Umbilical points on subspaces of Finslerian and Minkowskian spaces.

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In the theory of the subspaces of an Finslerian space, we can define many kinds of umbilicalities. We use the covariant derivatives on a subspace which differ from the induced or intrinsic ones. § 1 is devoted to the definitions of the covariant derivatives and the deductions of the **Gauss-Codazzi-Ricci's** equations. In § 2, we obtain some properties about the umbilicalities. Especially **THEOREM 4** is somewhat interesting. This is related to the isotropic curvature which is the proper notion in Finslerian geometry but has the formal resemblance to the fact that the totally umbilical subspace of a Riemannian space of constant curvature is also of constant curvature. Umbilicalities on a subspace of a Minkowskian space are studied in § 3.

§ 1. We consider an n-dimensional Finslerian space F_n with line element $(x^h, \dot{x}^h)$ and fundamental metric function $F(x, \dot{x})$ where the small Latin indices run over the range $1, 2, \cdots, n$. The metric tensor on F_n is given by the components $g_{ji}(x, \dot{x}) = \dfrac{1}{2}\dfrac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^j \partial \dot{x}^i}$. About the notations, terminologies and some properties of F_n, we follow the H. Rund's book [5]. An m-dimensional subspace F_m of F_n can be represented locally by $x^h = x^h(u^\alpha)$ where the Greek indices run over the range $1, 2, \cdots, m$ and denote the quantities tangential to F_m. The components of the Finslerian metric tensor on F_m are denoted by $g_{\beta\alpha}(u, \dot{u}) = g_{ji}(x, \dot{x}) B^j_\beta B^i_\alpha$ with $x^h = x^h(u)$, $B^h_\alpha = \dfrac{\partial x^h}{\partial u^\alpha}$ and $\dot{x}^h = B^h_\alpha \dot{u}^\alpha$. Consider a system of homogeneous equations $C_h B^h_\alpha(u) = 0$ and we may find a system of its fundamental solutions $C^R_i(u)$ where the capital Latin indices run over the range $m+1, m+2, \cdots, n$ and denote the quantities normal to F_m. Now we employ the following notations:

$$g^{SR} = g^{ji} C^S_j C^R_i, \quad \|g_{SR}\| = \|g^{SR}\|^{-1}, \quad \|g^{\beta\alpha}\| = \|g_{\beta\alpha}\|^{-1},$$
$$B^\alpha_i = g^{\beta\alpha} g_{ji} B^j_\beta, \quad C^h_S = g_{SR} g^{ih} C^R_i. \tag{1}$$

Two matrices $\left\| \|B^h_\alpha\|, \|C^h_S\| \right\|$ and $\left\| {}^t\|B^\alpha_i\|, {}^t\|C^R_i\| \right\|$ constitute contravariant and covariant conjugate frames of F_n at every line element on F_m.

From E. Cartan's symmetric connexion coefficients $\Gamma^{*h}_{ji}(x, \dot{x})$ on F_n ([1] p. 16, [5] p. 70), we construct two connexions along F_m with coefficients

$$\Gamma^{*\alpha}_{\gamma\beta}(u, \dot{u}) = B^\alpha_h(u, \dot{u}) \left(\frac{\partial B^h_\beta(u)}{\partial u^\gamma} + \Gamma^{*h}_{ji}(x, \dot{x}) B^j_\gamma(u) B^j_\beta(u) \right) \tag{2}$$

and

$$\Gamma^{*R}_{S\beta}(u, \dot{u}) = C^R_h(u) \left(\frac{\partial C^h_S(u, \dot{u})}{\partial u^\beta} + \Gamma^{*h}_{ji}(x, \dot{x}) C^j_S(u, \dot{u}) B^j_\beta(u) \right). \tag{3}$$

In case of $\dot{x}^h = \xi^h(x)$, the coefficients $\Gamma^{*\alpha}_{\gamma\beta}$ were studied by some geometers ([3], [4], [5]). But H. Rund treated these for line elements on a hypersurface F_{n-1} ([6]). Following his methods, we will extend some of his discussions to the subspaces with general dimensions. To do this, we define the generalized covariant derivatives along F_m by

$$
\begin{aligned}
T^{\tilde{h}\tilde{\alpha}\tilde{R}}_{\tilde{i}\tilde{\beta}\tilde{S}\,||\,\gamma} = {} & \frac{\partial T^{\tilde{h}\tilde{\alpha}\tilde{R}}_{\tilde{i}\tilde{\beta}\tilde{S}}}{\partial u^\gamma} - \frac{\partial T^{\tilde{h}\tilde{\alpha}\tilde{R}}_{\tilde{i}\tilde{\beta}\tilde{S}}}{\partial \dot{x}^\delta}\Gamma^{*\delta}_{\varepsilon\gamma}\dot{u}^\varepsilon \\
& + \sum_{x=1}^{r} \Gamma^{*h_x}_{kj} B^j_\gamma T^{h_1\cdots h_{x-1}k\,h_{x+1}\cdots h_r\,\tilde{\alpha}\tilde{R}}_{\tilde{i}\,\tilde{\beta}\tilde{S}} \\
& - \sum_{x=1}^{s} \Gamma^{*k}_{i_x j} B^j_\gamma T^{\tilde{h}\qquad\qquad\qquad\tilde{\alpha}\tilde{R}}_{i_1\cdots i_{x-1}k\,i_{x+1}\cdots i_s\,\tilde{\beta}\tilde{S}} \\
& + \sum_{x=1}^{t} \Gamma^{*\alpha_x}_{\delta\gamma} T^{\tilde{h}\alpha_1\cdots\alpha_{x-1}\delta\,\alpha_{x+1}\cdots\alpha_t\,\tilde{R}}_{\tilde{i}\,\tilde{\beta}\,\tilde{S}} \\
& - \sum_{x=1}^{u} \Gamma^{*\delta}_{\beta_x\gamma} T^{\tilde{h}\qquad\tilde{\alpha}\qquad\tilde{R}}_{\tilde{i}\beta_1\cdots\beta_{x-1}\delta\,\beta_{x+1}\cdots\beta_u\,\tilde{S}} \\
& + \sum_{x=1}^{v} \Gamma^{*R_x}_{T\gamma} T^{\tilde{h}\tilde{\alpha}R_1\cdots R_{x-1}T\,R_{x+1}\cdots R^{a}}_{\tilde{i}\tilde{\beta}\qquad\tilde{S}} \\
& - \sum_{x=1}^{w} \Gamma^{*T}_{S_x\gamma} T^{\tilde{h}\tilde{\alpha}\qquad\tilde{R}}_{\tilde{i}\tilde{\beta}S_1\cdots S_{x-1}T\,S_{x+1}\cdots S_w}
\end{aligned}
\tag{4}
$$

