A Result on Integral Equation in a Banach Space

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1. Introduction and main result

Let E be a Banach space with the dual space E^*. The norms in E and E^* are denoted by $\| \|$.

It is our object in this paper to give a sufficient condition for the existence of the unique mild solution to the Cauchy problem of the from

(1. 1) $\quad u'(t)=Au(t)+f(t, u(t)), \quad 0<t\leqq T,$

(1. 2) $\quad u(0)=x, \quad x\in E,$

where A is a densely defined closed linear m-dissipative operator and $f(t, x)$ is strongly continuous mapping of $[0, \infty)\times E$ into E.

The main theorem we prove is as follows:

Theorem. Let A be a densely defined closed linear m-dissipative operator, and let $\{T(t); t\geqq 0\}$ be the strongly continuous semi-group of contraction operators which has A as its infinitesimal generator. Let $f(t, x)$ be a strongly continuous mapping of $[0, \infty)\times E$ into E which maps bounded sets into bounded sets. Suppose furthermore that there exists a real-valued locally integrable function $\alpha(t)$ defined on $[0, \infty)$ such that for $(t, x), (t, y)\in[0, \infty)\times E$ and some $x^*\in F(x-y)$

$$2Re\left(f(t, x)-f(t, y),\ x^*\right)\leqq\alpha(t)\|x-y\|^2,$$

where $F(x-y)$ denotes the set of all $z^*\in E^*$ such that $(x-y, z^*)=\|x-y\|^2=\|z^*\|^2$. Then (1.1) has a unique mild solution $u(t)$ defined on $[0, \infty)$.

2. Definitions

Definition 2. 1. An operator A with domain $D(A)$ is said to be dissipative provided that $Re(Ax-Ay, x^*)\leqq 0$ for all $x, y\in D(A)$ and some $x^*\in F(x-y)$. If in addition, $R(I-A)=E$, we say that A is m-dissipative.

Remark. It is well known that A is the infinitesimal generator of a strongly continuous semi-group $\{T(t); t\geqq 0\}$ of linear contractions on E if and only if A is densely defined closed linear m-dissipative, and in fact, satisfies

$$Re(Ax-Ay, x^*)\leqq 0 \qquad \text{for all } x, y\in D(A) \text{ and all } x^*\in F(x-y).$$

Definition 2. 2. A function u: $[0, T]\to E$ is called a mild solution of (1.1) and (1,2) if it admits the integral representation

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(2.1) $$u(t)=T(t)\,x+\int_0^t T(t-s)\,f\big(s,u(s)\big)\,ds\,.$$

3. Proof of the theorem

Theorem will be proved by means of the propositions which follow, each of which is under the hypothesis of the theorem.

Proposition 3.1. For any $x\in E$ there exist a positive number T and a continuous function $u(t\,;\,x)$: $[0,T]\to E$ such that $u(t\,;\,x)$ is a solution of (2.1) on $[0,T]$.

Proof. In view of the continuity of $f(t,x)$ there exist constants $r_0>0$, $T_1>0$, and $M>0$ such that $\|f(t,v)\|\leqq M$ for $(t,v)\in[0,T]\times S(x,r_0)$, where $S(x,r_0)$ denotes the closed sphere of center x with radius r_0.

Since $D(A)$ is dense in E there exists a sequence $x_n\in D(A)$ such that x_n converges to x. Let $v=T(t)\,x_n+w$. Then we can choose $T_2>0$ and a positive integer L such that if $n\geqq L$, $t\in[0,T_2]$ and $\|w\|\leqq T_2M$, then $v\in S(x,r_0)$ and so $\|f(t,v)\|\leqq M$.

