Note on the Approximation of Nonlinear Semi-Groups in Hilbert Space

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In the theory of linear semi-groups, the following Trotter-Kato Theorem is well known. [7]

Theorem.

Let $\{T^{(\sigma)}(t):\ t\geqq 0\}$ be a sequence of contraction liner semigroups of class (C_0) in a Banach space X and let $A^{(\sigma)}$ be the infinitesimal generator of $\{T^{(\sigma)}(t);\ t\geqq 0\}$.

Suppose that, for some λ_0 with $Re(\lambda_0)>0$,

$$\lim_{\sigma\to\infty} R(\lambda_0\,;\,A^{(\sigma)})\,x=J_{\lambda_0}\,x \text{ exists for all } x\in X$$

in such a way that the range $R(J_{\lambda_0})$ is dense in X.

Then J_{λ_0} is the resolvent of the infinitesimal generator A of a contraction linear semi-group

$$\{T(t)\,;\ t\geqq 0\} \text{ of class } (C_0) \text{ in } X \text{ and}$$

$$\lim_{\sigma\to\infty} T^{(\sigma)}(t)\,x=T(t)\,x \text{ for every } x\in X.$$

The above canvergence is uniform with respect to t on any compact interval of $(0, \infty)$.

This note has been motivated by the recent results by I. Miyadera [1], in which he treats a general theory of the convergence of nonlinear semi-groups in Banach spaces.

The object of the present note is to give the approximation theorem of this type in case of semi-groups of nonlinear contraction operators in Hilbert space.

1. Let H be a Hilbert space with inner product $<x, y>$ and let $\{T(t)\,;\ t\geqq 0\}$ be a family of nonlinear operators from H into itself satisfying the following conditions:

(1) $T(s+t)=T(s)\,T(t)$ for every $s, t\geqq 0$, and $T_0=I$ (the identity operator),

(2) $\lim_{t\downarrow 0} T(t)\,x=x$ for every $x\in H$,

(3) $\|T(t)\,x-T(t)\,y\|\leqq\|x-y\|$ for every $x, y\in H$.

We call such a family $\{T(t)\,;\ t\geqq 0\}$ a nonlinear contraction semi-proup.

The infinitesimal generator A of a nonlinear contraction semi-group $\{T(t)\,;\ t\geqq 0\}$ is defined by

$$Ax=\lim_{h\downarrow 0}\frac{T(h)\,x-x}{h},$$

that is, A is the operator whose domain is the set

$$D(A)=\left\{x\in \mathrm{H}\,;\ \lim_{h\downarrow 0}\frac{1}{h}(T(h)x-x)\text{ exists in } H\right\},\text{ and, for}$$

$$x\in D(A),\ Ax=\lim_{h\downarrow 0}\frac{1}{h}(T(h)x-x).$$

By virtue of the condition (3), A satisfies the following

$$Re\langle Ax-Ay,\quad x-y\rangle\leqq 0 \text{ for every } x, y\in D(A).$$

In general, an operator A satisfying the above condition is called dissipative.

For a multi-valued oparator A (that is, for an A which maps an element $x\in D(A)$ to a subset Ax of H), we also say that A is dissipative, if the following is satisfied:

$$Re\langle x'-y',\quad x-y\rangle\leqq 0 \text{ for any } x'\in Ax,\ y'\in Ay.$$

An operator A is said to be maximal dissipative if A is dissipaitve and if for each any dissipative extension of A coincides with A.

By Minty, for a multi-valued maximal dissipative operator A, $R(I-\lambda A)=H$ $\lambda>0$, where $R(I-\lambda A)$ denotes the range of $I-\lambda A$ [2].

Let A be a multi-valued maximal dissipative operator, then Ax is closed convex in H. Hence there exists a unique element $A^0x\in Ax$ such that $\|A^0x\|=\inf_{y\in Ax}\|y\|$ [1].

We call this A^0 the canonical restriction of A.

2. Lemma 1. [2]

Let A be the infinitesimal generator of a nonlinear contraction semi-group $\{T(t)\,;\ t\geqq 0\}$ on H. Then A is a densely defined single valued maximal dissipative operator.

