Some Note on the Representation Theorem for Nonlinear Semi-Groups in Hibbert Space

by Shigeo KATO

(Received September 26, 1968)

In [1] J. W. Neuberger obtained the following result: Let $\{T_t:\ 0\leqq t<\infty\}$ be a nonlinear contraction semi-group in a Hilbert space H.

If there exists a dense subset D in H such that for each $x\in D$, the strong derivative

$$\frac{d}{dt}T_t x=\lim_{h\to 0}\frac{1}{h}(T_{t+h}x-T_t x)$$

exists and is continuous for $t\geqq 0$, then for each $x\in H$ and each $t\geqq 0$

$$\lim_{n\to\infty}\overline{\lim_{\delta\downarrow 0}}\left(I-\frac{t}{n}\frac{T_\delta-I}{\delta}\right)^{-n}x=T_t x\,.$$

The purpose of this note is to give the representation theorem of this type for semi-groups of nonlinear contraction operators and the basic notions of maximal dissipative operators.

Let H be a Hilbert space with inner product $\langle x, y\rangle$ and let $\{T_t:\ 0\leqq t<\infty\}$ be a family of nonlinear operators from H into itself satisfying the following conditions:

(1) $T_{s+t}=T_s T_t$ for every $s,\, t\geqq 0$, and $T_0=I$,

(2) $\lim_{t\downarrow 0} T_t x=x$ for every $x\in H$,

(3) $\|T_t x-T_t y\|\leqq\|x-y\|$ for every $x, y\in H$.

We call such a family $\{T_t:\ 0\leqq t<\infty\}$ a nonlinear contraction semi-group. The infinitesimal generator A of a nonlinear contraction semi-group $\{T_t\}$ is defined by

$$Ax=\lim_{h\downarrow 0}\frac{T_h x-x}{h},$$

that is, A is the operator whose domain is the set $D(A)=\left\{x\in H:\ \lim_{h\downarrow 0}\frac{T_h x-x}{h}\right.$ exists in $\left.H\right\}$, and, for $x\in D(A)$, $Ax=\lim_{h\downarrow 0}\frac{T_h x-x}{h}$.

By virtue of the condition (3), A satisfies the following

$$Re\langle Ax-Ay,\, x-y\rangle\leqq 0 \quad \text{for every} \quad x, y\in D(A)\,.$$

In fact,

$$Re\left\langle \frac{T_h x - x}{h} - \frac{T_h y - y}{h}, x - y \right\rangle = \frac{1}{h}(Re\langle T_h x - T_h y, x-y\rangle - \|x-y\|)$$
$$\leqq \frac{1}{h}\|x-y\|(\|T_h x - T_h y\| - \|x-y\|) \leqq 0 \quad \text{by } (3).$$

By passing to the limit as $h \downarrow 0$, we obtain the desired inequality. We call an operator A to be dissipative if A satisfies

$$Re\langle Ax - Ay, x-y\rangle \leqq 0 \qquad \text{for every} \quad x, y \in D(A).$$

Hence the infinitesimal generator of a contraction semi-group is a dissipative operator. We say A is maximal dissipative if A is dissipative and if any dissipative extension of A coincides with A.

Concerning to a maximal dissipative operator, we give the following Theorem due to Minty (see [2]).

Theorem 1. Let A be a maximal dissipative operator, then $R(I-A)=H$.

Proof. Let $L_0\left(\frac{x-Ax}{\sqrt{2}}\right) = \frac{x+Ax}{\sqrt{2}}$ for $x \in D(A)$.

