On the Structural Analysis of the Langered Plate

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The stress distributions and deflections of the composite structure which is built up with a deck plate and two spandrel arches stiffening the plate at the edges, are disccussed.

In dividing the deck plate into long strips, and recomposing the structure so as to apply Finite Fourier Integration Transform[1], the authors obtained the solutions in simple forms, which is feasible to take the boundary conditions of the plate into account.

As the numerical examples, the structure which has two opposite edges simply supported and the other two edges rested on the stiffening frame works, and is subjected to uniform load, are computed.

Introduction

The plate structure of its deck plate discretely supported along the edge lines with vertical frame works, as shown in Fig. 1, is supposed to be a kind of a deck type bridge structure.

If the aspect ratio of the length and width is large, we can consider it as a Langer girder. However, if the aspect ratio is small, we may call it "Langered plate".

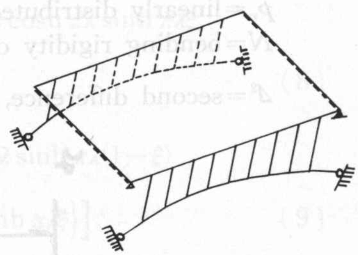

Fig. 1. Langered plate.

In the latter case, the transverse elastic behaviour becomes more complicated one than the former, and it may be interesting for the design to make the transverse elastic behaviour clear.

Therefore the long strip method based on the plate theory, is used for the stress analysis of the deck plate. So doing the bending moment, torsional moment and shearing force in the transverse direction can be expressed. On the long side edges, the boundary values are represented by the mean values over the short width.

In this way we find enough number of the equations of the equilibrium of forces about every nodes to solve the necessary displacements out.

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Basic Formulas

1) Finite Strip Method[2)]

Let us consider the case when a plate is divided into long strip elements, as shown in Fig. 2, and assuming that the deflection surface concerning a strip element is

$$w = w_r \cdot f^{(1)}(\eta) + w_{r+1} \cdot f^{(1)}(1-\eta) + \frac{M_r^* b^2}{N} \cdot f^{(3)}(\eta) + \frac{M_{r+1}^* \cdot b^2}{N} \cdot f^{(3)}(1-\eta) \tag{1}$$

where

$$f^{(1)}(\eta) = 1-\eta\,, \quad \eta = \frac{y}{b}\,, \quad M_r^* = -N\left(\frac{\partial^2 w}{\partial y^2}\right)_r, \quad f^{(3)}(\eta) = \frac{1}{6}\eta(1-\eta)(2-\eta)$$

the basic differential and difference equations are expressed as follows:

$$\frac{1}{b}\Delta_r^2 w_{r-1} + \frac{b}{6N}(6M_r^* + \Delta_r^2 M_{r-1}^*) = 0\,, \tag{2}$$

$$\frac{1}{b}\Delta_r^2 M_{r-1}^* - \frac{N}{b}\Delta_r^2 \ddot{w}_{r-1} + \frac{b}{6}(6\ddot{M}_r^* + \Delta_r^2 \ddot{M}_{r-1}^*) - \frac{bN}{6}(6\ddddot{w}_r + \Delta_r^2 \ddddot{w}_{r-1}) + p_r = 0 \tag{3}$$

where

p_r = linearly distributed load along the intersection,
N = bending rigidity of plate,
Δ^2 = second difference, and $\dot{f} = \frac{\partial f}{\partial x}$.

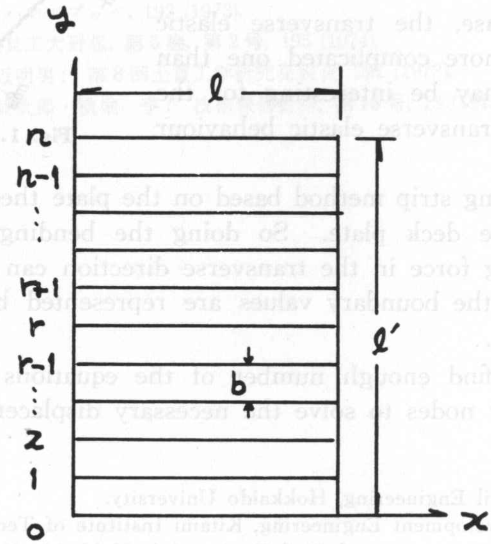

Fig. 2. Finite strip element.