Let $T=\mathrm{Min}\,\{T_1,T_2\}$. For any positive integer $n\geqq L$, let $t_0^n=0$ and $u_n(t_0^n)=x_n$. Inductively, for each positive integer i, define δ_i^n, t_i^n, $u_n(t_{i-1}^n)$ such that

(i) $0\leqq\delta_i^n$ and $t_{i-1}^n+\delta_i^n\leqq T$;

(ii) If $\|z-u_n(t_{i-1}^n)\|\leqq\delta_i^nM+\mathrm{Max}\left\{\|(T(t)-I)\,u_n(t_{i-1}^n)\|\,;\;0\leqq t\leqq\delta_i^n\right\}$,
then $\sup\left\{\|f(t,v)-f(t_{i-1}^n,u_n(t_{i-1}^n))\|\,;\;t_{i-1}^n\leqq t\leqq t_{i-1}^n+\delta_i^n\right\}\leqq 1/n$;

and δ_i^n is the largest number such that (i) and (ii) hold.
Define $t_i^n=t_{i-1}^n+\delta_i^n$ and for each $t\in[t_{i-1}^n,t_i^n]$ define

(3.1) $$u_n(t)=T(t-t_{i-1}^n)\,u_n(t_{i-1}^n)+\int_{t_{i-1}^n}^t T(t-s)\,f(t_{i-1}^n,u_n(t_{i-1}^n))\,ds.$$

It follows easily that for $t\in[t_{k-1}^n,t_k^n]$

(3.3) $$u_n(t)=T(t)\,x_n+\sum_{j=1}^{k-1}\int_{t_{j-1}^n}^{t_j^n} T(t-s)\,f(t_{j-1}^n,u_n(t_{j-1}^n))\,ds$$
$$+\int_{t_{k-1}^n}^t T(t-s)\,f(t_{k-1}^n,u_n(t_{k-1}^n))\,ds.$$

By the same argument as G. Webb [6], we see that $u_n(t)\in S(x,r_0)\cap D(A)$ and $T=t_N^n$ for some positive integer $N=N(n)$.

We will show that the sequence of continuous functions $\{u_n(t)\}$ converges uniformly to a function $u(t;x)$ from $[0,T]$ to E. Note that if $t\in(t_{i-1}^n,t_i^n)$, then $u_n(t)$ is differentiable at t and

(3.3) $$u_n'(t)=Au_n(t)+f(t_{i-1}^n,u_n(t_{i-1}^n))\,.$$

For each t, $s\,(t>s)$ in $[0,T]$ there exist i, k such that $t\in[t_{i-1}^n,t_i^n]$ and $s\in[t_{k-1}^n,t_k^n]$. Then

$$\|u_n(t)-u_n(s)\| \leqq \|(T(t)-T(s)x_n\| + \sum_{j=1}^{k-1}\int_{t_{j-1}^n}^{t_j^n} \|(T(t-\tau)-T(s-\tau))f(t_{j-1}^n, u_n(t_{j-1}^n))\|d\tau$$

$$+\int_{t_{k-1}^n}^{s} \|(Tt-\tau)-T(s-\tau))f(t_{k-1}^n, u_n(t_{k-1}^n))\|d\tau$$

$$+\int_{s}^{t_k^n} \|T(t-\tau)f(t_{k-1}^n, u_n(t_{k-1}^n))\|d\tau$$

$$+\sum_{j=k+1}^{i-1}\int_{t_{j-1}^n}^{t_j^n} \|T(t-\tau)f(t_{j-1}^n, u_n(t_{j-1}^n))\|d\tau$$

$$+\int_{t_{i-1}^n}^{t} \|T(t-\tau)f(t_{i-1}^n, u_n(t_{i-1}^n))\|d\tau$$

$$\leqq \|Ax_n\|(t-s)+2M\sum_{j=1}^{k-1}(t_j^n-t_{j-1}^n)+2M(s-t_{k-1}^n)+M(t_k^n-s)+M\sum_{j=k+1}^{i-1}(t_j^n-t_{j-1}^n)$$

$$+M(t-t_{i-1}^n)\leqq(\|Ax_n\|+2M)(t-s).$$

Therefore $\|u_n(t)-u_m(t)\|$ is uniformly Lipschitz continuous on $[0, T]$ with Lipscitz constant $\|Ax_n\|+\|Ax_m\|+4M$.

