Analysis of Two-way Multi-cell Plate

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The present paper discusses the stress problem of "The Two-Way Multi-Cell Plate" as shown in Fig. 1, and the problem is expressed by finite difference equations with respect to the displacement vector at the nodal point, which are solved by means of "Finite Fourier Integration Transforms".

It is proved that the basic difference equations tend to the differential equation for the bending of an orthotropic plate, by letting the interval of the nodal line be infinitely small.

For numerical examples, the two-way multi-cell plate of the four simply supported sides are taken into account.

Introduction

The hollow structure, whose top and bottom deck plates are transversely as well as longitudinally connected with one another by the thin web plates, is a kind of sandwich plate. We will name it "Two-Way Multi-Cell Plate".

In this paper, the stress problems of the two-way multi-cell plate will be considered. As the structure consists of many rectangular plate elements, the appropriate stiffness matrix should first be needed.

For this purpose, the displacement shear equation for the long strip element, which was proposed by S. G. Nomachi, is integrated step by step with the neglection of the smaller terms.

The crosswise web meets with the longitudinal ones, at the nodal points, where equations of equilibrium are established with the three components of displacement at the nodes, because the stiffness matrix relates the nodal displacement vector with the nodal force vector.

Fixing the web plates by equi-distant intervals in the x and y directions, respectively, we can write the equations in the form of the finite difference ones, which could be efficiently treated by means of "Finite Integration Transforms".

If we let the interval of the webs be infinitely small, keeping the average rigidity of the web as a constant, the solution by "Finite Integration Transforms"
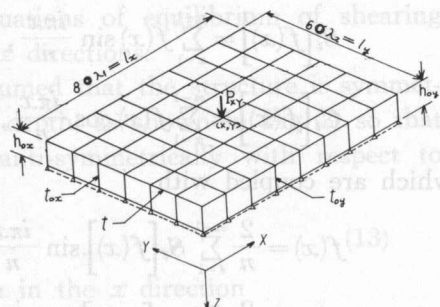

Fig. 1. Two-Way Multi-Cell Plate

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will lead us to an equivalent differential equation concerning the two-way multi-cell plate.

Basic Formulas

(1) Displacement-shear-equation

The following equations are obtained for a folded plate element as shown in Fig. 2.

$$T_{AB}(x) = \frac{N}{6}(2\ddot{u}_A + \ddot{u}_B) + \frac{1}{a}(\bar{S}_{AB} - \bar{S}_{BA}), \tag{1}$$

$$T_{BA}(x) = \frac{N}{6}(2\ddot{u}_B + \ddot{u}_A) + \frac{1}{a}(\bar{S}_{BA} - \bar{S}_{AB}), \tag{2}$$

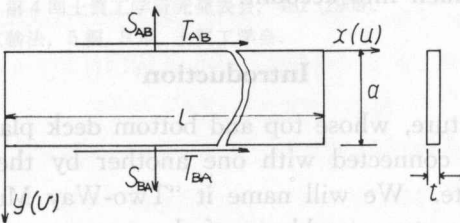$$\frac{1}{2}Gt(\dot{v}_A + \dot{v}_B) = \frac{Gt}{a}(u_A - u_B) + \frac{1}{a}(\bar{S}_{AB} - \bar{S}_{BA}). \tag{3}$$

Fig. 2.   Folded Plate Element

Let the letter T be the shear flow and letter S, the normal force per unit length.

$$\dot{u} = \frac{\partial u}{\partial x}, \qquad N = Eta, \qquad \bar{S} = \int S dx.$$

(2) Finite Fourier Integration Transform and their inverse formulas

Let us introduce the symbolic notations

$$\boldsymbol{S}_i\Big[f(x)\Big] = \sum_{x=1}^{n-1} f(x) \sin \frac{i\pi x}{n}, \tag{4}$$

$$\boldsymbol{C}_i\Big[f(x)\Big] = \sum_{x=1}^{n-1} f(x) \cos \frac{i\pi x}{n}, \tag{5}$$

which are coupled with

$$f(x) = \frac{2}{n} \sum_{i=1}^{n-1} \boldsymbol{S}_i\Big[f(x)\Big] \sin \frac{i\pi x}{n}, \tag{6}$$