2) **Finite Fourier Integration Transform**[1)]

Let us introduce the symbolic notation

$$\boldsymbol{S}_i\left[f(x)\right]=\sum_{x=1}^{n-1} f(x)\sin\frac{i\pi x}{n}, \tag{4}$$

of which inversion formula is

$$f(x)=\frac{2}{n}\sum_{i=1}^{n-1}\boldsymbol{S}_i\left[f(x)\right]\sin\frac{i\pi x}{n}. \tag{5}$$

Applying the above formulas to the sine transform, we have

$$\boldsymbol{S}_i\left[\Delta^2 f(x-1)\right]=-\sin\frac{i\pi}{n}\left\{(-1)^i f(n)-f(0)\right\}-D_i\boldsymbol{S}_i\left[f(x)\right], \tag{6}$$

where

$$D_i=2\left(1-\cos\frac{i\pi}{n}\right).$$

3) **Closed Form Solution Concerning Fourier Series**

Following formulas have been introduced by Dr. S. Iguchi[3)] :

$$\sum_m \frac{m}{(m^2+\lambda^2)^2}\sin m\pi\xi=\frac{\pi^2}{4\lambda\sinh^2\pi\lambda}\left\{\xi\cosh\pi\lambda\cdot\sinh\pi\lambda(1-\xi)\right.$$
$$\left.-(1-\xi)\sinh\pi\lambda\xi\right\}, \tag{7}$$

$$\sum_m \frac{(-1)^m m}{(m^2+\lambda^2)^2}\sin m\pi\xi=-\frac{\pi^2}{4\lambda\sinh^2\pi\lambda}\left\{(1-\xi)\cosh\pi\lambda\sinh\pi\lambda\xi\right.$$
$$\left.-\xi\sinh\pi\lambda(1-\xi)\right\}, \tag{8}$$

$$\sum_m \frac{m^3}{(m^2+\lambda^2)^2}\sin m\pi\xi=\frac{\pi}{4\cdot\sinh^2\pi\lambda}\left[2\sinh\pi\lambda\sinh\pi\lambda(1-\xi)\right.$$
$$\left.-\pi\lambda\left\{\xi\cosh\pi\lambda\sinh\pi\lambda(1-\xi)-(1-\xi)\sinh\pi\lambda\xi\right\}\right], \tag{9}$$

$$\sum_m \frac{(-1)^m m^3}{(m^2+\lambda^2)^2}\sin m\pi\xi=-\frac{\pi}{4\cdot\sinh^2\pi\lambda}\left[2\sinh\pi\lambda\sinh\pi\lambda\xi\right.$$
$$\left.-\pi\lambda\left\{(1-\xi)\cosh\pi\lambda\sinh\pi\lambda\xi-\xi\sinh\pi\lambda(1-\xi)\right\}\right], \tag{10}$$

$$\sum_m \frac{1}{m(m^2+\lambda^2)^2}\sin m\pi\xi=-\frac{\pi}{4\lambda^4\sinh^2\pi\lambda}\left[2\sinh\pi\lambda\sinh\pi\lambda(1-\xi)\right.$$
$$\left.+\pi\lambda\left\{\xi\cosh\pi\lambda\sinh\pi\lambda(1-\xi)-(1-\xi)\sinh\pi\lambda\xi\right\}\right]+\frac{\pi}{2\lambda^4}(1-\xi) \tag{11}$$