This implies that $\frac{d}{dt}\|u_n(t)-u_m(t)\|^2$ exists for a.e. $t\in[0, T]$. For each $t\in(0, T)$ there exist i, k such that $t\in(t_{i-1}^n, t_i^n)$ and $t\in(t_{k-1}^m, t_k^m)$. By Lemma 1.3 in [3] and (3.3) we have

$$\frac{d}{dt}\|u_n(t)-u_m(t)\|^2 = 2Re\Big(u_n'(t)-u_m'(t), x_{nm}^*(t)\Big)$$

$$= 2Re\Big(A(u_n(t)-u_m(t))+f(t_{i-1}^n, u_n(t_{i-1}^n))-f(t_{k-1}^m, u_m(t_{k-1}^m)), x_{nm}^*(t)\Big)$$

$$\leqq 2Re\Big(f(t_{i-1}^n, u_n(t_{i-1}^n))-f(t_{k-1}^m, u_m(t_{k-1}^m)), x_{nm}^*(t)\Big)$$

$$\leqq \alpha(t)\|u_n(t)-u_m(t)\|^2+2(1/n+1/m)\|u_n(t)-u_m(t)\|$$

$$\leqq\Big(\alpha(t)+1\Big)\|u_n(t)-u_m(t)\|^2+(1/n+1/m)^2$$

for a.e. $t\in[0, T]$ and some $x_{nm}^*(t)\in F(u_n(t)-u_m(t))$.

It follows that

$$\|u_n(t)-u_m(t)\|^2\leqq\|x_n-x_m\|^2 \exp\left[\int_0^t(\alpha(\tau)+1)\,d\tau\right]$$

$$+(1/n+1/m)^2\int_0^t \exp\left[\int_s^t(\alpha(\tau)+1)\,d\tau\right]ds$$

$$\leqq\|x_n-x_m\|^2 \text{ Max }\left\{\exp\left[\int_0^t(\alpha(\tau)+1)\,d\tau\right];\ 0\leqq t\leqq T\right\}$$

$$+(1/n+1/m)^2 \text{ Max }\left\{\int_0^t \exp\left[\int_s^t \alpha(\tau)+1)\,d\tau\right]ds;\ 0\leqq t\leqq T\right\}.$$

Thus $\{u_n(t)\}$ converges uniformly to a continuous function $u(t\,;\,x)$ on $[0, T]$.

Next we will show that the above $u(t\,;\,x)$ satisfies (2.1) on $[0, T]$. To show this, note that

$$\int_0^t T(t-s)f(s,u(s\,;\,x))\,ds=\sum_{j=1}^{k-1}\int_{t_{j-1}^n}^{t_j^n} T(t-s)f(s,u(s\,;\,x))\,ds$$
$$+\int_{t_{k-1}^n}^{t} T(t-s)f(s,u(s\,;\,x)\,ds$$

for $t\in[t_{k-1}^n, t_k^n]$.

Then we have by (3.2)

$$\|u_n(t)-\left\{T(t)\,x+\int_0^t T(t-s)f(s,u(s\,;\,x))\,ds\right\}\|\leqq\|T(t)\,x_n-T(t)\,x\|$$
$$+\sum_{j=1}^{k-1}\int_{t_{j-1}^n}^{t_j^n}\left\{\|f(t_{j-1}^n,u_n(t_{j-1}^n))-f(s,u_n(s))\|+\|f(s,u_n(s))-f(s,u(s\,;\,x))\|\right\}ds$$
$$+\int_{t_{k-1}^n}^{t}\left\{\|f(t_{k-1}^n,u_n(t_{k-1}^n))-f(s,u_n(s))\|+\|f(s,u_n(s))-f(s,u(s\,;\,x))\|\right\}ds$$
$$\leqq\|x_n-x\|+\left[1/n+\mathrm{Max}\left\{\|f(s,u_n(s))-f(s,u(s\,;\,x))\|\,;\,0\leqq t\leqq T\right\}\right]T\,.$$

Because of the uniform convergence of $\{u_n(t)\}$ to $u(t\,;\,x)$ on $[0, T]$, $C=\{u_n(t), u(t)\,;\,0\leqq t\leqq T, n=1,2,\cdots\}$ is a compact set in E. Since $f(t,x)$ is uniformly continuous on $[0, T]\times C$ we have

$$\mathrm{Max}\left\{\|f(s,u_n(s))-f(s,u(s\,;\,x))\|\,;\,0\leqq t\leqq T\right\}\longrightarrow 0 \text{ as } n\to\infty\,,$$

and hence 2.1 holds for $u(t\,;\,x)$.