$$f(x) = \frac{2}{n} \sum_{i=0}^{n} \boldsymbol{R}_i\Big[f(x)\Big] \cos \frac{i\pi x}{n}, \tag{7}$$

where

$$R_0\Big[f(x)\Big]=\frac{1}{2}\Big\{C_0\Big[f(x)\Big]+\frac{1}{2}f(n)+\frac{1}{2}f(0)\Big\},$$

$$R_i\Big[f(x)\Big]=C_i\Big[f(x)\Big]+\frac{1}{2}(-1)^if(n)+\frac{1}{2}f(0),\qquad\qquad(8)$$

$$R_n\Big[f(x)\Big]=\frac{1}{2}\Big\{C_n\Big[f(x)\Big]+\frac{1}{2}(-1)^xf(n)+\frac{1}{2}f(0)\Big\}.$$

$$(x,\, i=0,1,2\cdots n.)$$

For convenience sake, let us define the second difference and the modified difference as follows:

$$\Delta^2 f(x-1)=f(x+1)-2f(x)+f(x-1),$$

$$\Delta f(x)=f(x+1)-f(x-1).$$

Applying the above formulas to the sine and cosine transforms, we have

$$S_i\Big[\Delta^2 f(x-1)\Big]=-\sin\frac{i\pi}{n}\Big\{(-1)^if(n)-f(0)\Big\}-D_i\cdot S_i\Big[f(x)\Big],\qquad(9)$$

$$S_i\Big[\Delta f(x)\Big]=-2\cdot\sin\frac{i\pi}{n}\cdot R_i\Big[f(x)\Big],\qquad\qquad(10)$$

$$C_i\Big[\Delta^2 f(x-1)\Big]=(-1)^i\Delta f(n-1)-\Delta f(0)-D_iR_i\Big[f(x)\Big],\qquad(11)$$

$$C_i\Big[\Delta f(x)\Big]=-(-1)^i\Delta f(n-1)-\Delta f(0)$$

$$+\Big(1+\cos\frac{i\pi}{n}\Big)\Big\{(-1)^if(n)+f(0)\Big\}+2\cdot\sin\frac{i\pi}{n}\,S_i\Big[f(x)\Big],\qquad(12)$$

where

$$D_i=2\Big(1-\cos\frac{i\pi}{n}\Big).$$

Analysis of the Two-Way Multi-Cell Plate

We will begine with the three basic equations of equilibrium of shearing forces along the nodal line in the x, y and z directions.

To make the discussion simple, it is assumed that the structure is symmetrical with respect to the middle plane of the depth, as shown in Fig. 3, so that the deformations of the deck plate occur anti-symmetrically with respect to neutral plane.

$$u^0=-u^z,\qquad v^0=-v^z,\qquad w^0=w^z.\qquad\qquad(13)$$

First, we can get the following equation in the x direction

$$T_{Y,Y+1}(x)+T_{Y,Y-1}(x)+T_Y^{0z}(x)=0.\qquad\qquad(14)$$

Adjusting the subscripts of Eq. (2) and (3) to those of Fig. 3, and substituting them into Eq. (14), we obtain,

$$\left(\frac{2}{3}N_x+\frac{N_{0x}}{6}\right)\ddot{u}_Y+\frac{N_x}{6}(\ddot{u}_{Y+1}+\ddot{u}_{Y-1})-2G\left(\frac{t}{\lambda_2}+\frac{t_{0x}}{h_{0x}}\right)u_Y$$

$$+\frac{Gt}{\lambda_2}(u_{Y+1}+u_{Y-1})+\frac{Gt}{2}(\dot{v}_{Y+1}-\dot{v}_{Y-1})+Gt_{0x}\dot{w}_Y=0,\qquad(15)$$

which may be replaced by the abbreviate form:

$$K\ddot{u}+G_1u+G_2\dot{v}=0\qquad(16)$$

where

$$K=\left(\frac{2}{3}N_x+\frac{N_{0x}}{6}\right)_{\text{etc.}},\qquad G_1=-2G\left(\frac{t}{\lambda_2}+\frac{t_{0x}}{h_{0x}}\right)_{\text{etc.}},\qquad G_2=\frac{Gt}{2}_{\text{etc.}},$$

