Note on Convex Subgroups of Partially Ordered Abelian Groups

by Kiro ISOBE*

(Received April 30, 1975)

Let G be a partially ordered abelian group and generated by G^+.

For two convex subgroups H and K of G, looking at the question generally we cannot obtain that $H+K$ is convex. We show some examples in such a case and we investigate into the case that $H+K$ is convex.

1. Notation and terminology. All groups considered herein are abelian. Let G be a partially ordered abelian group and generated by G^+. We shall say that the order in G is *normal*, if a integer $n>0$ and $na\geq 0$ $(a\in G)$ implies $a\geq 0$. To every element $a\in G$ we define the notation P_a such that $P_a=(x: 0\leq x\leq b$ for every $b\geq a$ and $b\geq 0)$. It is clear that $a\leq b$ implies $P_a\subset P_b$. To a set $S\subset G$ we shall also say that S is *essentially convex*, if $a\in H$ implies $P_a\subset H$. Of course, if S is essentially convex, then S is convex. We shall say that the order in G is *semilattice*, if $P_a=(0)$ implies $a\leq 0$. Of course, if the order in G is lattice, then it is semilattice.

2. Some examples. Totality of all convex subgroups of G is lattice ordered set, namely, if H and K are two convex subgroups of G, then $H\wedge K=H\cap K$ and $H\vee K=\cap(X: X$ convex subgroup, $X\supset H+K)$. Generally $H\vee K$ do not coincide with $H+K$.

Example 1. Assuming that the order in G is normal, it is clear that the subgroup $(na: n$ integer$)$ is convex, if an element $a\in G$ cannot compare to 0. Let G be 2-dimensional integer group. $G=((m, n): m, n$ integers$)$.

We define $(m, n)\leq(m', n')$, if $m\leq m'$ and $n\leq n'$. We put $H=((-m, m): m$ integer$)$ and $K=((-n, 3n): n$ integer$)$. H and K are two convex subgroups of G but $H+K$ is not convex. In fact,

$$H+K=\big((-m-n, m+3n): m, n \text{ integers}\big).$$

We have $0<(1, 4)<(2, 6)=(6, -6)+(-4, 12)\in H+K$ but $(1, 4)\bar{\in} H+K$. Because, if $1=-m-n$ and $4=m+3n$, then $n=\dfrac{5}{2}$ is not an integer.

Example 2. In example 1, there is no strictly positive element in both H and K. We consider to the case that there is a strictly positive element in H or K. Let G be the same group in example 1. We put

$$H=\big((0, m): m \text{ integer}\big) \text{ and } K=\big((-2n, n): n \text{ integer}\big)$$

Thus H and K are two convex subgroups of G and H contains a strictly

* Department of Liberal Arts, Kitami Institute of Technology

positive element but $H+K$ is not convex. In fact,

$$H+K=\left((-2n, m+n) : m, n \text{ integers}\right).$$

Though $0<(1,0)<(2,0)=(0,1)+(-2\cdot(-1),-1)\in H+K$, we have $(1,0)\bar{\in} H+K$. Because, if $1=-2n$ and $0=m+n$, then $n=-\frac{1}{2}$ is not an integer.

Example 3. We consider to the case that there are strictly positive elements in two convex subgroups H and K of G both. Let G be 3-dimensional integer group. $G=((l, m, n) : l, m, n \text{ integers})$. We define $(l, m, n)\leq(l', m', n')$, if and only if, $l<l'$ and $m<m'$ or $l=l'$, $m=m'$ and $n\leq n'$. We put

$$\boldsymbol{a}=(1, -7, 1), \quad \boldsymbol{b}=(-1, 7, 0) \text{ and}$$

$$A=(k\boldsymbol{a} : k \text{ integer}), \quad B=(l\boldsymbol{b} : l \text{ integer}), \quad H=A+B.$$

Eurthermore we put $\boldsymbol{c}=(5, -1, 1)$, $\boldsymbol{d}=(-5, 1, 0)$ and $C=(m\boldsymbol{c} : m \text{ integer})$, $D=(n\boldsymbol{d} : n \text{ integer})$, $K=C+D$. H is convex subgroup. In fact, if

$$0\leq(x, y, z)\leq(k, -7k, k)+(-l, 7l, 0)=(k-l, -7k+7l, k)\in H,$$

then $k-l=0$ and $x=y=0$. Thus $(x, y, z)=(0, 0, z)=(z, -7z, z)+(-z, 7z, 0)\in H$.

