A Result on Nonlinear Semi-groups

by Shigeo KATO

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Let E be a Banach space and let $\{T(t): t\geqq 0\}$ be a family of nonlinear operators from a subset D of E into itself satisfying the following conditions:

(1) $T(s+t)=T(s)T(t)$ for $s, t\geqq 0$, and $T(0)=I$ (the identity operator),

(2) $\lim_{t\downarrow 0} T(t)x=x$ for $x\in D$,

(3) $\| T(t)x-T(t)y\| \leqq \|x-y\|$ for $x, y\in D$ and $t\geqq 0$.

We simply call such a family $\{T(t); t\geqq 0\}$ a nonlinear contraction semi-group.

We define the infinitesimal generator A of a nonlinear contraction semi-group $\{T(t); t\geqq 0\}$ by

$$Ax=\lim_{t\downarrow 0}\frac{1}{t}\Big(T(t)x-x\Big),$$

if the right side exists.

We say a multi-valued nonlinear operator A with domain $D(A)$ a d-operator for convenience if there exists a sufficiently small positive constant σ_0 such that for each $x, y\in D(A)$

$$\langle x-y,\ u-v;\ \sigma_0\rangle \leqq 0 \qquad \text{for } u\in Ax,\ v\in Ay,$$

where $\langle x-y,\ u-v;\ \sigma_0\rangle=\dfrac{1}{\sigma_0}\Big(\|x-y+\sigma_0(u-v)\|-\|x-y\|\Big)$.

If A is a d-operator then it is easy to see that A is dissipative, that is, $Re(u-v, f)\leqq 0$ for $f\in F(x-y)$, where F denotes the duality map from E into the dual space E^* of E.

An operator A is said to be a maximal d-operator if A is a d-operator and if any extension of A which is a d-operator coincides with A.

In this note we consider the following general nonlinear Cauchy problem

$$\frac{d}{dt}u(t)\in Au(t), \qquad u(0)=x\in D(A),$$

where A is a multi-valued d-operator in a reflexive Banach space.

We say $u(t)$ is a solution of the above equation if $u(t)$ is strongly absolutely continuous on any finite interval with $u(0)=x$, and if

$$u(t)\in D(A), \quad \frac{d}{dt}u(t)\in Au(t) \qquad \text{for } a.e.\ t\geqq 0.$$

1. Lemmas

Let x, y be in E and $\sigma>0$.

We define $\langle x, y\,;\sigma\rangle$ and $\langle x, y\rangle$ as follows:

$$\langle x, y\,;\sigma\rangle=\frac{1}{\sigma}(\|x+\sigma y\|-\|x\|),\qquad \langle x, y\rangle=\lim_{\sigma\downarrow 0}\langle x, y\,;\sigma\rangle\,.$$

Lemma 1. (see [3], [4]).

For any x, y and z in E,

(1) $|\langle x, y\rangle|\leqq\|y\|$, $\langle x, x\rangle=\|x\|$,

(2) $\langle x, y+z\rangle\leqq\langle x, y\rangle+\langle x, z\rangle$,

(3) $\langle x, ay\rangle=a\langle x, y\rangle$ $(a\geqq 0)$,

(4) $\langle x, ax+y\rangle=a\|x\|+\langle x, y\rangle$ $(a=\text{real})$,

(5) $\frac{1}{2}\big(\langle x, y\rangle-\langle x, -y\rangle\big)\leqq\langle x, y\rangle$,

(6) $\langle x, y\,;\sigma\rangle$ is a non-decreasing function of $\sigma>0$.

Lemma 2. ([9]).

Let $f(t)$ be a E-valued function defined on a real interval J such that $\frac{d}{dt}f(t)$ and $\frac{d}{dt}\|f(t)\|$ exist for *a. e.* $t\in J$.