Lemma 2. [1, 4]

Let A be a densely defined multi-valued maximal dissipative operator. Then A generates a nonlinear contraction semi-group $\{T(t)\,;\ t\geqq 0\}$ on H whose infinitesimal generator is A^0, the canonical restriction of A.

Put $A_n=n\left(\left(I-\dfrac{1}{n}A\right)^{-1}-I\right)$ for $n=1, 2, \cdots,$.

Then A_n generates a nonlinear contraction semi-group $\{T_n(t)\,;\ t\geqq 0\}$ on H which satisfies the following conditions.

(1) $T_n(t)x\in C_H^1[0,\infty)$ for $t\geqq 0$ and $x\in H$, where $C_H^1[0,\infty)$ denotes the set of all strongly continuously differentiable H-valued functions on $[0,\infty)$.

(2) $\dfrac{d}{dt}T_n(t)x=A_nT_n(t)x$ for $t\geqq 0$ and $x\in H$.

(3) $\lim_{n\to\infty}T_n(t)x=T(t)x$ for $t\geqq 0$ and $x\in H$.

The convergence (3) is uniform in t on any compact interval of $[0,\infty)$.

Lemma 3.

Let $\{T_n(t)\,;\ t\geqq 0\}$ be a sequence of nonlinear contraction semi-groups on H with the infinitetimal generators A_n and let $\{T(t)\,;\ t\geqq 0\}$ be a nonlinear contraction semi-group on H with the infinitesimal generator A.
Suppose that,

(a) $D(A_n)=D(A)=H$ for $n=1, 2, \cdots$, and
$T_n(t)\,x,\ T(t)\,x\in C^1_H[0, \infty)$ for $t\geqq 0$, n and $x\in H$,

(b) there exists a positive constant L such that
$\|A_n x-A_n y\|\leqq L\|x-y\|$ for each $x, y\in H$ and n,

(c) $\lim_{n\to\infty} A_n x=Ax$ for all $x\in H$.

Then

$$\lim_{n\to\infty} T_n(t)\,x=T(t)\,x \text{ for } t\geqq 0 \text{ and } x\in H.$$

The above convergence is uniform in t on any compact interval of $[0, \infty)$.

Proof

By the conditions (b), (c), the convergence $\lim_{n\to\infty} A_n x=Ax$ is uniform with respect to x on any compact subset of H and,

$$\|T_n(t)\,x-T(t)\,x\|^2=\int_0^t \frac{d}{ds}\|T_n(s)\,x-T(s)\,x\|^2 ds$$

$$=2\int_0^t Re<A_nT_n(s)\,x-AT(s),\quad T_n(s)\,x-T(s)\,x>ds$$

$$\leqq 2\int_0^t Re<A_n T(s)\,x-AT(s)\,x,\quad T_n(s)\,x-T(s)\,x>ds,$$

where we have used the dissipativity of A_n.

Since

$$T_n(s)\,x=x+\int_0^s A_nT_n(\tau)\,xd\tau \text{ and}$$

$$T(s)\,x=x+\int_0^s AT(\tau)\,xd\tau,$$

$$\sup_{\substack{0\leqq s\leqq t\\ n\geqq 1}}\|T_n(s)\,x-T(s)\,x\|\leqq t\sup_{n\geqq 1}(\|A_n x\|+\|Ax\|)<\infty.$$

Since

$\{T(s)\,x\,;\ 0\leqq s\leqq t\}$ is compact for any $t>0$,

$\|T_n(t)\,x-T(t)\,x\|^2\leqq t$ const. $\sup_{0\leqq s\leqq t}\|(A_n-A)\,T(s)\,x\|\to 0$

as $n\to\infty$.

The uniform convergence in t on any compact interval of $[0, \infty)$ is obvious from the above estimate.

3. The main result is the following

Theorem

Let $\{T^{(\sigma)}(t)\,;\ t\geqq 0\}$ be a sequence of nonlinear contraction semi-groups on H

and let $A^{(\sigma)}$ be the infinitesimal generator of $\{T^{(\sigma)}(t)\,;\ t\geqq 0\}$ such that the range of $I-\lambda_0 A^{(\sigma)}=H$ for some $\lambda_0>0$ and for all σ.