Then

$$\left\| L_0\left(\frac{x-Ax}{\sqrt{2}}\right) - L_0\left(\frac{y-Ay}{\sqrt{2}}\right)\right\|^2 = \left\| \frac{x+Ax}{\sqrt{2}} - \frac{y+Ay}{\sqrt{2}}\right\|^2$$
$$= \frac{1}{2}\left(\|x-y\|^2 + \|Ax-Ay\|^2 + 2Re\langle Ax-Ay, x-y\rangle\right)$$
$$\leqq \frac{1}{2}\left(\|x-y\|^2 + \|Ax-Ay\|^2 - 2Re\langle Ax-Ay, x-y\rangle\right)$$
$$= \left\| \frac{x-Ax}{\sqrt{2}} - \frac{y-Ay}{\sqrt{2}}\right\|^2.$$

Thus, L_0 is a Lipschitz continuous operator defined on $\frac{1}{\sqrt{2}}R(I-A)$ with Lischitz constant 1. By Minty, L_0 has an extension L to all of H which has the same Lipschitzian property. We define $\tilde{A}\left(\frac{z+Lz}{\sqrt{2}}\right) = \frac{z-Lz}{\sqrt{2}}$ for $z \in H$.

Then, $-\tilde{A}$ is dissipative and is an extension of A. In fact,

$$Re\left\langle -\tilde{A}\left(\frac{z_1+Lz_1}{\sqrt{2}}\right) + \tilde{A}\left(\frac{z_2+Lz_2}{\sqrt{2}}\right), \quad \frac{z_1+Lz_1}{\sqrt{2}} - \frac{z_2+Lz_2}{\sqrt{2}}\right\rangle$$
$$= -\frac{1}{2}\langle z_1 - Lz_1 - (z_2 - Lz_2), \quad z_1 - z_2 + (Lz_1 - Lz_2)\rangle$$
$$= -\frac{1}{2}(\|z_1 - z_2\|^2 - \|Lz_1 - Lz_2\|^2) \leqq 0.$$

Let $x \in D(A)$, then $z = \frac{x-Ax}{\sqrt{2}} \in \frac{1}{\sqrt{2}}R(I-A)$.

Since $L_0(z) = L(z) = \frac{x+Ax}{\sqrt{2}}$,

$$-\tilde{A}\left(\frac{z+Lz}{\sqrt{2}}\right)=-\tilde{A}\left(\frac{1}{\sqrt{2}}\left(\frac{x-Ax}{\sqrt{2}}+\frac{x+Ax}{\sqrt{2}}\right)\right)=-\tilde{A}x$$

$$=-\frac{z-Lz}{\sqrt{2}}=-\frac{1}{\sqrt{2}}\left(\frac{x-Ax}{\sqrt{2}}-\frac{x+Ax}{\sqrt{2}}\right)=Ax.$$

Since A is maximal, $-\tilde{A}=A$ and

$$D(A)=D(\tilde{A})=\left\{\frac{z+Lz}{\sqrt{2}}:\ z\in H\right\}.$$

Let $z\in H$ be given arbitrary, and put $x=\dfrac{z+Lz}{\sqrt{2}}$, then $x\in D(A)$, and

$$\frac{1}{\sqrt{2}}(I-A)x=\frac{1}{\sqrt{2}}(I+\tilde{A})\left(\frac{z+Lz}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}\left(\frac{z+Lz}{\sqrt{2}}+\frac{z-Lz}{\sqrt{2}}\right)=z.$$

Hence $\dfrac{1}{\sqrt{2}}R(I-A)=H$, and so $R(I-A)=H$.

Lemma 1. Let A be a maximal dissipative operator. Then for every positive λ, $R(I-\lambda A)=H$.

Proof. By Theorem 1 $R(I-A)=H$. By virtue of the dissipativity of A, $(I-A)^{-1}$ exists and is Lipschitz continuous with Lipschitz constant 1.

Let $x\in H$ be given. We seek $y\in D(A)$ such that $x=y-Ay$. Consider the mapping $z\to Pz=(I-A)^{-1}\left(\dfrac{1}{\lambda}x-\dfrac{1-\lambda}{\lambda}z\right)$ of H into H.

Then

$$\|Pz_1-Pz_2\|\leqq\frac{1-\lambda}{\lambda}\|z_1-z_2\|.$$

For $\lambda>\dfrac{1}{2}$, P is a strict contraction, and has a unique fixed point y. Hence we obtain that $R(I-\lambda A)=H$ for $\lambda>\dfrac{1}{2}$.