Then

$$\frac{d}{dt}\|f(t)\|=\frac{1}{2}\Big(\langle f(t),\ \frac{d}{dt}f(t)\rangle-\langle f(t),\ -\frac{d}{dt}f(t)\rangle\Big)\quad\text{for } a.e.\, t\in J.$$

Proof.

If we denote $D^+f(t)$ and $D^-f(t)$ respectively the right and left derivatives of $f(t)$, then we have

$$\Big|\frac{1}{\sigma}\Big(\|f(t+\sigma)\|-\|f(t-\sigma)\|\Big)-\langle f(t),\ D^+f(t)\,;\sigma\rangle+\langle f(t),\ -D^-f(t)\,;\sigma\rangle\Big|$$

$$=\frac{1}{\sigma}\Big|\|f(t+\sigma)\|-\|f(t-\sigma)\|-\|f(t)+\sigma D^+f(t)\|+\|f(t)-\sigma D^-f(t)\|\Big|$$

$$\leqq\Big\|\frac{1}{\sigma}\Big(f(t+\sigma)-f(t)\Big)-D^+f(t)\Big\|+\Big\|\frac{1}{\sigma}\Big(f(t-\sigma)-f(t)\Big)+D^-f(t)\Big\|$$

$$\to 0\ \text{ as }\ \sigma\downarrow 0\qquad\text{for }\ a.e.\, t\in J.$$

Thus we have

$$D^+\|f(t)\|+D^-\|f(t)\|=\langle f(t),\ D^+f(t)\rangle-\langle f(t),-D^-f(t)\rangle\quad\text{for } a.e.\, t\in J,$$

so that

$$\frac{d}{dt}\|f(t)\|=\frac{1}{2}\Big(\langle f(t),\ \frac{d}{dt}f(t)\rangle-\langle f(t),\ -\frac{d}{dt}f(t)\rangle\Big)\qquad\text{for } a.e.\, t\in J.$$

Lemma 3.

Let A be a d-operator with domain $D(A)$ such that $R(I-\lambda_0 A)=E$ for some $\lambda_0>0$.

Then

(1) $J_\lambda=(I-\lambda A)^{-1}$ exists on E for any $\lambda>0$, and

$$\|J_\lambda x-J_\lambda y\|\leqq\|x-y\| \quad \text{for } x, y\in E.$$

(2) For x in $D(A)$, $A_\lambda x\in AJ_\lambda x$ and $\|A_\lambda x\|\leqq |||Ax|||$,

$$\text{where } A_\lambda=\frac{1}{\lambda}(J_\lambda-I) \text{ and } |||Ax|||=\inf\left\{\|y\|\,;\, y\in Ax\right\}.$$

(3) $\langle x-y,\ A_\lambda x-A_\lambda y\rangle\leqq 0$ for $x, y\in E$.

Proof.

(1) Since $R(I-\lambda_0 A)=E$ for some $\lambda_0>0$ $R(I-\lambda A)=E$ for all $\lambda>0$. (see [1]).

If $z\in(I-\lambda A)x\cap(I-\lambda A)y$, then $z=x-\lambda u=y-\lambda v$ for some $u\in Ax$ and $v\in Ay$. Since

$\langle x-y, u-v\rangle=\langle x-y, \frac{1}{\lambda}(x-y)\rangle=\frac{1}{\lambda}\|x-y\|\leqq 0$ by Lemma 1 (3), we have $x=y$.

Hence $(I-\lambda A)x\cap(I-\lambda A)y=\emptyset$ for $x\neq y$.

Set $J_\lambda x=u$ and $J_\lambda y=v$. Then $x=u-\lambda z_1$, $y=v-\lambda z_2$ for some $z_1\in Au$ and some $z_2\in Av$.

Thus we have

$$\begin{aligned}\|u-v\|&=\langle u-v, u-v\rangle=\langle u-v, x-y+\lambda(z_1-z_2)\rangle\\&\leqq\langle u-v, x-y\rangle+\lambda\langle u-v, z_1-z_2\rangle\leqq\langle u-v, x-y\rangle\leqq\|x-y\|,\end{aligned}$$

and this implies $\|J_\lambda x-J_\lambda y\|\leqq\|x-y\|$.