Proposition 3. 2. Let $x, y\in E$. If $u(t\,;\,x)$ and $v(t\,;\,y)$ satisfy (2.1) on $[0, T_u]$ and $[0, T_v]$, respectively, then

$$\|u(t\,;\,x)-v(t\,;\,y)\|^2\leqq\|x-y\|^2\exp\left[\int_0^t(\alpha(\tau)+1)\,d\tau\right] \tag{3.4}$$

for $0\leqq t\leqq \mathrm{Min}\,\{T_u, T_v\}$.

Consequently, the solution of (2.1) is unique.

Proof. Let the sequences $\{x_n\}$ and $\{y_n\}$ be as in the proof of Proposition 3.1. For each positive integer n let $\{t_i^n\}_{i=0}^n$ be a partition of $[0, \mathrm{Min}\,\{T_u, T_v\}]$ and define for $t\in[t_{k-1}^n, t_k^n]$,

$$u_n(t\,;\,x)=T(t)\,x_n+\sum_{j=1}^{k-1}\int_{t_{j-1}^n}^{t_j^n} T(t-s)f(s,u(t_{j-1}^n\,;\,x)\,ds$$
$$+\int_{t_{k-1}^n}^{t} T(t-s)f(s,u(t_{k-1}^n\,;\,x))\,ds\,,$$

and $v_n(t\,;\,y)$ similarly.

Then, for $t\in(t_{k-1}^n, t_k^n)$,

$$u_n'(t\,;\,x)=Au_n(t\,;\,x)+f(t_{k-1}^n, u(t_{k-1}^n\,;\,x))\,,$$

and

$$v_n'(t\,;\,y)=Av_n(t\,;\,y)+f(t_{k-1}^n, v(t_{k-1}^n\,;\,y))\,.$$

Furthermore $\{u_n(t\,;\,x)\}$ and $\{v_n(t\,;\,y)\}$ converge uniformly to $u(t\,;\,x)$ and $v(t\,;\,y)$

respectively as the mesh of $\{t_i^n\}$ goes to zero with n.

By the same argument as in Proposition 3.1, we see that for a.e. $t\in(t_{k-1}^n, t_k^n)$ and some $x_n^*(t)\in F(u_n(t;x)-v_n(t;y))$,

$$\frac{d}{dt}\|u_n(t;x)-v_n(t;y)\|^2$$

$$=2Re\Big(A(u_n(t;x)-v_n(t;y))+f(t_{k-1}^n, u(t_{k-1}^n;x))-f(t_{k-1}^n, v(t_{k-1}^n;y)),\ x_n^*(t)\Big)$$

$$\leqq\alpha(t)\|u_n(t;x)-v_n(t;y)\|^2$$

$$+2\|u_n(t;x)-v_n(t;y)\|\Big\{\|f(t,u_n(t;x))-f(t_{k-1}^n, u(t_{k-1}^n;x))$$

$$+\|f(t,v_n(t;y))-f(t_{k-1}^n, v(t_{k-1}^n;y))\|\Big\}$$

$$\leqq(\alpha(t)+1)\|u_n(t;x)-v_n(t;y)\|^2+2\Big\{\|f(t,u_n(t;x))-f(t_{k-1}^n, u(t_{k-1}^n;x))\|^2$$

$$+\|f(t,v_n(t;y))-f(t_{k-1}^n, v(t_{k-1}^n;y))\|^2\Big\},$$

and it follows that, for $t\in[t_{k-1}^n, t_k^n]$,

$$\|u_n(t;x)-v_n(t;y)\|^2\leqq\|x_n-y_n\|^2\exp\left[\int_0^t(\alpha(\tau)+1)\,d\tau\right]$$

$$+2\sum_{j=1}^{k-1}\int_{t_{j-1}^n}^{t_j^n}\Big\{\|f(s,u_n(s;x))-f(t_{j-1}^n, u(t_{j-1}^n;x))\|^2$$

$$+\|f(s,v_n(s;y))-f(t_{j-1}^n, v(t_{j-1}^n;y))\|^2\Big\}\exp\left[\int_s^t(\alpha(\tau)+1)\,d\tau\right]ds$$

$$+2\int_{t_{k-1}^n}^t\Big\{\|f(s,u_n(s;x))-f(t_{k-1}^n, u(t_{k-1}^n;x))\|^2$$

$$+\|f(s,v_n(s;y))-f(t_{k-1}^n, v(t_{k-1}^n;y))\|^2\Big\}\exp\left[\int_s^t(\alpha(\tau)+1)\,d\tau\right]ds.$$

By going to the mesh of $\{t_i^n\}$ tend to zero with n, we have

$$\|u(t;x)-v(t;y)\|^2\leqq\|x-y\|^2\exp\left[\int_0^t(\alpha(\tau)+1)\,d\tau\right].$$

The uniqueness of the solution of (2.1) follows at once.