Let $B=\lambda A$, $\dfrac{1}{2}<\lambda\leqq 1$, then the above mentioned argument implies that $R(I-\lambda B)=R(I-\lambda^2 A)=H$.

Hence we obtain that $R(I-\lambda^n A)=H$ for $\dfrac{1}{2}<\lambda\leqq 1$, and $n=1,2,\cdots$ by repeating this process.

For σ, $0<\sigma\leqq 1$, there exist a suitable λ, $\dfrac{1}{2}\lambda\leqq 1$ and n such that $\sigma=\lambda^n$.

Thus we obtain that $R(I-\lambda A)=H$ for every $\lambda>0$.

By Lemma 1, $J_\lambda=\left(I-\dfrac{1}{\lambda}A\right)^{-1}$ and $A_\lambda=AJ_\lambda=\lambda(J_\lambda-I)$ are well defined everywhere for $\lambda>0$.

Lemma 2. Let A be a maximal dissipative operator and $\lambda>0$.

Then:

(1) $\|J_\lambda x - J_\lambda y\| \leqq \|x-y\| \qquad \|A_\lambda x - A_\lambda y\| \leqq \lambda\|x-y\|$

for $x, y \in H$ and A_λ is a dissipative operator.

(2) $\|A_\lambda x\| \leqq \|Ax\|$ for $x \in D(A)$.

Proof.

(1) Set $J_\lambda x = z_1$ and $J_\lambda y = z_2$. Then

$$\|x-y\|^2 = \|z_1 - z_2 - \frac{1}{\lambda}(Az_1 - Az_2)\|^2$$

$$= \|z_1 - z_2\|^2 + \frac{1}{\lambda^2}\|Az_1 - Az_2\|^2 - \frac{2}{\lambda} Re\langle Az_1 - Az_2,\ z_1 - z_2\rangle$$

$$\geqq \|z_1 - z_2\|^2 + \frac{1}{\lambda^2}\|Az_1 - Az_2\|^2 .$$

Since $Az_1 = AJ_\lambda x = A_\lambda x$ and $Az_2 = AJ_\lambda y = A_\lambda y$, we obtain both $\|J_\lambda x - J_\lambda y\| \leqq \|x-y\|$ and $\|A_\lambda x - A_\lambda y\| \leqq \lambda\|x-y\|$.

(2)

$$\|A_\lambda x\| \leqq \|\lambda(J_\lambda - I)x\| = \lambda\|J_\lambda x - J_\lambda\left(I - \frac{1}{\lambda}A\right)x\|$$

$$\leqq \lambda\|x - \left(I - \frac{1}{\lambda}A\right)x\| = \|Ax\| \quad \text{for } xD(A).$$

Lemma 3. For each $x \in \overline{D(A)}$, the closure of $D(A)$,

$$J_\lambda x \longrightarrow x \quad \text{as} \quad \lambda \longrightarrow \infty .$$

Proof. $\|J_\lambda x - x\| = \frac{1}{\lambda}\|A_\lambda x\| \leqq \frac{1}{\lambda}\|Ax\|$ by Lemma 2 (2) for $x \in D(A)$.

Hence $J_\lambda x \to x$ as $\lambda \to \infty$ for $x \in D(A)$.

Since J_λ is contraction the result can be extended to all $x \in \overline{D(A)}$.

Lemma 4. Let A be a maximal dissipative operator and $\lambda > 0$.

Then:

(1) If $D(A) \ni x_\lambda \to x$. as $\lambda \to \infty$ and $\{Ax_\lambda\}$ is bounded, then $x_0 \in D(A)$ and $\omega - \lim_{\lambda\to\infty} Ax_\lambda = Ax$ (We denote by $\omega - \lim$ weak convergence).

(2) Let $x_\lambda \to x_0$ as $\lambda \to \infty$ and $\{A_\lambda x_\lambda\}$ is bounded, then $x_0 \in D(A)$ and $\omega - \lim_{\lambda\to\infty} A_\lambda x_\lambda = Ax_0$.

(3) If $x \in D(A)$, then $A_\lambda x \to Ax$ as $\lambda \to \infty$.

Proof.