(2) See [1], [6].

(3) Since $A_\lambda x-A_\lambda y=\frac{1}{\lambda}(J_\lambda x-J_\lambda y)-\frac{1}{\lambda}(x-y)$, we have

$$\langle x-y, A_\lambda x-A_\lambda y\rangle\leqq\frac{1}{\lambda}(\langle x-y, J_\lambda x-J_\lambda y\rangle-\|x-y\|)\leqq 0.$$

Lemma 4.

Let A be a d-operator with domain $D(A)$ such that $R(I-\lambda_0 A)=E$ for some $\lambda_0>0$.

Then

(1) A is a maximal d-operator.

(2) Ax_0 is closed and convex for each $x_0\in A$.

Proof.

(1) Let A_1 be an extension of A which is a d-operator.

If there exists a $x_0\in D(A_1)$ such that $x_0\notin D(A)$, then by Lemma 3 (1)

$$(I-\lambda_0 A_1)x_0\cap(I-\lambda_0 A)x=\emptyset \quad \text{for all } x\in D(A).$$

Since $R(I-\lambda_0 A)=E$ we have the contradiction $(I-\lambda_0 A_1)x_0\cap E=\emptyset$.

If there exists a $x_0 \in D(A)$ such that $A_1 x_0$ contains properly Ax_0, then there exists a $y_0 \in A_1 x_0$ such that $y_0 \bar{\in} Ax_0$.

Since $x_0 - \lambda_0 y_0 \bar{\in} (I - \lambda_0 A) x_0$, $x_0 - \lambda_0 y_0 \bar{\in} (I - \lambda_0 A) x$ for all $x \in D(A)$.

This contradicts also the fact $R(I - \lambda_0 A) = E$.

(2) Since A is a maximal d-operator by (1) we see that

$$Ax_0 = \left\{ z \, ; \, \langle x - x_0, \, y - z \, ; \, \sigma_0 \rangle \leqq 0 \quad \text{for all } x \in D(A) \text{ and } y \in Ax \right\}$$

The closedness of Ax_0 is obvious.

Let $y_1, y_2 \in Ax_0$ and let a, b be positive numbers such that $a + b = 1$. Then by Lemma I (2)

$$\langle x - x_0, \, y - (a y_1 + b y_2) \rangle \leqq a \langle x - x_0, \, y - y_1 \rangle + b \langle x - x_0, \, y - y_2 \rangle \leqq 0$$

$$\text{for all } x \in D(A) \text{ and } y \in Ax.$$

If $a y_1 + b y_2 \in Ax_0$, we define A_1 by $A_1 x = Ax$ for $x \neq x_0$ and $A_1 x_0 = Ax_0 \cup \{a y_1 + b y_2\}$. Then $A_1 x_0$ contains properly Ax_0. This contradicts the maximality of A. (see the proof of (1)).

Let A be a d-operator with domain $D(A)$ such that $R(I - \lambda_0 A) = E$ for some $\lambda_0 > 0$. Then by Lemma 3 (1) $J_n = \left(I - \dfrac{1}{n} A \right)^{-1}$ and $A_n = n(J_n - I)$ are everywhere defined on E.

Since $\|A_n x - A_n y\| \leqq 2n \|x - y\|$ for $x, y \in E$, the differential equation

$$\frac{d}{dt} u_n(t) = A_n u_n(t), \quad u_n(0) = x \in E,$$

has a unique global sloution $u_n \in C_E^1[0, \infty)$, where $C_E^1[0, \infty)$ denotes the set of all strongly continuously differentiable E-valued functions defined on $[0, \infty)$.

By using Lemma 2 and Lemma 3 (2) we will deduce some estimates for $u_n(t)$ and $u_n'(t)$, where $u_n'(t)$ denotes the strong derivative of $u_n(t)$ with respect to t.