$$\sum_m \frac{(-1)^m}{m(m^2+\lambda^2)^2}\sin m\pi\xi=\frac{\pi}{4\cdot\lambda^4\sinh^2\pi\lambda}\left[2\cdot\sinh\pi\lambda\sinh\pi\lambda\xi\right.$$
$$\left.+\pi\lambda\left\{(1-\xi)\cosh\pi\lambda\sinh\pi\lambda\xi-\xi\sinh\pi\lambda(1-\xi)\right\}\right]-\frac{\pi}{2\lambda^4}\xi, \tag{12}$$

$$\sum_m \frac{1}{(m^2+\lambda^2)^2}\cos m\pi\xi = \frac{\pi}{4\lambda^3}\left[\frac{1}{\sinh^2\pi\lambda}\left[\sinh\pi\lambda\cosh\pi\lambda(1-\xi)\right.\right.$$

$$\left.\left. +\pi\lambda\left\{\cosh\pi\lambda\cosh\pi\lambda(1-\xi)-(1-\xi)\sinh\pi\lambda\sinh\pi\lambda(1-\xi)\right\}\right]-\frac{\pi\lambda}{2}\right] \tag{13}$$

The Analysis of the Langered Plate

Let us consider a structure of which two opposite sides $y=0$ and $y=l'$ are simply supported and the other two sides $x=0$ and $x=l$ are supported through the vertical members to the chord members of the structure.

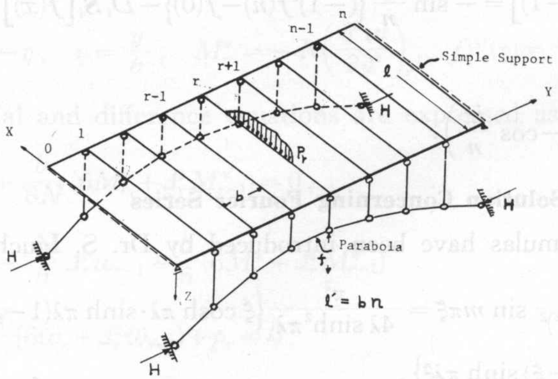

Fig. 3. Langered plate.

Assuming that the shape of the chord members is parabolic and that the structure together with external loads are symmetrical with respect to the midspan of the plate, the deformation will occur symmetrically with respect to the midspan.

Applying the Finite Fourier Transform in the x direction and the Finite Fourier Integration Transform in the y direction to Eqs. (2) and (3), we have

$$\left[\frac{D_i}{b}+b\left(1-\frac{D_i}{6}\right)\left(\frac{m\pi}{l}\right)^2\right]\boldsymbol{S}_i[\widetilde{M}^*_{rm}]+\left[\frac{N}{b}D_i\left(\frac{m\pi}{l}\right)^2\right.$$
$$\left.+bN\left(1-\frac{D_i}{6}\right)\left(\frac{m\pi}{l}\right)^4\right]\boldsymbol{S}_i[\widetilde{w}_{rm}]=b\left(1-\frac{D_i}{6}\right)\left(\frac{m\pi}{l}\right)\beta_m\boldsymbol{S}_i[M^*_{r0}]$$
$$+\left\{\frac{N}{b}D_i\left(\frac{m\pi}{l}\right)+bN\left(1-\frac{D_i}{6}\right)\left(\frac{m\pi}{l}\right)^3\right\}\beta_m\cdot\boldsymbol{S}_i[w_{r0}]$$
$$-bN\left(1-\frac{D_i}{6}\right)\left(\frac{m\pi}{l}\right)\beta_m\boldsymbol{S}_i[\ddot{w}_{r0}]+\int_0^l\boldsymbol{S}_i[p_r]\sin\frac{m\pi x}{l}\,dx\,, \tag{14}$$

$$\boldsymbol{S}_i[\widetilde{w}_{rm}]=\frac{b^2}{ND_i}\left(1-\frac{D_i}{6}\right)\cdot\boldsymbol{S}_i[\widetilde{M}^*_{rm}] \qquad (i\neq 0) \tag{15}$$

where

$$\left[\tilde{f}(x)\right]=\int_0^l f(x)\sin\frac{m\pi x}{l}\,dx\,,\quad \beta_m=1-(-1)^m\,,$$