Put

$$R(n\,;\ A^{(\sigma)})=\left(I-\frac{1}{n}A^{(\sigma)}\right)^{-1}\quad\text{for } \sigma, n=1,2,\cdots,$$

and suppose that,

(a) $\lim_{\sigma\to\infty} R(n\,;\ A^{(\sigma)})\,x=J_n x$ for each $x\in H$ and $n=1,2,\cdots,$

(b) there exists a n_0 such that $R(J_{n_0})$ is dense in H,

(c) $\bigcap_{\sigma=1}^{\infty} D(A^{(\sigma)})\supset R(J_{n_0})$, and $\{A^{(\sigma)}\,;\ \sigma=1,2,\cdots\}$ is pointwise bounded for $R(J_{n_0})$, that is, $\sup_{\sigma\geqq 1}\|A^{(\sigma)}x\|=M_x<\infty$ for each $x\in R(J_{n_0})$.

Then,

(i) $R(J_n)$ is denes in H for $n=1,2.\cdots$, and $R(J_n)=R(J_m)$ for $n, m=1,2,\cdots$.

(ii) there exists a multi-valued maximal dissipative operator A such that

$$J_n=\left(I-\frac{1}{n}A\right)^{-1}\quad\text{for } n=1,2,\cdots.$$

Moreover, A generates a nonlinear contraction semi-group $\{T(t)\,;\ t\geqq 0\}$ on H whose infinitesimal generator is A^0 and

$$\lim_{\sigma\to\infty} T^{(\sigma)}(t)\,x=T(t)\,x \text{ for each } x\in H \text{ and } t\geqq 0.$$

The above convergence is uniform in t on any compact interval of $[0, \infty)$.

Proof

(i) By the dissipativity of $A^{(\sigma)}$ and $R\left(I-\frac{1}{n}A^{(\sigma)}\right)=H$, $R(n\,;\ A^{(\sigma)})=\left(I-\frac{1}{n}A^{(\sigma)}\right)^{-1}$ is well defined everyhere for $n=1,2,\cdots$.

Since

$$\|R(n\,;\ A^{(\sigma)})\,x-R(n\,;\ A^{(\sigma)})\,y\|\leqq\|x-y\| \text{ for each } x, y\in H \text{ and } n, \sigma=1,2,\cdots,$$

by passing to the limit as $\sigma\to\infty$, we obtain

$$\|J_n x-J_n y\|\leqq\|x-y\| \text{ for eoch } x, y\in H \text{ and } n. \tag{3. 1}$$

For each $x\in H$

$$\begin{aligned}&\left(I-\frac{1}{n}A^{(\sigma)}\right)R(m\,;\ A^{(\sigma)})\,x\\ &=R(m\,;\ A^{(\sigma)})\,x+\frac{m}{n}\left(I-\frac{1}{m}A^{(\sigma)}\right)R(m\,;\ A^{(\sigma)})\,x-\frac{m}{n}R(m\,;\ A^{(\sigma)})\,x\\ &=\left(1-\frac{m}{n}\right)R(m\,;\ A^{(\sigma)})\,x+\frac{m}{n}\,x.\end{aligned}$$

Hence

$$(3.2)\quad R(m\,;\,A^{(\sigma)})\,x = R(n\,;\,A^{(\sigma)})\left\{\left(1-\frac{m}{n}\right)R(m\,;\,A^{(\sigma)})\,x+\frac{m}{n}\,x\right\}$$

for each $x\in H$.