Lemma 5.

Let $x \in D(A)$ and $t_0 > 0$. Then

$$\|u_n'(t)\| \leqq \|\!|Ax|\!\| \quad \text{and} \quad \|u_n(t)\| \leqq \|x\| + t \|\!|Ax|\!\| \qquad \text{for } n \geqq 1 \text{ and } t \in [0, t_0].$$

Proof.

Set $F(t\,;\, n, h) = \|u_n(t+h) - u_n(t)\|$, where $0 < h < t_0$.

Since $F(t\,;\, n, h)$ is Lipschiz continuous in t with $\|u_n(t+h) - u_n(t)\|$, $F'(t\,;\, n, h)$ exists for $a.e.\, t \in [0, t_0]$ and by Lemma 2 and Lemma 1 (5), we have

$$F'(t\,;\, n, h) \leqq \langle u_n(t+h) - u_n(t), \, A_n u_n(t+h) - A_n u_n(t) \rangle \leqq 0.$$

Hence $F(t\,;\, n, h) \leqq F(0\,;\, n, h)$, that is,

$$\|u_n(t+h) - u_n(t)\| \leqq \|u_n(h) - u_n(0)\|.$$

By dividing the above inequality by $h > 0$ and letting $h \downarrow 0$, we have

$$\|u_n'(t)\| \leqq \|u_n'(0)\| = \|A_n x\| \leqq \|\!|Ax|\!\| .$$

Since $u_n(t) = x + \int_0^t A_n u_n(s) ds$ for $t \in [0, t_0]$, it follows that

$$\|u_n(t)\| \leqq \|x\| + t\|\!|Ax|\!\| .$$

Lemma 6.

For each $t_0 > 0$, the strong limit $u(t) = \lim_{n\to\infty} u_n(t)$ exists uniformly for $t \in [0, t_0]$. $u(t)$ is Lipschiz continuous on $[0, t_0]$ with $u(0) = x$.

Proof.

Set $G(t; n, m) = \|u_n(t) - u_m(t)\|$ for $t \in [0, t_0]$.
Since $G(t; n, m)$ is Lipschiz continuous in t with $\|u_n(t) - u_m(t)\|$, $G'(t; n, m)$ exists for $a.e.\, t \in [0, t_0]$ and

$$G'(t; n, m) \leqq \langle u_n(t) - u_m(t), \quad A_n u_n(t) - A_m u_m(t) \rangle \qquad \text{for } a.e.\, t \in [0, t_0] .$$

Since $\langle x, y ; \sigma \rangle$ is a nondecreasing function of $\sigma > 0$ and $\langle x, y \rangle = \inf_{\sigma > 0} \langle x, y ; \sigma \rangle$, if follows that

$$\begin{aligned}
G'(t; n, m) &\leqq \langle u_n(t) - u_m(t), \quad A_n u_n(t) - A_m u_m(t), \sigma_0 \rangle \\
&\leqq \frac{1}{\sigma_0} \Big\{ \Big\| J_n u_n(t) - J_m u_m(t) + \sigma_0 \Big(A_n u_n(t) - A_m u_m(t) \Big) \Big\| \\
&\quad + \frac{1}{n} \|A_n u_n(t)\| + \frac{1}{m} \|A_m u_m(t)\| \Big\} \\
&= \langle J_n u_n(t) - J_m u_m(t), \quad A_n u_n(t) - A_m u_m(t); \sigma_0 \rangle \\
&\quad + \frac{1}{\sigma_0} \Big(\frac{1}{n} \|A_n u_n(t)\| + \frac{1}{m} \|A_m u_m(t)\| + \|J_n u_n(t) - J_m u_m(t)\| - \|u_n(t) - u_m(t)\| \Big) \\
&\leqq \frac{2}{\sigma_0} \Big(\frac{1}{n} \|A_n u_n(t)\| + \frac{1}{m} \|A_m u_m(t)\| \Big) \\
&\leqq \frac{2}{\sigma_0} \Big(\frac{1}{n} + \frac{1}{m} \Big) \|\!|Ax|\!\| .
\end{aligned}$$