(1)

Since $\{Ax_\lambda\}$ is bounded we may choose a subsequence $\{x_{\lambda'}\}$ of $\{x_\lambda\}$ such that

$$\omega-\lim_{\lambda'\to\infty} Ax_{\lambda'}=y .$$

For every $x\in D(A)$ we have

$$\langle Ax-y_0,\, x-x_0\rangle=\lim_{\lambda'\to\infty}\langle Ax-Ax_{\lambda'},\, x-x_{\lambda'}\rangle\leqq 0$$

Hence $x_0\in D(A)$ by the maximal dissipativity o f A.
We must show that $y_0=Ax_0$.
For every $x\in D(A)$ we have

$$\|x-x_0-(Ax-y_0)\|\geqq\|x-x_0\| .$$

In fact,

$$\begin{aligned}\|x-x_0-(Ax-y_0)\|^2&=\|x-x_0\|^2+\|Ax-y_0\|^2-2\,Re\langle x-x_0,\, Ax-y_0\rangle\\ &\geqq\|x-x_0\|^2 .\end{aligned}$$

We put $x_1=(I-A)^{-1}(x_0-y_0)$. Then $x_1\in D(A)$ and $x_1-Ax_1=x_0-y_0$.
Since $\|x_1-x_0-(Ax_1-y_0)\|\geqq\|x_1-x_0\|$ we have $x_0=x_1$, and so $Ax_1=Ax_0=y_0$.
Since y_0 was the weak limit of an arbitrary weak convergent subsequence of $\{Ax_\lambda\}$, we have $\omega-\lim_{\lambda\to\infty} Ax_\lambda=Ax_0$.

(2)

$$\|J_\lambda x_\lambda-x_\lambda\|=\frac{1}{\lambda}\|A_\lambda x_\lambda\|\to 0 \text{ as } \lambda\to\infty \text{ by the boundedness of } \{A_\lambda x_\lambda\}.$$

Hence $J_\lambda x_\lambda\to x_0$ as $\lambda\to\infty$.
Since $\{AJ_\lambda x_\lambda\}=\{A_\lambda x_\lambda\}$ is bounded, we have

$$x_0\in D(A) \quad\text{and}\quad \omega-\lim_{\lambda\to\infty} A_\lambda x_\lambda=Ax_0 \quad\text{by (1)}.$$

(3)

$$\|A_\lambda x\|\leqq\|Ax\| \quad\text{for}\quad x\in D(A) \quad\text{by Lemma 2 (2) and}$$

$$\omega-\lim_{\lambda\to\infty} A_\lambda x=Ax \quad\text{by (2)}.$$

Since Ax is the weak limit of $\{A_\lambda x\}$,

$$\|Ax\|\leqq\varliminf_{\lambda\to\infty}\|A_\lambda x\|\leqq\varlimsup_{\lambda\to\infty}\|A_\lambda x\|\leqq\|Ax\| .$$

Thus $\|Ax\|=\lim_{\lambda\to\infty}\|A_\lambda x\|$, and so $A_\lambda x\to Ax$ as $\lambda\to\infty$.

Lemma 5. Let A be the infinitesimal generator of a contraction semi-group $\{T_t\}$ and $x_0\in D(A)$.
Then $T_t x_0$ is uniformly Lipschitz continuous in t on $[0, \infty]$, that is, there exists a positive constant M such that

$$\|T_t x_0-T_s x_0\|\leqq M|t-s| \qquad \text{for every, } s,\, t\geqq 0.$$

Proof. Without loss of generality we assume that $s<t$. We devide $[s, t]$ into n equal parts and put

$$t_i^{(n)}=s+\frac{i(t-s)}{n} \qquad i=0, 1, 2, \cdots, n .$$

Since $T_t x_0$ is uniformly continuous in $[s, t]$, for an arbitrary $\varepsilon>0$ there exists a sufficiently large integer n_0 such that

$$\|T_{t_i^{(n_0)}}x_0 - T_{t_{i-1}^{(n_0)}}x_0\| < \varepsilon \quad \text{for} \quad i=1,2,\cdots,n_0, \text{ and}$$

$$\left\|\frac{T_{h_0}x_0 - x_0}{h_0}\right\| \leqq \lim_{h\downarrow 0}\left\|\frac{T_h x_0 - x_0}{h}\right\| + \varepsilon = \|Ax_0\| + \varepsilon,$$

where $h_0 = \dfrac{t-s}{n_0}$.

