Notes on a conformal transformation and a special concircular scalar field in a Riemannian manifold[1)]

by Minoru ISU[3)] and Shigeo KATÔ[4)]

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Introduction

Recently one of the present authors has investigated some properties of a Riemannian manifold M^n which admits an infinitesimal conformal transformation ξ^i satisfying an equation;

$$\underset{\xi}{\mathscr{L}} g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i = 2\rho g_{ij},$$

where $\underset{\xi}{\mathscr{L}}$ means the Lie derivative with respect to ξ^i. In the previous paper [1][2)] we gave the necessary and sufficient conditions that ρ is the special concircular scalar field in M^n $(n \geqq 3)$ admitting a proper conformal Killing vector field ξ^i thinking of an open set U such that $\rho \neq 0$ at any point of it. But we can prove that ρ is the special concircular scalar field in an Einstein space of dimension $n \geqq 3$ admitting a proper conformal Killing vector field ξ^i without thinking of an open set U such that $\rho \neq 0$ at any point of it (see [2], [3]). In §1 we obtain the necessary and sufficient conditions that ρ is the special concircular scalar field in case of the scalar curvature $R = \text{const.} \neq 0$ of M^n admitting a proper conformal Killing vector field ξ^i. In §2 we shall give some remark when M^n is a compact orientable Riemannian manifold.

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§1. The necessary and sufficient conditions.

Let M^n be an n-dimensional Riemannian manifold of class C^r $(r \geqq 3)$ which has local coordinates x^i and admits a conformal killing vector field ξ^i. Then we can derive some well-known formulas which are used later (see [4])

$$\underset{\xi}{\mathscr{L}} R_{jk} = -(n-2)\nabla_k \rho_j - g_{jk} \nabla_i \rho^i, \tag{1.1}$$

$$\nabla_i \rho^i = -\frac{1}{2(n-1)}(2\rho R + \underset{\xi}{\mathscr{L}} R), \tag{1.2}$$

where ∇_i is the operator of covariant differentiation with respect to g_{ij}, R_{jk} the Ricci tensor, R the scalar curvature and $\rho^i = g^{ij}\rho_j$.

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1) See [1][2)].
2) Numbers in brackets refer to the references at the end of the paper.
3) Department of Mathematics, Faculty of Education Fukushima University.
4) Department of Liberal Arts, Kitami Institute of Technology.

We call a nonconstant scalar field ρ in M^n a concircular scalar field or a special concircular scalar field, if it satisfies the eqation,

$$\nabla_i \nabla_j \rho = \alpha g_{ij} \tag{1.3}$$

or

$$\nabla_i \nabla_j \rho = \sigma \rho g_{ij}, \quad \sigma \text{ non-zero constant, respectively.} \tag{1.4}$$

First, substituting (1.2) into (1.1), we obtain

$$\nabla_k \rho_j = \frac{2\rho R + \underset{\xi}{\mathscr{L}} R}{2(n-1)(n-2)} g_{jk} - \frac{1}{n-2} \underset{\xi}{\mathscr{L}} R_{jk}. \tag{1.5}$$

From (1.3) and (1.5), we have

$$\underset{\xi}{\mathscr{L}} R_{jk} = \frac{-2(n-1)(n-2)\alpha + 2\rho R + \underset{\xi}{\mathscr{L}} R}{2(n-1)} g_{jk}. \tag{1.6}$$

Putting

$$\beta = \frac{-2(n-1)(n-2)\alpha + 2\rho R + \underset{\xi}{\mathscr{L}} R}{2(n-1)} \tag{1.7}$$

we have

$$\underset{\xi}{\mathscr{L}} R_{jk} = \beta g_{jk}. \tag{1.8}$$

From (1.1) and (1.8), we obtain

$$\beta g_{jk} = -(n-2)\nabla_k \rho_j - g_{jk} \nabla_i \rho^i$$

and multiplying both sides of it by g^{jk} and summing with respect to j and k, we get

$$\beta = \frac{2\rho R + \underset{\xi}{\mathscr{L}} R}{n} \tag{1.9}$$

from (1.2).

Hence (1.8) becomes

$$\underset{\xi}{\mathscr{L}} R_{jk} = \frac{2\rho R + \underset{\xi}{\mathscr{L}} R}{n} g_{jk}. \tag{1.10}$$

From (1.7) and (1.9), we get

$$\alpha = -\frac{2\rho R + \underset{\xi}{\mathscr{L}} R}{2n(n-1)}. \tag{1.11}$$

Conversely, if (1.10) and (1.11) are satisfied, we can easily prove that ρ satisfies (1.3).

Therefore we have the following

PROPOSITION 1. *If M^n $(n \geqq 3)$ admits a proper conformal Killing vector field ξ^i, the necessary and sufficient conditions that ρ is a concircular scalar field are* (1.10) *and* (1.11).