and by virtue of the procedure introduced in literature [2], Eq. (16) can be expressed by the following difference equation,

$$\frac{K}{\lambda}\Delta_x^2u_{x-1}+\frac{G_1\lambda}{6}(\Delta_x^2u_{x-1}+6u_x)+\frac{G_2}{2}\Delta_xv_x=0,\qquad(17)$$

where

$$\Delta_x^2f(x-1)=f(x+1)-2\cdot f(x)+f(x-1),$$
$$\Delta_xf(x)=f(x+1)-f(x-1).$$

Then, Eq. (15) becomes

$$\left(\frac{N_x}{6\lambda_1}+\frac{Gt\lambda_1}{6\lambda_2}\right)\Delta_x^2\Delta_Y^2u_{x-1,Y-1}+\left(\frac{N_x}{\lambda_1}+\frac{N_{0x}}{6\lambda_1}-\frac{Gt_{0x}\lambda_1}{3h_{0x}}\right)\Delta_x^2u_{x-1,Y}$$

$$+\frac{Gt\lambda_1}{\lambda_2}\Delta_Y^2u_{x,Y-1}-2\frac{Gt_{0x}\lambda_1}{h_{0x}}u_{xY}+\frac{Gt}{4}\Delta_x\Delta_Yv_{x,Y}+\frac{Gt_{0x}}{2}\Delta_xw_{x,Y}=0$$

$$(18)$$

Similarly in the y direction, we can write as follows:

$$\left(\frac{N_y}{6\lambda_2}+\frac{Gt\lambda_2}{6\lambda_1}\right)\Delta_x^2\Delta_Y^2v_{x-1,Y-1}+\left(\frac{N_y}{\lambda_2}+\frac{N_{0y}}{6\lambda_2}-\frac{Gt_{0y}\lambda_2}{3h_{0y}}\right)\Delta_Y^2v_{x,Y-1}$$

$$+\frac{Gt\lambda_2}{\lambda_1}\Delta_x^2v_{x-1,Y}-2\frac{Gt_{0y}\lambda_2}{h_{0y}}v_{xY}+\frac{Gt}{4}\Delta_x\Delta_Yu_{xY}+\frac{Gt_{0y}}{2}\Delta_Yw_{x,Y-1}=0.$$

$$(19)$$

Along the intersection of the crosswise ribs in the z direction, we can obtain the third equation of equilibrium of shearing forces,
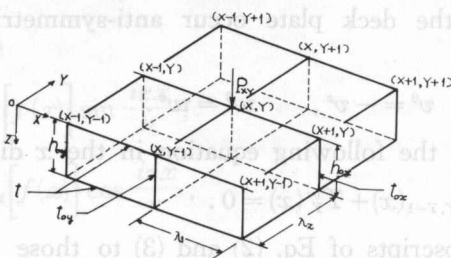

Fig. 3.   Force acting on the plate

$$\frac{Gt_{0x}h_{0x}}{\lambda_1}\Delta_x^2 w_{x-1,Y}+\frac{Gt_{0y}h_{0y}}{\lambda_2}\Delta_Y^2 w_{x,Y-1}-Gt_{0x}\Delta_x u_{x,Y}-Gt_{0y}\Delta_Y v_{XY}$$

$$=\frac{2}{\lambda_1}\left[\bar{\bar{p}}\left(X,\frac{X+1}{Y}\right)\Big|_0^{\lambda_1}-\bar{\bar{p}}\left(X,\frac{X-1}{Y}\right)\Big|_0^{\lambda_1}\right]$$

$$+\frac{2}{\lambda_2}\left[\bar{\bar{p}}\left(Y,\frac{X}{Y+1}\right)\Big|_0^{\lambda_2}-\bar{\bar{p}}\left(Y,\frac{X}{Y-1}\right)\Big|_0^{\lambda_2}\right]-P_{XY}, \tag{20}$$

where p=uniform load distributed upper side of the rib element, and
    P=concentrated load at nodal point.