Since $t_i^{(n_0)} = s + ih_0$, $i=0,1,\cdots,n_0$,

$$\begin{aligned}\|T_t x_0 - T_s x_0\| &\leqq \sum_{i=1}^{n_0}\|T_{t_i^{(n_0)}}x_0 - T_{t_{i-1}^{(n_0)}}x_0\| = \sum_{i=1}^{n_0}\|T_{s+ih_0}x_0 - T_{s+(i-1)h_0}x_0\| \\ &\leqq \sum_{i=1}^{n_0}\|T_{ih_0}x_0 - T_{(i-1)h_0}x_0\| \leqq n_0\|T_{h_0}x_0 - x_0\| \\ &\leqq n_0 h_0(\|Ax_0\| + \varepsilon) = (t-s)(\|Ax_0\| + \varepsilon).\end{aligned}$$

Since ε can be chosen arbitrarily small,

$$\|T_t x_0 - T_s x_0\| \leqq (t-s)\|Ax_0\|.$$

Theorem 2. Let A be the infinitesimal generator of a contraction semi-group $\{T_t\}$. Then A is a maximal dissipative operator.

Proof. By Zorn's Lemma A has a maximal dissipative extension $\tilde{A}$. Then $\tilde{A}$ generates a unique nonlinear contraction semi-group $\{\tilde{T}_t\}$ which satisfies the following conditions:

(a) For every $x\in D(\tilde{A})$, $\tilde{T}_t x \in D(\tilde{A})$ for $t\geqq 0$ and $\tilde{T}_t x$ is uniformly Lipschitz continuous in t.

(b) $\tilde{A}$ is the infinitesimal generator of $\{\tilde{T}_t\}$ (see [3], [4] and [5]).

By (a) for every $x\in D(\tilde{A})$ $\tilde{T}_t x$ has a right derivative $\dfrac{d^+}{dt}\tilde{T}_t x$ for $t\geqq 0$, and $\dfrac{d^+}{dt}\tilde{T}_t x = \tilde{A}\tilde{T}_t x$ by (b).

Moreover the uniform Lipschitz continuity in t of $\tilde{T}_t x$ implies that the strong derivative $\dfrac{d}{dt}\tilde{T}_t x$ exists for $a.e.\, t\geqq 0$, and hence

$$\frac{d^+}{dt}\tilde{T}_t x = \frac{d}{dt}\tilde{T}_t x = \tilde{A}\tilde{T}_t x \quad \text{for} \quad a.e.\, t\geqq 0.$$

By Lemma 5, $\dfrac{d}{dt}T_t x$ exists for $a.e.\, t\geqq 0$ for $x\in D(A)$.

Hence for $x\in D(A)$ $T_t x\in D(A)$ and $\dfrac{d}{dt}T_t x = AT_t x$ for $a.e.\, t\geqq 0$. For every $x\in D(A)$

$$\begin{aligned}\|T_t x - \tilde{T}_t x\|^2 &= \int_0^t \frac{d}{dt}\|T_s x - \tilde{T}_s x\|^2 ds \\ &= 2\int_0^t Re\left\langle \frac{d}{ds}T_s x - \frac{d}{ds}\tilde{T}_s x,\ T_s x - \tilde{T}_s x\right\rangle ds\end{aligned}$$

$$= 2\int_0^t Re\langle \tilde{A}T_s x - \tilde{A}\tilde{T}_s x,\ \ T_s x - \tilde{T}_s x\rangle ds \leqq 0\,,$$

Where we have used the dissipativity of $\tilde{A}$. Thus we have $T_t x = \tilde{T}_t x$ for every $x \in D(A)$.