Now, in case of the scalar curvature $R=\text{const.}\neq 0$, we wet

(1. 10₁) $$\underset{\xi}{\mathcal{L}} R_{jk}=\frac{2\rho R}{n} g_{jk},$$

(1. 11₁) $$\alpha=-\frac{R}{n(n-1)}\rho,$$

from (1. 10) and (1. 11) respectively.

Hence we have the following

PROPOSITION 2. *In case of the scalar curvature $R=const.\neq 0$ of M^n admitting a proper conformal Killing vector field ξ^i, a concircular scalar field ρ is a special concircular scalar field.*

COROLLARY 1. *In case of the scalar curvature $R=const.\neq 0$ of M^n admitting a proper conformal Killing vector field ξ^i, the necessary and sufficient condition that ρ is a special concircular scalar field is* (1. 10₁).

PROOF. This is proved evidently from Propositions 1 and 2. Therefore we can prove Corollary 2 in [1] without thinking of an open set U such that $\rho\neq 0$ at any point of it, that is,

COROLLARY 2. *If $M^n (n\geqq 3)$ admits a proper conformal Killing vector field ξ^i and the scalar curvature $R\neq 0$ is constant, then the necessary and sufficient condition that ρ is the special concircular scalar field is $\underset{\xi}{\mathcal{L}} T_{ij}=0$,* where $T_{ij}=R_{ij}-\frac{R}{n} g_{ij}$.

§ 2. Some remark.

Let (g_{ij}) be the symmetric matrix of the positive definite metric on M^n. The following theorem is well known (K. Yano and M. Obtata [5], [6]).

THEOREM A. *If a compact Riemannian manifold M^n of dimension $n\geqq 2$ with $R=const.$ admits an infinitesimal nonisometric conformal transformation ξ^i : $\underset{\xi}{\mathcal{L}} g_{ji}=2\rho g_{ji}$, $\rho\neq const.$, and if one of the following conditions is satisfied, then M^n is isometric to a sphere.*

(2. 1) *The vector field ξ^i is a gradient of a scalar.*

(2. 2) $R_i^h \rho^i = k\rho^h$, *k being a constant.*

(2. 3) $\underset{\xi}{\mathcal{L}} R_{ji}=\alpha g_{ji}$, *$\alpha$ being a scalar field.*

Its proof is introduced from that there exists ρ such that

(2. 4) $$\nabla_j \rho_i = -\frac{R}{n(n-1)}\rho g_{ij}.$$

If M^n is an Einstein space admitting a conformal Killing vector field ξ^i, ρ satisfies (2. 4) from Corollary 2. Therefore ρ^i is a conformal Killing vector field and ρ^i is a gradient of a scalar ρ. Furthermore ρ^i is an infinitesimal nonisometric conformal Killing vector because ξ^i is so [7]. Therefore in an Einstein space admitting a conformal Killing vector field ξ^i, ρ^i satisfies the conditions of Theorem A exchanging from ξ^i into ρ^i in (2. 1). An Einstein space evidently satisfies (2. 2) and (2. 3). Since from Corollary 1, (2. 3) is equivalent to (2. 4). Differentiating (2. 4) covariantly with respect to x^k and substituting the resulting equation into the Ricci identity

$$\nabla_k \nabla_j \rho_i - \nabla_j \nabla_k \rho_i = -R^l_{ijk} \rho_l ,$$

we have

$$\frac{R}{n(n-1)}(\rho_k g_{ij} - \rho_j g_{ik}) = R^l_{ijk} \rho_l ,$$

so that multiplying both sides of it by g^{ij} and summing with respect to i and j, we get

$$R^l_k \rho_l = \frac{R}{n} \rho_k .$$

Therefore in case of $n \geqq 3$ we have the following diagram

$$(2.1) \Longrightarrow (2.4) \Longleftrightarrow (2.2) \Longleftrightarrow (2.3)$$

without thinking of an open set U such that $\rho \neq 0$ at any point of it.

References

[1] M. ISU: A conformal transformation and a special concircular scalar field in a Riemannian manifold, to appear in Hokkaido Math. Jour.

[2] T. KOYANAGI: On a certain property of a Riemannian space admitting a special concircular scalar field, Jour. Fac. Sci. Hokkaido Univ., 22 (1972), 154–157.

[3] H. KÔJYÔ: On conformal Killing tensors of a Riemannian manifold of constant curvature, Hokkaido Math. Jour. 2 (1973), 236–242.

[4] K. YANO: The theory of Lie derivatives and its applications, (North-Holland, 1957).

[5] K. YANO and M. OBATA: Sur le groupede transformations conformes d'une variété de Riemann dont le scalaire de courbure est constant, C. R. Acad. Sc. Paris, 260 (1965), 2698–2700.

[6] K. YANO: Integral formulas in Riemannian Geometry, (Dekker, 1970).

[7] M. ISU: Notes on isometry of Riemannian manifolds to spheres, to appear in Science Reports of Foculty of Education, Fukushima Univ., No. 24 (1974).