Making use of Eqs. (9), (10), (11) and (12), and performing the Finite Fourier Integration Transform on Eqs. (18), (19) and (20), we have

$$\tilde{u}=R_m S_i\big[u(x,y)\big], \qquad \tilde{v}=S_m R_i\big[v(x,y)\big], \qquad \tilde{w}=S_m S_i\big[w(x,y)\big], \tag{21}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} \tilde{u} \\ \tilde{w} \\ \tilde{v} \end{bmatrix}=\begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \\ \tilde{P}_3 \end{bmatrix}, \tag{22}$$

where

$$a_{11}=\frac{1}{6}\left(\frac{N_x}{\lambda_1}+\frac{Gt\lambda_1}{\lambda_2}\right)D_i D_m-\left(\frac{N_x}{\lambda_1}+\frac{N_{0x}}{6\lambda_1}-\frac{Gt_{0x}\lambda_1}{3h_{0x}}\right)D_m-\frac{Gt\lambda_1}{\lambda_2}D_i-2\frac{Gt_{0x}\lambda_1}{h_{0x}},$$

$$a_{12}=Gt_{0x}\sin\frac{m\pi}{n}, \qquad a_{13}=Gt\sin\frac{m\pi}{n}\sin\frac{i\pi}{k}, \qquad a_{12}=a_{21},$$

$$a_{22}=-G\left(\frac{t_{0x}h_{0x}}{\lambda_1}D_m+\frac{t_{0y}h_{0y}}{\lambda_2}D_i\right), \qquad a_{23}=2Gt_{0y}\sin\frac{i\pi}{k},$$

$$a_{31}=a_{13}, \qquad a_{32}=a_{23},$$

$$a_{33}=\frac{1}{6}\left(\frac{N_y}{\lambda_2}+\frac{Gt\lambda_2}{\lambda_1}\right)D_i D_m-\left(\frac{N_y}{\lambda_2}+\frac{N_{0y}}{6\lambda_2}-\frac{Gt_{0y}\lambda_2}{3h_{0y}}\right)D_i-\frac{Gt\lambda_2}{\lambda_1}D_m-2\frac{Gt_{0y}\lambda_2}{h_{0y}},$$

$$D_m=2\left(1-\cos\frac{m\pi}{n}\right), \qquad D_i=2\left(1-\cos\frac{i\pi}{k}\right),$$

$$N_x=Et\lambda_2, \qquad N_{0x}=Et_{0x}h_{0y}, \qquad N_y=Et\lambda_1,$$

$$N_{0y}=Et_{0y}h_{0y}$$

and nodal force in the x, y and z directions are found in $\tilde{P}_1$, $\tilde{P}_2$ and $\tilde{P}_3$, respectively.

Derivation of differential equation

When we take the nodal length λ_1 and λ_2 to be infinitely small in Eq. (22), what kind of differential equation will be found?

Taking $\tilde{P}_1=\tilde{P}_3=0$, Eq. (22) yields

$$\det|a|\cdot\tilde{w}=(a_{11}a_{33}-a_{13}a_{31})\tilde{P}_2. \tag{23}$$

Deviding the nodal line into infinite number of pieces, is equal to letting n and k be infinity, which enables us to write as follows:

$$D_i \doteqdot \left(\frac{i\pi}{k}\right)^2 - \frac{1}{12}\left(\frac{i\pi}{k}\right)^4, \qquad \sin\frac{i\pi}{k} \doteqdot \left(\frac{i\pi}{k}\right) - \frac{1}{6}\left(\frac{i\pi}{k}\right)^3,$$

$$D_m \doteqdot \left(\frac{m\pi}{n}\right)^2 - \frac{1}{12}\left(\frac{m\pi}{n}\right)^4, \qquad \sin\frac{m\pi}{n} \doteqdot \left(\frac{m\pi}{n}\right) - \frac{1}{6}\left(\frac{m\pi}{n}\right)^3,$$

$$l_x = n\lambda_1 \text{ and } l_y = k\lambda_2.$$

Substituting the above in (23) and neglecting the terms of higher order, we have

$$\det |a| \doteqdot \left[\frac{2t_1}{t}\left\{a_y\left(2+\frac{t_1}{3t}\right)+a_x\right\}-2\frac{t_{0x}^2}{t^2\lambda_1^2}\right]D_x^4 + \left[\frac{1}{t}\left\{2a_y t_1 + 2a_x t_1\left(2+\frac{t_2}{3t}\right)\right.\right.$$