This leads the following

$$(3.3)\quad R(R(m\,;\,A^{(\sigma)})) = R(R(n\,;\,A^{(\sigma)})),$$

$$(3.4)\quad n(I-(R(n\,;\,A^{(\sigma)}))^{-1})\,x = m(I-(R(m\,;\,A^{(\sigma)}))^{-1})\,x$$

for $x\in R(R(n\,;\,A^{(\sigma)}))$, where $(R(n\,;\,A^{(\sigma)}))^{-1}$ are multi-valued operators defined by

$$(R(n\,;\,A^{(\sigma)}))^{-1}\,x = \{y\,;\,R(n\,;\,A^{(\sigma)})\,y = x\}\,.$$

Since

$\lim_{\sigma\to\infty} R(n\,;\,A^{(\sigma)})\,x = J_n x$ for $x\in H$ and n, we have

$$(3.5)\quad J_n x = J_m\left\{\left(1-\frac{m}{n}\right)J_n x+\frac{m}{n}\,x\right\}\ \text{for all } x\in H.$$

Consequently

$$(3.6)\quad R(J_n) = R(J_m)\,,$$

$$(3.7)\quad n(I-J_n^{-1})\,x = m(I-J_m^{-1})\,x \text{ for all } x\in D\,,$$

where D_0 is the set $R(J_n)$ independent of n.

(ii) We define A by

$$(3.8)\quad Ax = n(I-J_n^{-1})\,x \text{ for } x\in D_0\,.$$

Then A is a multi-valued maximal dissipative operator. In fact, for each $x, y\in H$

$$Re\langle R(n\,;\,A^{(\sigma)})\,x-R(n\,;\,A^{(\sigma)})\,y\,,\quad (I-R(n,\,A^{(\sigma)}))\,x-(I-R(n\,;\,A^{(\sigma)}))\,y\rangle$$
$$= Re\langle R(n\,;\,A^{(\sigma)})\,x-R(n\,;\,A^{(\sigma)})\,y\,,$$
$$-\frac{1}{n}(A^{(\sigma)}R(n\,;\,A^{(\sigma)})\,x-A^{(\sigma)}R(n\,;\,A^{(\sigma)})y\rangle$$
$$= -\frac{1}{n}Re\langle A^{(\sigma)}R(n\,;\,A^{(\sigma)})\,x-A^{(\sigma)}R(n\,;\,A^{(\sigma)})\,y\,,$$
$$R(n\,;\,A^{(\sigma)})\,x-R(n\,;\,A^{(\sigma)})\,y\rangle\geqq 0\,,$$

where we have used the dissipativity of $A^{(\sigma)}$.

Hence,

$$(3.9)\quad Re\langle R(n\,;\,A^{(\sigma)})\,x-R(n\,;\,A^{(\sigma)})\,y\,,$$
$$x-y\rangle\geqq\|R(n\,;\,A^{(\sigma)})\,x-R(n\,;\,A^{(\sigma)})\,y\|$$

for each $x, y\in H$.

By passing to the limit as a$\to\infty$, we have

$$(3.10)\quad Re\langle J_n x-J_n y,\, x-y\rangle\geqq\|J_n x-J_n y\|^2 \text{ for all } x, y\in H\,.$$

For any $x, x'\in D_0 = D(A)$, and for $y\in Ax$, and $y'\in Ax'$

there exist z and z' such that

$$y = n(n-z) \text{ and } y' = n(x'-z').$$

$$Re<y-y',\ x-x'> = nRe<X-x'-(z-z'),\ x-x'>$$

$$= n(\|x-x'\|^2 - Re<z-z',\ x-x'>)$$

$$\leqq n(\|x-x'\|^2 - \|J_n z - J_n z'\|) = 0 \quad \text{by (3. 10)}$$

Thus A is a dissipative operator.

Since

$$Ax = n(I - J_n^{-1})\, x \text{ for } x \in D_0 \text{ and } n,$$

$$(3.\ 11) \quad J_n x = \left(I - \frac{1}{n} A\right)^{-1} x \text{ for all } x \in H \text{ and } n.$$

This follows that A is maximal dissipative.

(ii)

By Lemma 2, A generates a nonlinear contraction semi-group $\{T(t)\,;\ t \geqq 0\}$ on H whose infinitesimal generator is A^0.

We put $A_n = n(J_n - I)$ for $n = 1, 2, \cdots$.