Since $D(A)$ is dense in H (see [5]),

$$T_t x = \tilde{T}_t x \qquad \text{for every} \quad x \in H\,.$$

Thus $A \supset \tilde{A}$. However, $\tilde{A} \supset A$ and $A = \tilde{A}$ follows.

Theorem 3. Let A be the infinitesimal generator of a contraction semi-group $\{T_t:\ 0 \leqq t < \infty\}$.

If $\{x_\lambda\} \in D(A)$ and $x_\lambda \to x_0 \in D(A)$ as $\lambda \to \infty$ imply

$$\overline{\lim_{\lambda\to\infty}} \|Ax_\lambda\| \leqq \|Ax_0\|\,.$$

Then:

(i) For each $x \in D(A)$ the strong derivative $\dfrac{d}{dt} T_t x$ of $T_t x$ exists for $t \geqq 0$ and is continuous on $[0, \infty]$.

(ii) $T_t x$ is represented by

$$T_t x = \lim_{\lambda\to 0} (I - \lambda A)^{-\left[\frac{t}{\lambda}\right]} x$$

for $t \geqq 0$ and for each $x \in H$, where the convergence is uniform for any compact set in $[0, \infty]$ for each fixed $x \in H$.

Proof.

(i) By Theorem 2, for each $x \in D(A)$ $T_t x$ has a right derivative $\dfrac{d^+}{dt} T_t x$ and $\dfrac{d^+}{dt} T_t x = AT_t x$ for $t \geqq 0$.

For every $t_0 \geqq 0$, let $t \to t_0$, then

$$\overline{\lim_{t\to t_0}} \|AT_t x\| \leqq \|AT_{t_0} x\|$$

for each $x \in D(A)$ by the assumption. Hence $AT_t x \to AT_{t_0} x$ as $t \to t_{t_0}$ by Lemma 4 (1).

Thus, since the right derivative $\dfrac{d^+}{dt} T_t x$ of $T_t x$ exists and continuous in t everywhere, the strong derivative $\dfrac{d}{dt} T_t x$ also exists and is continuous in t everywhere.

(ii) We put $\left[\dfrac{t}{\lambda}\right] = n_{\lambda,t}$.

Then we have the following estimate for each $x \in D(A)$:

$$\|(I - \lambda A)^{-n_{\lambda,t}} x - T_t x\| = \left\| \left[(I - \lambda A)^{-1}\right]^{n_{\lambda,t}} x - \left[T_{\frac{t}{n_{\lambda,t}}}\right]^{n_{\lambda,t}} x \right\|$$

$$\leqq \sum_{K=1}^{n_{\lambda,t}} \left\| \left[(I - \lambda A)^{-1}\right]^{n_{\lambda,t} - K - 1} T_{\frac{(K-1)t}{n_{\lambda,t}}} x - \left[(I - \lambda A)^{-1}\right]^{n_{\lambda,t} - K} T_{\frac{Kt}{n_{\lambda,t}}} x \right\|$$

$$\leqq \sum_{K=1}^{n_{\lambda,t}} \| (I-\lambda A)^{-1} T_{\frac{(K-1)t}{n_{\lambda,t}}} x - T_{\frac{Kt}{n_{\lambda,t}}} x \|$$

$$\leqq \sum_{K=1}^{n_{\lambda,t}} \| T_{\frac{(K-1)t}{n_{\lambda,t}}} x - (I-\lambda A) T_{\frac{Kt}{n_{\lambda,t}}} x \|$$

$$\leqq \sum_{K=1}^{n_{\lambda,t}} \| T_{\frac{Kt}{n_{\lambda,t}}} x - T_{\frac{(K-1)t}{n_{\lambda,t}}} x - \lambda \frac{d}{dt} T_{\frac{Kt}{n_{\lambda,t}}} x \|$$

$$\leqq \sum_{K=1}^{n_{\lambda,t}} \left\| \int_{\frac{(K-1)t}{n_{\lambda,t}}}^{\frac{Kt}{n_{\lambda,t}}} \left(\frac{d}{ds} T_s x - \frac{\lambda n_{\lambda,t}}{t} \frac{d}{dt} T_{\frac{Kt}{n_{\lambda,t}}} x \right) ds \right\|$$

