# A Certain Expansion of Positive Cone of Partially Ordered Linear Spaces

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Let E be a partially ordered linear space in a most general sense. We define the set P in E such that

$$P=\{x:\ x<y \text{ implies } y\geqq 0\}.$$

In section 1, we shall consider about some necessary conditions which yield to that $P=E^+$. In section 2, we shall investigate about some properties of P. And assuming that P satisfies conditions of positive cone of partially ordered linear space, we shall make properties of ordering which is introduced by P clear. Finally we shall get necessary and sufficient condition of that $P=E^+$.

## 1. Some necessary conditions for $P=E^+$

If $\bigcap_{x>0} x=0$, then $\bigcap_{x>a} x=a$ for every $a\in E$. Because $a=\bigcap_{x>0} x+a=\bigcap_{x>0}(x+a)=\bigcap_{y>a} y$.

(1) If $\bigcap_{x>0} x=0$, then $P=E^+$.

It is as in the following. It is clear that $E^+\subset P$. Conversely if $a\in P$, then $a<x$ concludes $x\geq 0$. From existence of $\bigcap_{x>a} x(=a)$, we have $0=\bigcap_{x>0} x\leq \bigcap_{x>a} x=a$ and $P\subset E^+$.

(2) If E is vector lattice, then $\bigcap_{x>0} x=0$.

It is as in the following. We put $a=a^+-a^-\leq x$ for every $x>0$. If $a^+>0$, then $a=a^+-a^-<\frac{1}{2}a^+$. This fact is not in agreement with $a^+=a\cup 0$. Consequently we have $a^+=0$ and $a\leq 0$.

(1) and (2) yield the following theorem.

**Theorem 1.** *If* E *is vector lattice, then* $P=E^+$.

*The inverse of Theorem 1. is not established.*

*Example 1.* Let E be the collection of all differentiable real functions in an interval $(a, b)$. For $f, g\in E$ we define $f\geq g$ if $f(x)\geq g(x)$, $(a<x<b)$. E is not vector lattice[1)] but $P=E^+$.

The following relation is obvious.

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1) See [1].

(3) If E is Archimedean, then $\bigcap_{x>0} x=0$[2].

(1) and (3) yield the following theorem.

**Theorem 2.** *If* E *is Archimedean, then* $P=E^+$.

The inverse of Theorem 2. is not established.

*Example 2.* Let E be $R^2$ with lexicographic ordering.

E is not Archimedean but E is vector lattice (with total ordering) and $P=E^+$.

## 2. Some properties of P

Let A be a subset of E. A point $a$ is an internal point[3] of A if, given $x$ in E, there exists $\delta>0$ such that $a+\lambda x\in A$ whenever $|\lambda|\leq\delta$.

**Theorem 3.** *If a point $a$ is an internal point of* P, *then* $a\geq 0$.

*Proof.* We put $0<p\in E$. To $p$ there exists $\delta>0$ such that $a+\lambda p\in P$ whenever $|\lambda|\leq\delta$. Now we put $0<\lambda\leq\delta$ and we have $P\ni a-\lambda p<a$. Consequently $a\geq 0$.

**Theorem 4.**

i) $E^+\subset P$.

ii) $P\ni a, b$ *implies* $a+b\in P$.

iii) $P\ni a$ *and* $\alpha>0$ *imply* $\alpha a\in P$.

*Proof.* i) is clear. If $a, b\in P$ and $a+b<x$, then $a<x-b$.

Since $a\in P$ we have $x-b\geq 0$ and $b\leq x$. If $b=x$, then $a+b<b$ and $a<0$. This fact is not in agreement with $a\in P$. Consequently $b<x$ and $x\geq 0$. From the definition of P we have ii). Finally we put $a\in P$ and $\alpha>0$. If $\alpha a<x$, then $a<\frac{1}{\alpha}x$ and $\frac{1}{\alpha}x\geq 0$.

Consequently $x\geq 0$ and $\alpha a\in P$.

It is not established that P satisfies the conditions of positive cone in partially ordered linear space.

*Example 3.* Let E be $R^2$. We introduce ordering to E such that $E^+=\{(x, y): (x, y)=(0, 0) \text{ or } x>0\}$. We have $P=\{(x, y): x\geq 0\}$.

Consequently P do not satisfy the condition that $a, -a\in P$ implies $a=0$.