Solving this differential inequality we have

$$G(t; n, m) \leqq G(0; n, m) + \frac{2t}{\sigma_0} \Big(\frac{1}{n} + \frac{1}{m} \Big) \|\!|Ax|\!\| \leqq \frac{2t_0}{\sigma_0} \Big(\frac{1}{n} + \frac{1}{m} \Big) \|\!|Ax|\!\| .$$

Hence $u_n(t)$ is convergent uniformly on $[0, t_0]$.

Set $u(t) = \lim_{n\to\infty} u_n(t)$ for $t \in [0, t_0]$. Then the Lipschiz continuity of $u(t)$ on $[0, t_0]$ is obvious from the same property of $u(t)$. The proof is complete.

Since A_n is independent of t, we can rewrite

$$u_n(t) = T(t, n) x \qquad \text{for} \quad x \in D(A) .$$

Since

$$\frac{d}{dt}\|T(t,n)x-T(t,n)y\| \leqq \langle T(t,n)x-T(t,n)y\,,\ A_nT(t,n)x$$
$$-A_nT(t,n)y\rangle \leqq 0 \qquad \text{for}\quad a.e.\ t\geqq 0\,,\ \text{ we have}$$
$$\|T(t,n)x-T(t,n)y\| \leqq \|x-y\|\,.$$

We set $T(t)x = \lim_{n\to\infty} T(t,n)x$ for $x\in D(A)$.

Then it is easy to see that $\{T(t);\, t\geqq 0\}$ is a nonlinear contraction semi-group on $D(A)$.

If A is a d-operator with domain $D(A)$ in a reflexive Banach space E such that $R(I-\lambda_0 A)=E$ for some $\lambda_0>0$, then Ax_0 is closed and convex for each $x_0\in D(A)$ by Lemma 4 (2).

We define the canonical restriction A^0 of A by $A^0x=\{u\,;\, u\in Ax$ and $\|u\|=\|Ax\|\}$ for $x\in D(A)$.

Since Ax is strongly closed and convex Ax is closed in the weak topology of E and hence $A^0x\neq\emptyset$, for E is reflexive.

An operator A is said to be demiclosed if the following condition is satisfied: if $x_n\in D(A)$, $n=1,2,\cdots,$, $x_n\to x_0\in E$ and if there are $y_n\in Ax_n$ such that $y_n\rightharpoonup y_0$ (we denote by $\rightharpoonup$ weak convergence), then $x_0\in D(A)$ and $y_0\in Ax_0$.

2. Theorems

We now state the following theorems. (see also [10]).

Theorms 1.

Let A be a demiclosed d-operator with domain $D(A)$ in a reflexive Banach space E such that $R(I-\lambda_0 A)=E$ for some $\lambda_0>0$.

Then there exists a unique nonlinear contraction semi-group $\{T(t);\, t\geqq 0\}$ on $D(A)$ which satisfies the following differential equation:

$$\frac{d}{dt}u(t)\in A^0u(t) \qquad \text{with } u(0)=x\in D(A)\,.$$

Proof.

Since the existence of a nonlinear contraction semi-group $\{T(t);\, t\geqq 0\}$ on $D(A)$ has been proved in Lemma 6, we only have to show that $T(t)x$ satisfies the above differential equation.

Note that $\{A_nT(\cdot,n)x\}$ is a bounded set in $L^2_E[0,t_0]$ for each $t_0>0$, where $L^2_E[0,t_0]$ denotes the set of all square integrable E-valued measurable functions on $[0,t_0]$.

Thus some subsequence of $\{A_nT(\cdot,n)x\}$ converges to an element z in $L^2_E[0,t_0]$ with respect to the weak topology $\sigma(L^2_E[0,t_0],\, L^2_E[0,t_0]^*)$.