$$\leqq \sum_{K=1}^{n_{\lambda,t}} \frac{t}{n_{\lambda,t}} \sup \left\{ \left\| \frac{d}{ds} T_s x - \frac{\lambda n_{\lambda,t}}{t} \frac{d}{dt} T_{\frac{Kt}{n_{\lambda,t}}} x \right\| : \frac{(K-1)t}{n_{\lambda,t}} \leqq s \leqq \frac{Kt}{n_{\lambda,t}} \right\}$$

$$\leqq t \max_{1 \leqq K \leqq n_{\lambda,t}} \sup \left\{ \left\| \frac{d}{ds} T_s x - \frac{\lambda n_{\lambda,t}}{t} \frac{d}{dt} T_{\frac{Kt}{n_{\lambda,t}}} x \right\| : \frac{(K-1)t}{n_{\lambda,t}} \leqq s \leqq \frac{Kt}{n_{\lambda,t}} \right\}$$

$$\leqq t \Bigg[\max_{1 \leqq K \leqq n_{\lambda,t}} \sup \left\{ \left\| \frac{d}{ds} T_s x - \frac{d}{dt} T_{\frac{Kt}{n_{\lambda,t}}} x \right\| : \frac{(K-1)t}{n_{\lambda,t}} \leqq s \leqq \frac{Kt}{n_{\lambda,t}} \right\} + \max_{1 \leqq K \leqq n_{\lambda,t}} \left(1 - \frac{\lambda n_{\lambda,t}}{t} \right) \left\| \frac{d}{dt} T_{\frac{Kt}{n_{\lambda,t}}} x \right\| \Bigg].$$

Since $\left\| \frac{d}{dt} T_{\frac{Kt}{n_{\lambda,t}}} x \right\|$ is bounded as $\lambda \to 0$, and $1 - \frac{\lambda n_{\lambda,t}}{t} = \frac{t - \lambda n_{\lambda,t}}{t} < \frac{\lambda}{t}$

$$\max_{1 \leqq K \leqq n_{\lambda,t}} \left(1 - \frac{\lambda n_{\lambda,t}}{t} \right) \left\| \frac{d}{dt} T_{\frac{Kt}{n_{\lambda,t}}} x \right\| \to 0 \quad \text{as } \lambda \to 0 .$$

Since $\frac{d}{ds} T_s x$ is uniformly continuous in $[0, t]$

$$\max_{1 \leqq K \leqq n_{\lambda,t}} \sup \left\{ \left\| \frac{d}{ds} T_s x - \frac{d}{dt} T_{\frac{Kt}{n_{\lambda,t}}} x \right\| : \frac{(K-1)t}{n_{\lambda,t}} \leqq s \leqq \frac{Kt}{n_{\lambda,t}} \right\} \to 0 \quad \text{as } \lambda \to 0 .$$

Since $D(A)$ is dense in H and $(I-\lambda A)^{-\left[\frac{t}{\lambda}\right]}$ is contraction we have

$$\lim_{\lambda \to 0} (I-\lambda A)^{-\left[\frac{t}{\lambda}\right]} x = T_t x \qquad \text{for each} \quad x \in H .$$

The uniform convergence in t on any compact set of $[0, \infty]$ is obvious from the above estimate.

References

[1] J. W. Neuberger: An exponential fomular for one-parameter semi-groups of nonlinear trans formations, J. Math. Soc. Japan, 18 (1966).

[2] G. J. Minty: Monotone (nonlinear) operators in Hilbert space, Duke Math. J., 29 (1962).

[3] T. Kato: Nonlinear semi-groups and evolution equations, J. Math. Soc. Japan, 19 (1967).

[4] Y. Komura: Nonlinear semi-groups in Hilbert space, J. Math. Soc. Japan, 19 (1967).

[5] Y. Komura: Differentiability of nonlinear semi-groups, to appear.

[6] S. Ôharu: Note on the representation of semi-groups of nonlinear operators, Proc. Japan Acad., 42 (1966).