Then, by Lemma 2 A_n generates a nonlinear contraction semi-group $\{T_n(t)\,;\ t \geqq 0\}$ on H which satisfies the following conditions ;

$$\mathrm{T}_n(t)\, x \in C_H^1[0, \infty) \text{ and } \frac{d}{dt}\ T_n(t)\, x = A_n T_n(t)\, x \text{ for all } x \in H \text{ and } t \geqq 0.$$

Moreover

$$(3.\ 12) \quad \lim_{n \to \infty} T_n(t)\, x = T(t)\, x \text{ for all } x \in H.$$

The convergence (3. 12) is uniform in t on any compact interval.

Put

$$A_n^{(\sigma)} x = A^{(\sigma)} R(n\,;\ A^{(\sigma)})\, x = n(R(n\,;\ A^{(\sigma)}) - I)\, x$$

for $x \in H$.

Then,

$$(3.\ 13) \quad \lim_{\sigma \to \infty} A_n^{(\sigma)} x = A_n x = n(J_n - I)\, x \text{ and}$$

$$(3.\ 14) \quad \|A_n^{(\sigma)} x - A_n^{(\sigma)} y\| \leqq n\|x-y\| \text{ for each } x, y \in H \text{ and } n, \sigma.$$

Let $\{T_n^{(\sigma)}(t)\,;\ t \geqq 0\}$ be the sequence of nonlinear contraction semi-groups on H which are generated by $A_n^{(\sigma)}$.

Since

$$(3.\ 15) \quad \|A_n^{(\sigma)} T_n^{(\sigma)}(t)\, x\| \leqq \|A_n^{(\sigma)} x\| \leqq A^{(\sigma)} x\| \leqq M_x, \text{ and}$$

$$(3.\ 16) \quad \|A_n T_n(t)\, x\| \leqq \|A_n x\| \leqq \|A^0 x\| \text{ for any } x \in D_0,$$

$$\lim_{\sigma \to \infty} T_n^{(\sigma)}(t)\, x = T_n(t)\, x \text{ for any } x \in D_0 \text{ and } t \geqq 0,$$

and the convergence is uniform in t on any compact interval of $[0, \infty)$ by Lemma 3. [4]

Since

D_0 is dense in H and $T_n^{(\sigma)}$ are contraction,

(3. 17) $\lim\limits_{\sigma\to\infty} T_n^{(\sigma)}(t)\,x = T_n(t)\,x$ holds good for all $x\in H$.

In other hand, by Lemma 2

(3. 18) $\lim\limits_{\sigma\to\infty} T_{(n)}^{(\sigma)}(t)\,x = T_n(t)\,x$ for all $x\in H$ and σ.

The convergence (3. 18) is uniform with respect to σ and t on any compact interval of $[0, \infty)$.

In fact, for each $x\in D_0$

$$\|T_n^{(\sigma)}(t)\,x - T_m^{(\sigma)}(t)\,x\|^2 = \int_0^t \frac{d}{ds}\|T_n^{(\sigma)}(s)\,x - T_m^{(\sigma)}(s)\,x\|^2 ds$$

$$= 2\int_0^t Re\langle A_n^{(\sigma)} T_n^{(\sigma)}(s)\,x - A_m^{(\sigma)} T_m^{(\sigma)}(s)\,x\,,\quad T_n^{(\sigma)}(s)\,x - T_m^{(\sigma)}(s)\,x\rangle ds$$

$$= 2\int_0^t Re\langle A_n^{(\sigma)} T_n^{(\sigma)}(s)\,x - A_m^{(\sigma)} T_m^{(\sigma)}(s)\,x,$$

$$R(n\,;\,A^{(\sigma)})\,T_n^{(\sigma)}(s)\,x - R\,(m\,;\,A^{(\sigma)})\,T_m^{(\sigma)}(s)\,x\rangle ds$$

$$+2\int_0^t Re\langle A_n^{(\sigma)} T_n^{(\sigma)}(s)\,x - A_m^{(\sigma)} T_m^{(\sigma)}(s)\,x\,,$$

$$\frac{1}{m} A_m^{(\sigma)} T_m^{(\sigma)}(s)x - \frac{1}{n} A_n^{(\sigma)} T_n^{(\sigma)}(s)\,x\rangle ds$$