Substituting Eq. (15) to Eq. (14) and performing the Fourier inverse transform to the result, we obtain

$$\boldsymbol{S}_i[M^*_{rx}] = \frac{2l^3}{a_{10}\pi^4}\left[\sum_m \frac{a_{11}\, m\cdot\beta_m}{\left(m^2+\frac{a_{20}}{2}\right)^2}\sin\frac{m\pi x}{l}\cdot\boldsymbol{S}_i[M^*_{r0}]\right.$$

$$+\sum_m \frac{a_{12}m^3+a_{13}m}{\left(m^2+\frac{a_{20}}{2}\right)^2}\cdot\beta_m\sin\frac{m\pi x}{l}\cdot\boldsymbol{S}_i[w_{r0}]$$

$$\left.+\sum_m \frac{\int_0^b \boldsymbol{S}_i[p_r]\sin\frac{m\pi x}{l}\,dx}{\left(m^2+\frac{a_{20}}{2}\right)^2}\sin\frac{m\pi x}{l}\right] \tag{16}$$

where

$$a_{10}=\frac{b^3}{D_i}\left(1-\frac{D_i}{6}\right)^2,\quad a_{20}=\frac{2b}{a_{10}\left(\frac{\pi}{l}\right)^2}\left(1-\frac{D_i}{6}\right),$$

$$a_{11}=b\left(\frac{\pi}{l}\right)\left(1-\frac{D_i}{6}\right)(1-\nu),\quad a_{12}=b\left(\frac{\pi}{l}\right)^2 N\left(1-\frac{D_i}{6}\right),\quad \text{and}$$

$$a_{13}=\frac{N}{b}\left(\frac{\pi}{l}\right)D_i\,.$$

Applying the equations of closed form, we obtain

$$\boldsymbol{S}_i[M^*_{rx}]=A_0\Big[\big\{P(\xi)+P(1-\xi)\big\}\boldsymbol{S}_i[M^*_{r0}]$$

$$+\big\{Q_1(\xi)+Q_1(1-\xi)+Q_2(\xi)+Q_2(1-\xi)\big\}\boldsymbol{S}_i[w_{r0}]+S(\xi)\Big] \tag{17}$$

where

$$A_0=\frac{2l^3}{a_{10}\pi^4},\quad P(\xi)=\sum_m \frac{a_{11}m}{\left(m^2+\frac{a_{20}}{2}\right)^2}\sin\frac{m\pi x}{l},\quad \xi=\frac{x}{l},$$

$$P(1-\xi)=\sum_m \frac{a_{11}m(-1)^m}{\left(m^2+\frac{a_{20}}{2}\right)^2}\sin\frac{m\pi x}{l},\quad Q_1(\xi)=\sum_m \frac{a_{12}m^3}{\left(m^2+\frac{a_{20}}{2}\right)^2}\sin\frac{m\pi x}{l},$$

$$Q_1(1-\xi)=\sum_m \frac{a_{12}m^3(-1)^m}{\left(m^2+\frac{a_{20}}{2}\right)^2}\sin\frac{m\pi x}{l},\quad Q_2(\xi)=\sum_m \frac{a_{13}m}{\left(m^2+\frac{a_{20}}{2}\right)^2}\sin\frac{m\pi x}{l},$$

$$Q_2(1-\xi)=\sum_m \frac{a_{13}m(-1)^m}{\left(m^2+\frac{a_{20}}{2}\right)^2}\sin\frac{m\pi x}{l},\quad \text{and}$$

$$S(\xi)=\sum_m \frac{\int_0^l \boldsymbol{S}_i[p_r]\sin\frac{m\pi x}{l}\,dx}{\left(m^2+\frac{a_{20}}{2}\right)^2}\sin\frac{m\pi x}{l}\,.$$