$$+2a_y t_2\left(2+\frac{t_1}{3t}\right)+2a_x t_2\right\}-2\frac{t_{0x}^2}{t^2\lambda_1^2}\left(2+\frac{t_2}{3t}\right)-2\frac{t_{0y}^2}{t^2\lambda_1^2}\left(2+\frac{t_1}{3t}\right)\Big]D_x^2 D_y^2$$

$$+4\left\{\frac{t_2}{t}a_x a_y - a_x\frac{t_{0y}^2}{\lambda_1^2 t^2}\right\}D_y^2 + \left[\frac{2t_2}{t}\left\{a_x\left(2+\frac{t_2}{3t}\right)+a_y\right\}-2\frac{t_{0y}^2}{t^2\lambda_1^2}\right]D_y^4$$

$$+4\left\{\frac{t_1}{t}a_x a_y - a_y\frac{t_{0x}^2}{t^2\lambda_2^2}\right\}D_x^2 \tag{24}$$

where

$$t_1 = \frac{t_{0x}h_{0x}}{\lambda_2}, \qquad t_2 = \frac{t_{0y}h_{0y}}{\lambda_1}, \qquad a_x = \frac{t_{0x}}{h_{0x}t\lambda_2}, \qquad a_y = \frac{t_{0y}}{h_{0y}t\lambda_1}$$

$$D_x = \left(\frac{m\pi}{l_x}\right) \text{ and } D_y = \left(\frac{i\pi}{l_y}\right).$$

On the other hand, the terms of the right side of (23) yield

$$\frac{(a_{11}a_{33}-a_{31}a_{13})}{Gt\lambda_1\lambda_2}\widetilde{P}_2 = \frac{4t_{0x}t_{0y}}{Gt^3 h_{0x}h_{0y}\lambda_1^2\lambda_2^2}\widetilde{P}_2 \doteqdot \frac{4t_{0x}t_{0y}}{Gt^3 h_{0x}h_{0y}\lambda_1\lambda_2}\cdot q. \tag{25}$$

Dividing Eq. (24) by the coefficient of q in (25), and letting λ_1 and λ_2 be small, we found that the coefficients with D_x^2 and D_y^2 become zero, and the coefficient with D_x^4 reduces to

$$B_x = Gth_{0x}^2 + \frac{Gt_{0x}h_{0x}^3}{6\lambda_2} \doteqdot E\left(\frac{1}{2}th_{0x}^2 + \frac{t_{0x}h_{0x}^3}{12\lambda_2}\right). \tag{26}$$

Evidently, B_x is a averaged moment of inertia of rib plate together with the upper and the lower flange plate about neutral axis.

In the same way B_y and B_{xy} are obtained as follows:

$$B_y = E\left(\frac{1}{2}th_{0y}^2 + \frac{t_{0y}h_{0y}^3}{12\lambda_1}\right),$$

$$B_{xy} = Gt(h_{0x}^2 + h_{0y}^2).$$

Thus, we can obtain a differential equation from the equations of two-way multi-cell plate, and that is the differential equation of orthotropic plate,

$$B_x \frac{\partial^4 w}{\partial x^4} + B_{xy} \frac{\partial^4 w}{\partial x^2 \partial y^2} + B_y \frac{\partial^4 w}{\partial y^4} = q. \tag{27}$$

Numerical Examples

The case when a lateral concentrated load acts on the center of the two-way multi-cell plate with four simply supported sides, is calculated to illustrate the numerical results, which are compared with those of the orthotropic plate theory.

The computation was carried out by FACOM 230–60 in Hokkaido University.