For convenience we assume that $\{A_nT(\cdot,n)x\}$ converges to z weakly in $L^2_E[0,t_0]$.

Let C_t be the set of all weak limit of a subsequence of $\{A_nT(t,n)x\}$ for each fixed $t\in[0,t_0]$ and let $[C_t]$ be the smallest closed convex extension of C_t.

We will show that $T(t)x\in D(A)$ for all $t\in[0, t_0]$ and $z(t)\in AT(t)x$ for $a.e.\ t\in[0, t_0]$.

For each $w\in C_t$ there exists a subsequence $\{A_{n_m}T(t, n_m)x\}$ such that

$$A_{n_m}T(t, n_m)x \rightharpoonup w \quad \text{as} \quad m\to\infty .$$

Since $J_{n_m}T(t, n_m)x\to T(t)x$, $J_{n_m}T(t, n_m)x\in D(A)$ *and* $A_{n_m}T(t, n_m)x\in AJ_{n_m}T(t, n_m)x$, it follows by the demiclosedness of A that

$$T(t)x\in D(A) \quad \text{and} \quad w\in AT(t)x .$$

Note that $AT(t)x$ is closed convex for each $t\in[0, t_0]$, thus we have

$$[C_t]\subset AT(t)x \qquad \text{for each} \quad t\in[0, t_0].$$

Since $\{A_nT(\cdot, n)x\}$ converges to z weakly in $L^2_E[0. t_0]$,
z is the strong limit in $L^2_E[0, t_0]$ of a sequence of elements of the type $\sum_i a_iA_{n+i}T(\cdot, n+i)x$, where $\{a_i\}$ is a finite set of nonnegative numbers such that $\sum_i a_i=1$. (see [5], [3]).

Thus we can seek a subsequence of such a sequence converging to $z(t)$ strongly in E for $t\in[0, t_0]-\Omega$, where Ω is a set of measure zero.

Let B be any strongly closed convex set containing C_t for each fixed $t\in[0, t_0]-\Omega$.

Let U be any open convex neighbourhood of 0 in the weak topology of E, then there exists a open convex neighbourhood V of 0 in the same topology of E such that $V+V\subset U$.

Since $B+V$ is open convex in the weak topology of E there is a n_0 such that $A_nT(t, n)x\in B+V$ for $n\geqq n_0$, by recalling the definition of C_t.

Thus the convex combinations of the type $\sum_i a_iA_{n+i}T(t, n+i)x$ belong to $B+V$ for $n\geqq n_0$.
Hence $z(t)\in(B+V)^{-w}$, where $(B+V)^{-w}$ denotes the closure of $B+V$ with respect to the weak topology of E.

Note that $(B+V)^{-w}\subset(B+V)+V\subset B+U$, it follows that $z(t)\in B+U$.
Thus we see $z(t)\in B^{-w}$, for U is arbitrary.

Since B is a strongly closed convex set, B is also a closed convex set in the weak topology of E.
Therefore we obtain $z(t)\in[C_t]$, since B is an arbitrary strongly closed convex set containing C_t.

Since $\|A_nT(t, n)x\|\leqq|\!|\!|Ax|\!|\!|$, the norm of a convex combination of $A_nT(t, n)x$'s is also $\leqq|\!|\!|Ax|\!|\!|$.

It follows that $\|z(t)\|\leqq|\!|\!|Ax|\!|\!|$ for $t\in[0, t_0]-\Omega$, and thus $z(t)$ is Bochner integrable on $[0, t_0]$.

Since $L^2_E[0, t_0]^*=L^2_{E^*}[0, t_0]$, (see [2]), and since

$$\left(T(t, n)x,\ x^*\right)=(x, x^*)+\int_0^t\left(A_nT(s, n)x,\ x^*\right)ds \ \text{for}\ x^*\in E^*, \text{and}\ t\in[0, t_0],$$

we have by going to $n\to\infty$

$$\left(T(t)x, x^*\right) = (x, x^*) + \int_0^t \left(z(s), x^*\right) \mathrm{ds},$$

where we have used the fact that $A_n T(\cdot, n)x$ converges to z weakly in $L^2_E[0, t_0]$. (Note that $f(t) = x^* \in L^2_{E^*}[0, t_0]$).