Proposition 3. 3. For any $x\in E$, the solution of (2.1) exists on $[0,\infty)$.

Proof. It follows from Propositions 3.1., 3.2. that there exists a unique local solution $u(t;x)$ of (2.1) on some interval $[0,\rho)$. We may assume that $[0,\rho)$ is a maximal interval of existence of $u(t;x)$. We have only to show that $\rho<\infty$ leads to a contradiction.

We define $u_n(t;x)$ on $[0,T]$ as in the proof of Proposition 3.2, where T is an arbitrary number such that $0<T<\rho$.

Then, for a.e. $t\in(t_{k-1}^n, t_k^n)$ and some $x_n^*(t)\in F(x_n(t))$,

$$\frac{d}{dt}\|u_n(t;x)\|^2=2Re\Big(Au_n(t;x)+f(t_{k-1}^n, u(t_{k-1}^n;x)),\ x_n^*(t)\Big)$$

$$\leqq\alpha(t)\|u_n(t\,;\,x)\|^2+2\|f(t,0)\|\,\|u_n(t\,;\,x)\|$$
$$+2\|f(t,u_n(t\,;\,x))-f(t_{k-1}^n,u(t_{k-1}^n\,;\,x))\|\,\|u_n(t\,;\,x)\|$$
$$\leqq(\alpha(t)+2)\|u_n(t\,;\,x)\|^2+\|f(t,0)\|^2+\|f(t,u_n(t\,;\,x))-f(t_{k-1}^n,u(t_{k-1}^n\,;\,x))\|^2.$$

Thus we have, for $t\in[t_{k-1}^n,t_k^n]$,

$$\|u_n(t\,;\,x)\|^2\leqq\|x_n\|^2\exp\left[\int_0^t(\alpha(\tau)+2)\,d\tau\right]+\int_0^t\|f(s,0)\|^2\exp\left[\int_s^t(\alpha(\tau)+2)\,d\tau\right]ds$$
$$+\sum_{j=1}^{k-1}\int_{t_{j-1}^n}^{t_j^n}\|f(s,u_n(s\,;\,x))-f(t_{j-1}^n,u(t_{j-1}^n\,;\,x))\|^2\exp\left[\int_s^t(\alpha(\tau)+2)\,d\tau\right]ds$$
$$+\int_{t_{k-1}^n}^t\|f(s,u_n(s\,;\,x))-f(t_{k-1}^n,u(t_{k-1}^n\,;\,x))\|^2\exp\left[\int_s^t(\alpha(\tau)+2)\,d\tau\right]ds.$$

Consequently we obtain, for $t\in[0,T]$,

$$\|u(t\,;\,x)\|^2\leqq\|x\|^2\exp\left[\int_0^t(\alpha(\tau)+2)\,d\tau\right]+\int_0^t\|f(s,0)\|^2\exp\left[\int_s^t(\alpha(\tau)+2)\,d\tau\right]ds.$$

Thus we obtain the boundedness of $u(t;\,x)$ on $[0,\rho)$. If $h,h'>0$ such that $h\geqq h'$ and $\rho-h\geqq0$, then by (2.1)

$$\|u(\rho-h\,;\,x)-u(\rho-h'\,;\,x)\|\leqq\|Ax\|(h-h')$$
$$+\int_0^{\rho-h}\|(T(\rho-h-s)-T(\rho-h'-s))f(s,u(s\,;\,x))\|ds$$
$$+\int_{\rho-h}^{\rho-h'}\|T(\rho-h'-s)f(s,u(s\,;\,x))\|ds\to0\ \text{as}\ h,h'\to0\,.$$

Thus, $\lim_{t\to\rho}u(t\,;\,x)$ exists and so, by Proposition 3.1, $u(t\,;\,x)$ can be continued to the right of ρ, which contradicts the assumption on ρ.

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