Boundary Conditions

The boundary condition concerning M_{xy} and Q_x along the edge line $x=0$, are expressed as :

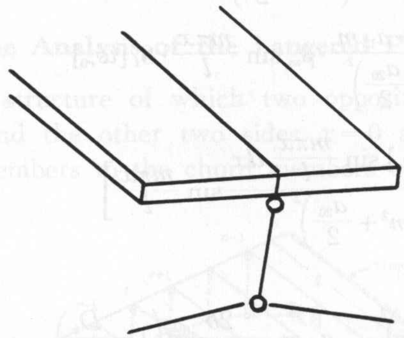

Fig. 4. Boundary condition.

$$M_{xy} = -N(1-V)\left(\frac{\partial^2 w}{\partial x \partial y}\right)_{x=0} = 0\,, \quad \text{and} \tag{18}$$

$$\bar{Q}_{r\cdot r+1}(0) + \bar{Q}_{r\cdot r-1}(0) = S_{vr}(0)\,. \tag{19}$$

where

$$\bar{Q}_{r\cdot r+1}(0) = \int_0^b Q_{r\cdot r+1}(0)\,dy \quad \text{and}$$

$$Q_x = -N(\varDelta \dot{w})$$

Eq. (18) can be rewritten by Eq. (1) as

$$\frac{D_i}{b}\boldsymbol{S}_i[\dot{w}_{r0}] - \frac{b}{N}\left(1-\frac{D_i}{6}\right)\boldsymbol{S}_i[\dot{M}^*_{r0}] = 0 \tag{20}$$

And we obtain from Eq. (19)

$$b\left(1-\frac{D_i}{6}\right)\left(\boldsymbol{S}_i[\dot{M}^*_{r0}] - N\boldsymbol{S}_i[\dddot{w}_{r0}]\right)$$

$$+\frac{b^3}{360}(30-7D_i)\left(\boldsymbol{S}_i[\dddot{M}^*_{r0}] - N\boldsymbol{S}_i[\ddddot{w}_{r0}]\right) = \boldsymbol{S}_i\left[S_{vr}(0)\right] \tag{21}$$

Substituting Eq. (17) into Eqs. (20) and (21), we have two equations of boundary conditions concerning $\boldsymbol{S}_i[M^*_{r0}]$ and $\boldsymbol{S}_i[w_{r0}]$:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{S}_i[M^*_{r0}] \\ \boldsymbol{S}_i[w_{r0}] \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 + \dfrac{1}{A_0}\boldsymbol{S}_i\left[S_{vr}(0)\right] \end{bmatrix} \tag{22}$$

where

$$A_{11} = \dot{P}(0) + \dot{P}(1)\,, \quad A_{12} = \dot{Q}_1(0) + \dot{Q}_1(1) + \dot{Q}_2(0) + \dot{Q}_2(1)\,,$$

$$A_{21}=b\left(1-\frac{D_i}{6}\right)\left\{\dot{P}(0)+\dot{P}(1)\right\}+\left\{-\frac{b^3}{D_i}\left(1-\frac{D_i}{6}\right)^2+\frac{b^3}{360}(30-7D_i)\right\}$$
$$\times\left\{\dddot{P}(0)+\dddot{P}(1)\right\}-\frac{b^5}{360D_i}(30-7D_i)\left(1-\frac{D_i}{6}\right)\left\{\overset{\cdot\cdot\cdot\cdot\cdot}{P}(0)+\overset{\cdot\cdot\cdot\cdot\cdot}{P}(1)\right\},$$