Thus we obtain by the Bochner integrability of $z(t)$

$$T(t)x = x + \int_0^t z(s)ds \qquad \text{for} \quad t \in [0, \mathrm{t}_0].$$

It follows that $\frac{d}{dt} T(t)x$ exists and

$$\frac{d}{dt} T(t)x = z(t) \in AT(t)x \qquad \text{for} \quad a.e.\ t \in [0, t_0].$$

Next we will show that $\frac{d}{dt} T(t)x \in A^0 T(t)x$ for $a.e.\ t \in [0, t_0]$. To prove this we note for each fixed $t \in [0, t_0]$,

$$\frac{d}{ds} \|T(s)x - T(t)x\| \leqq \langle T(s)x - T(t)x, \ \frac{d}{ds} T(s)x \rangle$$

$$\leqq \langle T(s)x - T(t)x, y \rangle \leqq \|y\|,$$

for each $y \in AT(t)x$ and $a.e.\ \mathrm{s} \in [0, t_0]$.
Thus

$$\frac{d}{ds} \|T(s)x - T(t)x\| \leqq |||AT(t)x||| \qquad \text{for} \quad a.e.\ s \in [0, t_0].$$

By integrating the above inequalitty and noting $\frac{d}{dt} T(t)x \in AT(t)x$ for $a.e.\ t \in [0, t_0]$, we obtain

$$\frac{d}{dt} T(t)x \in A^0 T(t)x \qquad \text{for} \quad a.e.\ t \in [0, t_0].$$

Therefore $T(t)x$ is a solution of the differential equation

$$\frac{d}{dt} u(t) \in A^0 u(t) \quad \text{with} \quad \mathrm{u}(0) = x,$$

since t_0 is arbitrary.

If there is another nonlinear contraction semi-group $\{S(t), t \geqq 0\}$ on $D(A)$ satisfying the above differential equation, then for each $x \in D(A)$

$$\frac{d}{dt} \|T(t)x - S(t)x\| \leqq \langle T(t)x - S(t)x, \ \frac{d}{dt} T(t)x - \frac{d}{dt} S(t)x \rangle \leqq 0 \qquad \text{for} \quad a.e.\ t \geqq 0.$$

The uniqueness follows by integrating the above inequality.

Theorem 2.

In Theorem I assume further that E is uniformly convex. Then A^0 is the strong infinitesimal generator of $\{T(t), t \geqq 0\}$.

Proof.

Let $t_n \in [0, t_0] - \Omega$ and $t_n \downarrow 0$.

Then $T(t_n)x \to x$, $z(t_n) \in AT(t_n)x$, and $\|z(t_n)\| \leqq \|A^0 x\|$.

Hence there exists a subsequence $\{z(t_{n_m})\}$ such that $z(t_{n_m}) \to y$ for some y. By the demiclosedness of A we have $y \in Ax$.

Since

$$\|A^0 x\| \leqq \|y\| \leqq \varliminf_{m\to\infty} \|z(t_{n_m})\| \leqq \varlimsup_{m\to\infty} \|z(t_{n_m})\| \leqq \|A^0 x\| \quad \text{and since}$$

E is uniformly convex, it follows that

$$z(t_{n_m}) \to A^0 x \quad \text{strongly as} \quad m \to \infty.$$

Thus $z(t)$ may be made continuous from the right at zero by setting $z(0) = A^0 x$ and redefining $z(t) = A^0 x$ on Ω.

Since $\quad T(t)x = x + \int_0^t z(s)ds,\quad$ we have

$$\lim_{t\downarrow 0} \frac{1}{t}\left(T(t)x - x\right) = z(0) = A^0 x.$$

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