On a uniqueness theorem for the differential equation $u'=f(t, u)$ in a Banach space

by Shigeo KATO*

(Received September 25, 1974)

Let E be a Banach space with norm $\| \ \|$, and let U be an open set in E. We consider the nonlinear abstract Cauchy problem

$$(D) \qquad u' = f(t, u), \quad u(0) = u_0 \in U.$$

Here f is a continuous E-valued mapping defined on $[0, T] \times U$. We say u is a solution of (D) on $[0, T]$ with $u(0) = u_0$ if u is continuous on $[0, T]$, differentiable on $(0, T)$ and if $u(t) \in U$, $u'(t) = f(t, u(t))$ for $t \in (0, T)$.

It is our object in this paper to give a sufficient condition for the uniqueness of solutions of (D).

Our result is a straightforward extension into a general Banach space of that of J. M. Bownds and J. B. Diaz [1].

For u, v in E we define $\langle u, v \rangle$ by

$$\langle u, v \rangle = \lim_{h \to -0} \frac{1}{h} (\|u + hv\| - \|u\|).$$

The above limit exists for each u, v in E since $\frac{1}{h}(\|u+hv\| - \|u\|)$ is bounded and nondecreasing as $h \uparrow 0$.

LEMMA. *Let I be an open interval and r a continuous function from I into E such that $D^- r(t)$ exists for all $t \in I$, where $D^- r(t)$ denotes the left derivative for $r(t)$. If $m(t) = \|r(t)\|$ for all $t \in I$, then $D^- m(t)$ exists and*

$$D^- m(t) = \langle r(t), D^- r(t) \rangle.$$

For a proof see [3].

THEOREM. *Suppose that f is continuously Fréchet differentiable on $(0, T) \times U$, and set*

$$V(t, u) = f_t(t, u) + f_u(t, u) f(t, u)$$

for $(t, u) \in (0, T) \times U$. Suppose furthermore that

$$\langle f(t, u) - f(t, v), \ V(t, u) - V(t, v) \rangle \leqq 0$$

for (t, u), $(t, v) \in (0, T) \times U$.
Then there exists at most one solution of (D).

PROOF. Suppose that there exist two solutions u and v of (D). Define

* Department of Liberel Arts, Kitami Institute of Technology.

$m(t)=\|f(t,u(t))-f(t,v(t))\|$ for $t\in[0,T]$. Then, $m(0)=0$, $m(t)\geqq 0$ for $t\in[0,T]$, and

$$D^- m(t)=\langle f(t,u(t))-f(t,v(t)),\ D^-(f(t,u(t))-f(t,v(t))\rangle$$
$$=\langle f(t,u(t))-f(t,v(t)),\ V(t,u(t))-V(t,v(t))\rangle$$

for $t\in(0,T)$, where we used the chain rule for Fréchet derivatives. Hence, the hypothesis of the theorem then implies that

$$D^- m(t)\leqq 0 \quad \text{for} \quad t\in(0,T).$$

Thus if follows that $m(t)\equiv 0$ on $[0,T]$ since m is nonincreasing and $m(0)=0$. Since u and v are solutions of (D), $u'(t)\equiv v'(t)$ on $(0,T)$, while the initial condition gives that $u(0)=v(0)$, therefore, $u(t)\equiv v(t)$ on $[0,T]$.

REMARK 1. If E is a Hilbert space with inner product $(\ ,\ )$, then we can easily see that

$$\langle u,v\rangle=\begin{cases}\operatorname{Re}(u,v)/\|u\| & (u\neq 0)\\ -\|v\| & (u=0),\end{cases}$$

and hence, our condition of the theorem becomes

$$\operatorname{Re}\big(f(t,u)-f(t,v),\ V(t,u)-V(t,v)\big)\leqq 0$$

for (t,u), $(t,v)\in(0,T)\times U$.

REMARK 2. If $E=R^n$, the n-dimensional Euclidean space, and U is an open set in R^n.

Suppose that $f\in C([0,T]\times\bar{U};\ R^n)\cap C^1((0,T)\times U;\ R^n)$, then for each $h=(h_1,\cdots,h_n)\in R^n$

$$f_u(t,u)\,h=(h_1,\cdots,h_n)\begin{pmatrix} f^1_{u_1}(t,u)\cdots f^n_{u_1}(t,u)\\ \vdots \qquad\qquad \vdots\\ f^1_{u_n}(t,u)\cdots f^n_{u_n}(t,u)\end{pmatrix},$$

where $f(t,u)=(f^1(t,u),\cdots,f^n(t,u))$, and $f^i_{u_j}(t,u)$ denotes the partial derivatives of $f^i(t,u)$ with respect to u_j.

Hence our condition of the theorem becomes

$$\sum_{i=1}^n\Big[f^i(t,u)-f^i(t,v)\Big]\Big[\Big(f^i_t(t,u)+\sum_{j=1}^n f^i_{u_j}(t,u)f^j(t,u)\Big)$$
$$-\Big(f^i_t(t,v)+\sum_{j=1}^n f^i_{v_j}(t,v)f^j(t,v)\Big)\Big]\leqq 0$$

for (t,u), $(t,v)\in(0,T)\times U$.

EXAMPLE. We consider the scalar differential equation

$$u'=f(t,u)=1+\frac{1}{1+\sqrt{u}} \qquad (0\leqq t\leqq T,\ u\geqq 0).$$

In this case

$$V(t,u)=\frac{-1}{2\sqrt{u}\,(1+\sqrt{u})^2} \qquad (0<t<T,\ u>0).$$

Hence, for (t, u), $(t, v) \in (0, T) \times (0, \infty)$,

$$\Big(f(t,u)-f(t,v)\Big)\Big(V(t,u)-V(t,v)\Big)$$
$$=\frac{-(\sqrt{u}-\sqrt{v})^2\,[1+u+v+\sqrt{uv}+2(\sqrt{u}+\sqrt{v})]}{2\sqrt{uv}\,[(1+\sqrt{u})(1+\sqrt{v})]^3}\leqq 0,$$

as required by the hypothesis of the theorem. The conclusion is that there is at most one solution $u(t)$ for which $u(t)>0$ when $t>0$.

References

[1] J. M. BOWNDS and J. B. DIAZ: On restricted uniqueness for systems of ordinary differential equations, Proc. Amer. Math. Soc. 37 (1973), 100–104.

[2] S. KATO: A Note on nonlinear differential equation in a Banach space, Proc. Japan Acad: 50 (1974), 313–316.

[3] S. KATO: Some remarks on nonlinear differential equations in Banach spaces, to appear in Hokkaido Math. Jour.

[4] H. MURAKAMI: On nonlinear ordinary and evolution equations, Funkcial. Ekvac. 9 (1966), 151–162.