$$A_{22}=b\left(1-\frac{D_i}{6}\right)\left\{\dot{Q}_1(0)+\dot{Q}_1(1)+\dot{Q}_2(0)+\dot{Q}_2(1)\right\}$$
$$+b^3\left\{-\frac{1}{D_i}\left(1-\frac{D_i}{6}\right)^2+\frac{1}{360}(30-7D_i)\right\}\left\{\dddot{Q}_1(0)+\dddot{Q}_1(1)+\dddot{Q}_2(0)+\dddot{Q}_2(1)\right\}$$
$$-\frac{b^5}{360D_i}(30-7D_i)\left(1-\frac{D_i}{6}\right)\left\{\overset{\cdot\cdot\cdot\cdot\cdot}{Q}_1(0)+\overset{\cdot\cdot\cdot\cdot\cdot}{Q}_1(1)+\overset{\cdot\cdot\cdot\cdot\cdot}{Q}_2(0)+\overset{\cdot\cdot\cdot\cdot\cdot}{Q}_2(1)\right\},$$

$$B_1=-\dot{S}(0),\quad \text{and}\quad B_2=-b\left(1-\frac{D_i}{6}\right)\dot{S}(0)$$
$$-b^3\left\{-\frac{1}{D_i}\left(1-\frac{D_i}{6}\right)^2+\frac{1}{360}(30-7D_i)\right\}$$
$$\times\dddot{S}(0)-\frac{b^5}{360D_i}(30-7D_i)\left(1-\frac{D_i}{6}\right)\overset{\cdot\cdot\cdot\cdot\cdot}{S}(0).$$

And we have

$$S_{vr}(0)=H\cdot C \tag{23}$$

where H = horizontal reaction at the end of frame work, and
$C=\tan\alpha_r-\tan\alpha_{r+1}$

, then we obtain

$$\boldsymbol{S}_i[w_{r0}]=\boldsymbol{M}_i+\boldsymbol{N}_i\cdot H, \tag{24}$$

$$\boldsymbol{S}_i[M^*_{r0}]=\boldsymbol{K}_i+\boldsymbol{L}_i\cdot\boldsymbol{S}_i\left[S_{vr}(0)\right],\quad\text{and} \tag{25}$$

$$w_{r0}=\frac{2}{n}\sum_{i=1}^{n-1}(\boldsymbol{M}_i+\boldsymbol{N}_iH)\sin\frac{i\pi r}{n} \tag{26}$$

where

$$\boldsymbol{M}_i=\frac{B_1}{A_{12}}-\frac{A_{11}}{A_{12}}\cdot\frac{B_1A_{22}-A_{12}B_2}{A_{11}A_{22}-A_{12}A_{21}},\quad \boldsymbol{N}_i=\frac{A_{11}}{A_{12}}\cdot\frac{A_{12}\cdot\boldsymbol{S}_i[C]}{A_0(A_{11}A_{22}-A_{12}A_{21})},$$

$$\boldsymbol{K}_i=\frac{B_1A_{22}-A_{12}B_2}{A_{11}A_{22}-A_{12}A_{21}},\quad\text{and}\quad \boldsymbol{L}_i=-\frac{A_{12}}{A_0}\,\frac{1}{A_{11}A_{21}-A_{12}A_{21}}.$$

From boundary condition for the lower chord members in the horizontal direction, we obtain[4)]

$$\sum_r\frac{H\cdot b\sec^3\alpha_r}{E\cdot A_r^{(S)}}+\sum_r\frac{H\cdot C}{E}\left(\frac{h_r}{A_r^{(V)}}-\frac{h_{r-1}}{A_{r-1}^{(V)}}\right)\tan\alpha_r+\sum_r w_{r0}\cdot C=0 \tag{27}$$

where $A_r^{(S)}$ = sectional area of chord member at $y=r$, and
$A_r^{(V)}$ = sectional area of vertical member at $y=r$.