$E = 34800 \text{ kg/cm}$, $\nu = 0.0$, $t = 3 \text{ mm}$, $t_{0x} = t_{0y} = 5 \text{ mm}$, $\lambda_1 = \lambda_2 = 10 \text{ cm}$,
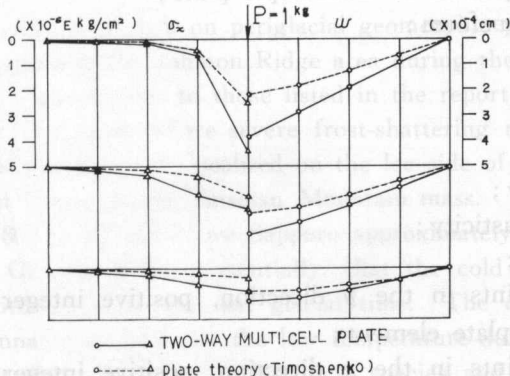
$h_{0x} = h_{0y} = 6 \text{ cm}$, $n = 8$, $k = 6$.

● —— ▲ TWO-WAY MULTI-CELL PLATE
○ ------ ▲ plate theory (Timoshenko)

Fig. 4. σ_x and w diagram

○ ------ ▲ Plate theory (Timoshenko)
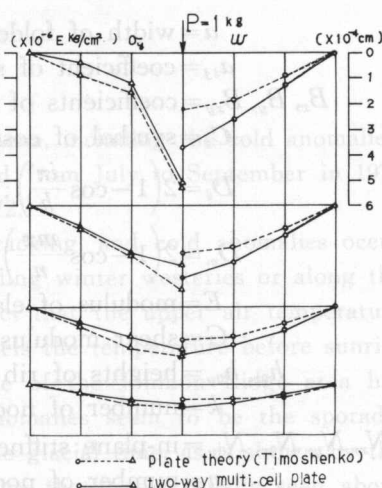○ —— ▲ two-way multi-cell plate

Fig. 5. σ_y and w diagram

Conclusions

In order to solve the stress problem of the two-way multi-cell plate, we would like to emphasize the followings ;

(a) The stiffness matrix is derived from the displacement shear equations of the long strip, by successive integration, which yields more precise expression of the stress by the nodal displacement together with a pair of adjacent ones.

(b) The equilibrium of forces at the nodal point is expressed by the simultaneous difference equations, which is treated by means of "Finite Fourier Integration Transforms".

(c) The method is not limited to the case of the two-way multi-cell plate, but is valid for any type of the spatial structure consisting of the rectangular plate elements.

The experiment checking the theory, will be carried out later.

References

1) S. G. Nomachi and K. G. Matsuoka: Applications of Finite Fourier Integration Transform for Structural Mechanics, Proc. of the 20th Japan National Congress for Applied Mechanics (1970).

2) S. G. Nomachi, K. G. Matsuoka and T. Ohshima: On the Stress Analysis of the Plate with Multi-Crosswise Ribs, Paper No. 120, the 5th Symposium of J. S. S. A., June, 1970.

3) S. G. Nomachi and T. Ohshima: Analysis of Two-Way Multi-Cell Plate, Paper No. 118, Section I, the 27th annual meeting of J. S. C. E., Oct., 1971.

Appendix. I.–Notation

a = width of folded plate element;

a_{ij} = coefficient of stiffness matrix;

B_x, B_y, B_{xy} = coefficients of the equation of orthotropic plate;

C_i = symbol of cosine transform;

$$D_i = 2\left(1 - \cos\frac{i\pi}{k}\right);$$

$$D_m = 2\left(1 - \cos\frac{m\pi}{n}\right);$$

E = modulus of elasticity;

G = shear modulus of elasticity;

h_{0x}, h_{0y} = heights of rib plate;

k = number of nodal points in the y direction, positive integer;

N_x, N_y, N_{0x}, N_{0y} = in-plane stiffness of plate element;

n = number of nodal points in the x direction, positive integer;

$\tilde{P}_1, \tilde{P}_2, l_3$ = horizontal and lateral load on the plate containing boundary values in the x, z and y directions, respectively;

R_i = symbol of modified cosine transform;

S_i = symbol of sine transform;

S = normal force in the folded plate element;

T = shear flow in the folded plate element;

t, t_{0x}, t_{0y} = thickness of flange plate and rib plates;

u, v, w = horizontal and vertical displacement component in the x, y and z directions, respectively;

X, Y, Z = coordinate axes for discrete number;

x, y, z = coordinate axes for continuous number;

λ_1, λ_2 = nodal length of plate element; and

ν = Poisson's ratio.