Some Properties of a Certain Partially Ordered Linear Space

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(Received April 30, 1970)

Introduction

Let R be a partially ordered linear space and we don't assume lattice order in R. The author [1] defined two sets P_a and N_a to every element $a \in R$ as the following:

$$(*) \qquad P_a = \{x : 0 \leq x \leq b \text{ for every } b, c \geq 0 \text{ and } a = b - c\}$$
$$N_a = P_{-a}.$$

We suppose in R the following postulate.

$$(**) \qquad P_a = \{0\} \text{ implies } a \leq 0.$$

In this paper we shall investigate some properties of such a partially ordered linear space.

Main results

(1) *If $a, b \geq 0$, $P_a \cap P_b = \{0\}$ and $c = a - b$ then we have not any element x such as $c = x - y$, $x, y \geq 0$ and $x < a$.*

Proof. If $x, y \geq 0$, $x < a$ and $c = x - y$ then $a - x = p > 0$, $c = x - y = (a - p) - y = a - b$. Hence $y = b - p \geq 0$ and $b \geq p$.

And we have $0 < p \in P_a \cap P_b$. This fact is inconsistent with the assumption.

(2) *If $a, b \geq 0$, $P_a \cap P_b = \{0\}$ and $c = a - b$ then $P_c = P_a$.*

Proof. Since $c \leq a$ we have $P_c \subset P_a$.

Putting $c = a - b = x - y$ for $x, y \geq 0$, we have $a - x = b - y$.

Since $a - x \leq a$ and $b - y \leq b$ we have $P_{a-x} \subset P_a$ and $P_{b-y} \subset P_b$.

Consequently $P_{a-x} = P_{b-y} \subset P_a \cap P_b = \{0\}$. And we have by (*) $a \leq x$. Hence $a \in P_c$ and $P_a \subset P_c$.

Since (2) we have the following (3) and (4).

(3) *If $a, b \geq 0$, $P_a \cap P_b = \{0\}$ and $c = a - b$ then $c = x - y$ for $x, y \geq 0$ implies $a \leq x$ and $b \leq y$.*

(4) *If $c = a - b$, $a, b \geq 0$ and $P_a \cap P_b = \{0\}$ then such elements a and b are determined uniquely.*

(5) *If $a, b \geq 0$ and $P_a \cap P_b = \{0\}$ then there exists $a \cup b$ and $a \cup b = a + b$.*

Proof. $a, b \leq a + b$. Moreover if $a, b \leq x$ then $a - b = (x - b) - (x - a)$. Consequently we have by (3) $a = x - b$, $a + b \leq x$ and $a \cup b = a + b$.

(6) *If $0 \leq x$, $y \leq a$ and x cannot be compared with y then there exists some*

element u such as $x<u\leq a$.

Proof. Hence by (**) there exists some element z such as $0<z\in P_{y-x}$ and we have $z\in P_{a-x}$ because $y-x\leq a-x$.

Putting $u=x+z$ we have $x<u\leq a$.

It is the same with the following (7) as with (6).

(7) *If* $0\leq x,\ y\leq a, b$ *and* x *cannot be compared with* y *then there exists some element* u *such as* $x<u\leq a, b$.

(8) *If* $a, b\geq 0,\ P_a\cap P_b=\{0\}$ *and* $0\leq x\leq a$ *then there exists* $(x+b)\cap a$ *and* $(x+b)\cap a=x$.

Proof. Now we have $x\leq a,\ x\leq a+b$. Moreover if $y\leq a$ and $y\leq x+b$ then $y-x\leq a$ and $y-x\leq b$. Consequently we have $P_{y-x}\subset P_a\cap P_b$ and hence $P_{y-x}=\{0\}$, $y\leq x$ and $(x+b)\cap a=x$.

(9) If $a, b\geq 0,\ 0\leq x\leq a+b$ and there exists $a\cap x$ then $x\in P_a+P_b$[1].

Proof. $x-(x\cap a)=x+(-a)\cup(-x)=(x-a)\cup 0$

$$(x-a)\cup 0-b=(x-a-b)\cup(-b)\leq 0$$

and we have $b\geq(x-a)\cup 0=x-(a\cap x)\in P_b$.

Since $a\cap x\in P_a$ we have $x=(a\cap x)+(x-a\cap x)\in P_a+P_b$.

Conversely we obtain the following (10) by (8).

(10) *If* $a, b\geq 0,\ 0\leq x\leq a+b,\ P_a\cap P_b=\{0\}$ *and* $x\in P_a+P_b$ *then there exists* $a\cap x$.

(11) *If* $a, b\geq 0,\ P_a\cap P_b=\{0\}$ *and* $c=x+y$ *for* $x\in P_a,\ y\in P_b$ *then such elements* x *and* y *are determined uniquely.*

Proof. Since $P_a\cap P_y=\{0\}$ we have by (8)

$$c\cap a=(x+y)\cap a=x.$$

Similarly

$$c\cap b=(x+y)\cap b=y.$$

Remaining questions

It is question that the following (12).

(12) *If* $a, b\geq 0,\ P_a\cap P_b=\{0\}$ *and* $0\leq x\leq a+b$ *then* $x\in P_a+P_b$.

Moreover we don't know whether $P_{a+b}\subset P_a+P_b$ or not generally.

Reference

[1] K. Isobe and T. Higashiyama: On Representation of Semi-ordered Linear Spaces, Mem. Kitami Inst. Tech. Vol. 2, No. 1, 1967.

[2] K. Isobe: Some Examples in Semi-ordered Linear Spaces, Mem. Kitami Inst. Tech. Vol. 2, No. 1, 1967.

[3] H. Nakano: Modern Spectral Theory, Tokyo Math. Book Ser. II, 1950.

1) $P_a+P_b=\{x+y : x\in P_a \text{ and } y\in P_b\}$