Substituting Eq. (26) to Eq. (27) yields

$$H=\frac{\frac{2C}{n}\sum\left(\sum_i \boldsymbol{M}_i\sin\frac{i\pi r}{n}\right)}{\sum_r\frac{EA_r^{(S)}}{b\sec^3\alpha_r}+\sum_r\frac{C}{E}\left(\frac{h_r}{A_r^{(V)}}-\frac{h_{r-1}}{A_{r-1}^{(V)}}\right)\tan\alpha_r+\frac{2C}{n}\sum_r\left(\sum_i \boldsymbol{N}_i\sin\frac{i\pi r}{n}\right)} \tag{28}$$

Numerical Examples

For the numerical calculation, the following values are taken as :

$E=2.1\times10^6\ \mathrm{kg/cm^2}$, $\nu=0.3$, $t=10\ \mathrm{cm}$,

$l=8.5\ \mathrm{m}$, $l'=18\ \mathrm{m}$, $b=1.5\ \mathrm{m}$, $n=12$,

$f=10\ \mathrm{m}$, $A^{(S)}=A^{(V)}=25\ \mathrm{cm^2}$, and

fully uniform load $q=1\ \mathrm{kg/cm^2}$.

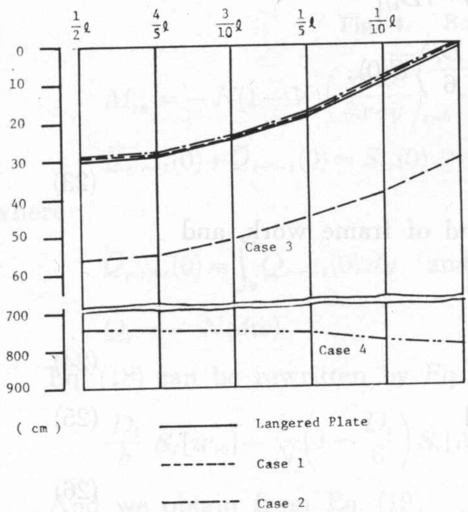

Fig. 5. w along $y=l'/2$

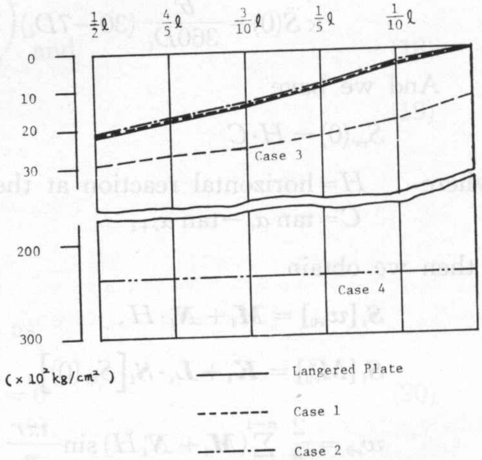

Fig. 6. σ_y along $y=l'/2$

The comparison of the results of the Langered plate, with plate theory, are shown in Fig. 5 and 6. Fig. 5 is for the deflection and Fig. 6 is for the stress.

Case 1 : Plate theory of which four sides are simply supported,

Case 2 : Plate theory of which two sides are simply supported and the other two sides are elastically supported by the beam $b\times h=10\times1000$ cm,

Case 3 : Plate theory of which two sides are simply supported and the other two sides are elastically supported by the beam $b\times h=10\times100$ cm, and

Case 4 : Plate theory of which two sides are simply supported and the other two sides are free from stresses.

Remarks

We can point out the following remarks:

1) Because we need not to pay any attention to the convergence problem of Fourier series, we can carry out the numerical calculation in short calculation time.

2) If we may replace the frame works with equivalent elastic beams, we can handls the analysis of Langered plate as the plate problem of which two opposite sides are simply supported and the other two sides are elastically supported.

3) If we use large values to the sectional area of frame members, the numerical results become equal to simply supported plate theory.

Acknowledgement

The computation was carried out by HITAC 8800/8700 in Tokyo University and FACOM 230-75 in Hokkaido University, and we used the application program of FEM plate bending which is stored in the Computor Center of Hokkaido University.

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