Similarly K is convex subgroup. It is clear that both of H and K have strictly positive elements.

We have $H+K=(k\boldsymbol{a}+l\boldsymbol{b}+m\boldsymbol{c} : k, l, m \text{ integers})$

$$=\left((k-l+5m, \ -7k+7l-m, \ k+m) : k, l, m, \text{ integers}\right), \text{ since } \boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}=\boldsymbol{d}.$$

Although $0<(2, 4, 0)<(4, 6, 0)=(5, -1, 1)-(1, -7, 1)=\boldsymbol{c}-\boldsymbol{a}\in H+K$, we have $(2, 4, 0)\bar{\in} H+K$. Because, if $2=k-l+5m$, $4=-7k+7l-m$ and $0=k+m$, then $k=-\frac{9}{17}$ is not an integer.

3. Theorem for $H\vee K=H+K$. Let G be semilattice ordered. We assume that two subgroups H and K of G are essentially convex and $H\cap K=(0)$.

Proposition. *If $0\leq h+k$, $h\in H$ and $k\in K$, then $0\leq h$ and $0\leq k$.*

Proof. Since $-h\leq k$, we have $P_h\subset P_k$ and $P_{-h}\subset H\cap K=(0)$. Thus $-h\leq 0$, consequently $h\geq 0$ and similarly $k\geq 0$.

Theorem. *$H+K$ is convex, if and only if, $0\leq x\leq h+k$, $h\in H$ and $k\in K$ implies $x\wedge h$ and $x\wedge k$ exist.*

Proof. We assume that $H+K$ is convex. If $0\leq x\leq h+k$, $h\in H$, and $k\in K$, then $0\leq x=x_1+x_2\leq h+k$, $x_1\in H$ and $x_2\in K$. Thus $0\leq(h-x_1)+(k-x_2)$, $h-x_1\in H$ and $k-x_2\in K$ and we have $x_1\leq h$ and $x_2\leq k$. It is clear that $x_1\leq x$ and $x_2\leq x$, since $x_1\geq 0$ and $x_2\geq 0$. On the other hand, if $y\leq h$ and $y\leq x=x_1+x_2$, then $y-x_1\leq h-x_1$, $y-x_1\leq x_2$ and $P_{y-x_1}\subset H\cap K=(0)$. Thus $y\leq x_1$, consequently $x_1=x\wedge h$ and similarly $x_2=x\wedge k$.

Conversely we assume that if $0\leq x\leq h+k$, $h\in H$ and $k\in K$, then $h\wedge x$ and $k\wedge x$ exist.

$$(x\wedge h)+(x\wedge k)=(2x)\wedge(h+x)\wedge(x+k)\wedge(h+k)\geq x.$$

On the other hand, since $h \wedge k = 0$, we have $h + k = h \vee k$.

$$x = (h+k) \wedge x = (h \vee k) \wedge x = (h \wedge x) \vee (k \wedge x) = (h \wedge x) + (k \wedge x).$$

Consequently $x = (x \wedge h) + (x \wedge k) \in H + K$.

4. Supplement. We consider to a case that a partially ordered abelian group is not normal.

Example 4. Let G be a totality of all integers. For two elements x and y of G we define $x \underset{*}{\leq} y$, if and only if, $x = y$ or $y - x$ is positive even integer.

Thus the element 'a positive odd integer n' cannot compare with 0 but $n = n = 2 \cdot n > 0$. Consequently G is not normal.

In example 4 G is not generated by G^+.

Example 5. Let G be 2-dimensional integer group. For two elements (m, n) and (m', n') we difine $(m, n) \leq (m', n')$, if and only if, $m < m'$ or $m = m'$ and $n \underset{*}{\leq} n'$ (sense in example 4). Thus for a positive odd integer n, the element $(0, n)$ cannot compare with 0 but $(0, 2n) = 2 \cdot (0, n) > 0$. Consequently G is not normal and generated by G^+.

References

[1] Paul Hill and Joe L. Mott: Embedding theorems and generalized discrete ordered abelian groups. Trans. Amer. Math. Soc. **Vol. 175**, 1973.

[2] K. Isobe: Some properties of a certain pertially ordered linear space. Mem. Kitami Inst. Tech. **Vol.** 3, No. 2, 1972.

[3] K. Isobe: Some examples in semi-ordered linear space. Mem. Kitami Inst. Tech. **Vol.** 2, No. 2, 1968.