sigma$ and $|y-b|<\sigma$, then

$$|r(x,y)-r(a,b)|\leqq|r(a,x)|\left|\frac{a-x}{a-b}\right|+|r(x,y)|\left|1-\frac{x-y}{a-b}\right|+|r(y,b)|\left|\frac{y-b}{a-b}\right| \leqq \mathrm{A}\left(\left|\frac{a-x}{a-b}\right|+\left|1-\frac{x-y}{a-b}\right|+\left|\frac{y-b}{a-b}\right|\right).$$

Thus, $\lim\limits_{(x,y)\to(a,b)} r(x,y)=r(a,b)$ and (iv) is established.

Furthermore, supposing that $x<c<y$,

$$r(x,y)=r(x,c)\frac{x-c}{x-y}+r(c,y)\frac{c-y}{x-y} \quad \text{(by (ii))}.$$

Putting $x=c-h$ and $y=c+k$, $h>0$, $k>0$ and

$$r(x,y)=r(x,c)\frac{h}{h+k}+r(c,y)\frac{k}{h+k}.$$

Putting $r(c-h,c)=r(c,c)+\varepsilon$ and $r(c,c+k)=r(c,c)+\varepsilon'$, $\varepsilon,\varepsilon'\to 0$ as $h,k\to 0$ (by (i) and (ii)). Consequently

$$r(x,y)=(r(c,c)+\varepsilon)\frac{h}{h+k}+(r(c,c)+\varepsilon')\frac{k}{h+k}=r(c,c)+\frac{\varepsilon h}{h+k}+\frac{\varepsilon' k}{h+k}.$$

Thus,

$$|r(x,y)-r(c,c)|\leqq\left|\frac{\varepsilon h}{h+k}\right|+\left|\frac{\varepsilon' k}{h+k}\right|<|\varepsilon|+|\varepsilon'|.$$

Hence, we have the following:

(B) $\lim\limits_{\substack{x,y\to c\\ x<c<y}} r(x,y)=r(c,c)$.

2. Rolle's theorem

We assume that $r(c, x) \equiv 0$ $(a \leqq x \leqq b)$. For $a \leqq x,\ y \leqq b$,

$$r(x, y)(x-y) = r(x, c)(x-c) + r(c, y)(c-y) = 0\,.$$

Thus, if $x \neq y$, then $r(x, y) = 0$ and since this result, if $x = y$, then

$$r(x, y) = r(y, y) = \lim_{x' \to y,\, a \leqq x' \leqq b} r(x', y) = 0\,.$$

Consequently, we obtain the following:

(C) if $r(c, x) \equiv 0$ $(a \leqq x \leqq b)$, then $r(x, y) \equiv 0$ $(a \leqq x, y \leqq b)$.

Now we assume $r(a, b) = 0$ $(a < b)$. We put $\lambda = \dfrac{b-a}{3}$. If $r(a, x) \equiv 0$ $(a+\lambda \leqq x \leqq b-\lambda)$, then $r(c, c') = 0$ $(a+\lambda \leqq c,\ c' \leqq b-\lambda)$ by (C). Otherwise, if there exists c_1 such that $r(a, c_1) \neq 0$ and $a+\lambda \leqq c_1 \leqq b-\lambda$, then $r(a, c_1) \neq 0$, $r(c_1, b) \neq 0$ and sign of $r(a, c_1)$ is different from sign of $r(c_1, b)$ because

$r(a, b)(a-b) = r(a, c_1)(a-c_1) + r(c_1, b)(c_1-b) = 0$. $r(c_1, x)$ is continuous function of x over $a \leqq x \leqq b$ by (iii) and (iv). Consequently, by virtue of previous result and midvalue theoerm of continuous functions, there exists c_1' such that $r(c_1, c_1') = 0$ and $a < c_1' < b$. Since $a+\lambda \leqq c_1 \leqq b-\lambda$ and $a < c_1' < b$, we have $|c_1' - c_1| < 2\lambda = \dfrac{2}{3}(b-a)$. We assume $c_1 < c_1'$ without loss of generality and doing similar step over and over again, we obtain a sequence that $a < c_1 < c_2 < \cdots < c_2' < c_1' < b$ such as $c_n' - c_n < \left(\dfrac{2}{3}\right)^n (b-a)$. Consequently, putting $c = \lim_{n\to\infty} c_n = \lim_{n\to\infty} c_n'$, $r(c, c) = 0$ by (B). Thus (v) is established.

Now, if we put $R(x, y) = r(x, y) - r(a, b)$, then $R(x, y)$ satisfies (i), (ii), (iii) and $R(a, b) = 0$. By virtue of (v), there exists c such that

$$R(c, c) = r(c, c) - r(a, b) = 0 \quad \text{and} \quad a < c < b\,.$$

Thus (vi) is established.

Now, let $s(x, y)$ satisfy the conditions (i), (ii), (iii) and $s(x, y) \neq 0$ $(a \leqq x \leqq b)$. If we put $R(x, y) = r(x, y) - \dfrac{r(a, b)}{s(a, b)} s(x, y)$, then $R(x, y)$ satisfies the conditions (i), (ii), (iii) and $R(a, b) = 0$. By virtue of (v), there exists c such that $R(c, c) = r(c, c) - \dfrac{r(a, b)}{s(a, b)} s(c, c) = 0$ and $a < c < b$.

Thus we obtain (vii).

References

[1] K. Isobe: Rolle の定理, 平均値の定理について (Some note on Rolle's theorem and mean value theorem) (unpublished).