$$\leqq 2\int_0^t Re\langle A_n^{(\sigma)} T_n^{(\sigma)}(s)x - A_m^{(\sigma)} T_m^{(\sigma)}(s)x\,,\ \frac{1}{m} A_m^{(\sigma)} T_m^{(\sigma)}(s) - \frac{1}{n} A_n^{(\sigma)} T_n^{(\sigma)}(s)x\rangle ds$$

$$\leqq 4\left(\frac{1}{n} + \frac{1}{m}\right) t\|A^{(\sigma)}x\|^2 \leqq 4\left(\frac{1}{n} + \frac{1}{m}\right) t\, M_x^2$$

For an arbitrary $\varepsilon>0$ and $T>0$ there exists a n_0, such that

$$\left(\frac{1}{n} + \frac{1}{m}\right) T < \frac{\varepsilon^2}{4M_x^2} \quad \text{for } n,\, m \geqq n_0\,.$$

Thus

$$\|T_n^{(\sigma)}(t)\,x - T_m^{(\sigma)}(t)\,x\| \leqq 2M_x\sqrt{\frac{1}{n} + \frac{1}{m}} \quad \text{for } m,\, n \geqq n_0\,,$$

$t\in[0, T]$ and $\sigma = 1, 2, \cdots$.

Since

D_0 is dense in H and $T_n^{(\sigma)}(t)$ are contraction the convergence (3. 18) is uniform with respect to σ and t on any compact interval of $[0, \infty)$ for all $x\in H$.

Now we can prove the convergence

(3. 19) $\lim\limits_{\sigma\to\infty} T^{(\sigma)}(t)\,x = T(t)\,x$ for all $x\in H$.

Let

$T>0$ be arbitratily fixed and let $x\in D$.

For each $\varepsilon>0$, there exists a n_0 such that

$$(3.20)\quad \|T^{(\sigma)}(t)x - T_n^{(\sigma)}(t)x\| < \frac{\varepsilon}{2} \text{ for } t \in [0, T], \quad n \geqq n_0$$

and σ by (3. 18), and

$$(3.21)\quad \|T_n(t)x - T(t)x\| < \frac{\varepsilon}{2} \text{ for } t \in [0, T] \text{ and } n \geqq n_0$$

by (3. 12.)

Thus, by (3. 20) and (3. 21),

$$(3.22)\quad \|T^{(\sigma)}(t)x - T(t)x\| < \varepsilon + \|T_n^{(\sigma)}(t)x - T_n(t)x\|$$
$$\text{for } t \in [0, T], \quad n \geqq n_0 \text{ and } \sigma = 1, 2, \cdots.$$

By passing to the limit as $\sigma \to \infty$, we have

$$\lim_{\sigma\to\infty} T^{(\sigma)}(t)x = T(t)x \text{ for all } x \in D_0 \text{ and } t \in [0, T].$$

Since D_0 is dense in H and $T^{(\sigma)}(t)$ are contraction

$$\lim_{\sigma\to\infty} T^{(\sigma)}(t)x = T(t)x \text{ holds good for any } x \in H \text{ and } t \in [0, T].$$

References

[1] I. Miyadera: On the convergence of nonlinear semi-groups, J. Math. Soc. Japan, 21 (1969).
[2] G. J. Minty: Monotone (nonlinear) operators in Hilbert space, Duke Math. J., 29 (1962).
[3] T. Kato: Nonlinear semi-groups and evolution eqations, J. Math. Soc. Japan, 19 (1967).
[4] Y. Komura: Nonlinear semi-groups in Hilbert space, J. Math. Soc. Japan, 19 (1967).
[5] Y. Komura: Differentiability of nonlinear semi-groups, J. Math. Soc. Japan, 21 (1969).
[6] S. Kato: Some note on the Representation theorem for nonlinear semi-groups in Hilbert space, Mem. Kitami Inst. Tech. Vol. 2, No. 3, 1968.
[7] K. Yoshida: Functional Analysis, Springer, 1965.