**Theorem 5.** *If a point $a$ is an internal point of* P, *thne $a$ is an internal point of* $E^+$.

*Proof.* Let $a$ be an internal point of P. We put $0<p\in E$.

To $p$ there exists $\delta>0$ such that $a+\lambda p\in P$ whenever $|\lambda|\leq\delta$.

Now we put $0<\lambda\leq\delta$ and we have $a-\lambda p\in P$ and $a-\frac{\lambda}{2}p\in P$.

To any element $x\in E$ there exists $\delta'>0$ such that $a+\mu x\in P$ whenever

2) See [2].

3) See [3].

$|\mu| \leq \delta'$. If $|\mu| \leq \frac{\delta'}{2}$, then $a+2\mu x \in \mathrm{P}$. And we have

$$(a-\lambda p)+(a+2\mu x)=2a+2\mu x-\lambda p \in \mathrm{P}, \qquad a+\mu x-\frac{\lambda}{2} p \in \mathrm{P}.$$

Since $a+\mu x-\frac{\lambda}{2} p<a+\mu x$ we have $a+\mu x \geq 0$. Consequently $a$ is an internal point of $\mathrm{E}^{+}$.

If P satisfies conditions of positive cone in partially ordered linear space, then we can introduce another ordering in E, namely, to elements $a$ and $b$ of E we define $a \geq b$ if $a-b \in \mathrm{P}$.

And E is partially ordered linear space with positive cone P. We denote such a partially ordered linear space by $\tilde{\mathrm{E}}$.

We get easily by Theorem 4. i) that $a \leq b$ in E implies $a \leq b$ in $\tilde{\mathrm{E}}$.

**Theorem 6.** *If we put* $\tilde{\mathrm{P}}=\{x:\ x<y \text{ in } \tilde{\mathrm{E}} \text{ implies } y \geq 0 \text{ in } \tilde{\mathrm{E}}\}$, *then* $\tilde{\mathrm{P}}=\mathrm{P}$.

*Proof.* It is clear that $\mathrm{P}=\tilde{\mathrm{E}}^{+} \subset \tilde{\mathrm{P}}$. We put $a \in \tilde{\mathrm{P}}$.

Assuming $a<x$ in E we have $a<x$ in $\tilde{\mathrm{E}}$. Since $x-a>0$ in E we have $a<a+\frac{1}{2}(x-a)$ in E and $a<a+\frac{1}{2}(x-a)$ in $\tilde{\mathrm{E}}$. Consequently $a+\frac{1}{2}(x-a) \in \tilde{\mathrm{E}}^{+}=\mathrm{P}$.

Since $a+\frac{1}{2}(x-a)<x$ in E, we have $x \geq 0$ in E. Hence $a \in \mathrm{P}$.

**Theorem 7.** *It is established that* $\bigcap_{x>0} x=0$ *in* $\tilde{\mathrm{E}}$.

Proof. It is clear that 0 is lower bound of every $x>0$ in $\tilde{\mathrm{E}}$. Let $a(\in \tilde{\mathrm{E}})$ be a lower bound of every $x>0$ in $\tilde{\mathrm{E}}$. $-a<x$ in $\tilde{\mathrm{E}}$ implies $0<a+x$ in $\tilde{\mathrm{E}}$. If $-a<x$, then we have $a \leq a+x$ in $\tilde{\mathrm{E}}$ and $0 \leq x$ in $\tilde{\mathrm{E}}$. Consequently $-a \in \tilde{\mathrm{P}}=\tilde{\mathrm{E}}^{+}$ and $a \leq 0$ in $\tilde{\mathrm{E}}$.

If $\mathrm{P}=\mathrm{E}^{+}$, then $\tilde{\mathrm{E}}^{+}=\mathrm{E}^{+}$ and $\tilde{\mathrm{E}}$ is same to E. Hence we have the following theorem.

**Thearem 8.** $\mathrm{P}^{+}=\mathrm{E}$, *if and only if*, $\bigcap_{x>0} x=0$.

**Reference**

[1] K. Isobe: Some Examples in Semi-ordered Linear Space. *Mem. Kitami Inst. Tech. Vol.* 2, *No.* 2 (1968), 327–330.

[2] G. Jameson: Ordered Linear Space. (1970), Springer-Verlag.

[3] T. Higashiyama: Some Notes on Semi-ordered Linear Spaces. *Jour. Hokkaido Univ. Education Vol. 22, No. 2* , (*Sec. II A*), (1972) 69–71.