for the components $T^{\tilde{h}\tilde{\alpha}\tilde{R}}_{\tilde{i}\tilde{\beta}\tilde{S}} = T^{h_1\cdots h_r\,\alpha_1\cdots\alpha_t\,R_1\cdots R_v}_{i_1\cdots i_s\,\beta_1\cdots\beta_u\,S_1\cdots S_w}(u, \dot{u})$ of a tensor over F_m where each of the waved indices stands for a coraesponding set of indices. Our definition may be easily justified with transformation laws of Γ^{*h}_{ji}, $\Gamma^{*\alpha}_{\gamma\beta}$ and $\Gamma^{*R}_{S\beta}$ but slightly differs from H. Rund's one ([6]). It is interesing that these covariant derivatives do not in general consist with intrinsic or induced ones. And it is one reason why we will investigate the subspace F_m with connexion coefficients $\Gamma^{*\alpha}_{\gamma\beta}$. Corresponding to Γ^{*h}_{ji}, $\Gamma^{*\alpha}_{\gamma\beta}$ and $\Gamma^{*R}_{S\beta}$ we define the three curvature tensors by the following systems of equations:

$$
\begin{aligned}
K^h_{k\cdot ji} &= 2\left(\frac{\partial\Gamma^{*h}_{k[j}}{\partial x^{i]}} - \frac{\partial\Gamma^{*h}_{k[j}}{\partial\dot{x}^{|m|}}\Gamma^{*m}_{i]l}\dot{x}^l + \Gamma^{*l}_{k[j}\Gamma^{*h}_{i]l}\right), \\
K^\alpha_{\delta\cdot\gamma\beta} &= 2\left(\frac{\partial\Gamma^{*\alpha}_{\delta[\gamma}}{\partial u^{\beta]}} - \frac{\partial\Gamma^{*\alpha}_{\delta[\gamma}}{\partial\dot{u}^{|\zeta|}}\Gamma^{*\zeta}_{\beta]\varepsilon}\dot{u}^\varepsilon + \Gamma^{*\varepsilon}_{\delta[\gamma}\Gamma^{*\alpha}_{\beta]\varepsilon}\right), \\
K^R_{S\cdot\beta\alpha} &= 2\left(\frac{\partial\Gamma^{*R}_{S[\beta}}{\partial u^{\alpha]}} - \frac{\partial\Gamma^{*R}_{S[\beta}}{\partial\dot{u}^{|\delta|}}\Gamma^{*\delta}_{\alpha]\gamma}\dot{u}^\gamma + \Gamma^{*T}_{S[\beta}\Gamma^{*R}_{|T|\alpha]}\right), \\
K_{kjih} &= g_{lj}K^l_{k\cdot ih},\quad K_{\delta\gamma\beta\alpha} = g_{\varepsilon\gamma}K^\varepsilon_{\delta\cdot\beta\alpha}.
\end{aligned}
\tag{5}
$$

Applying the formula (4) to B^h_α and C^h_S, we obtain the generalized Euler-Schouten's tensors with components

$$
H^h_{\beta\alpha} = B^h_{\alpha||\beta} = \frac{\partial B^h_\alpha}{\partial u^\beta} + \Gamma^{*h}_{ji}B^j_\beta B^i_\alpha - \Gamma^{*\gamma}_{\beta\alpha}B^h_\gamma = H^h_{\alpha\beta} \tag{6}
$$

and

$$
L^h_{\beta S} = C^h_{S||\beta} = \frac{\partial C^h_S}{\partial u^\beta} - \frac{\partial C^h_S}{\partial\dot{u}^\gamma}\Gamma^{*\gamma}_{\beta\alpha}\dot{u}^\alpha + \Gamma^{*h}_{ji}C^j_S B^i_\beta - \Gamma^{*R}_{S\beta}C^h_R. \tag{7}
$$

Then we see that $H^h_{\beta\alpha}$ and $L^h_{\beta S}$ are respectively normal and tangential to F_m with respect to the index h, that is,

$$H^h_{\beta\alpha} = C^h_R H^R_{\beta\alpha} \text{ and } L^h_{\beta S} = B^h_\alpha L^\alpha_{\beta S}. \tag{8}$$

It can be also shown that neither $g_{ji||\beta}$ nor $g_{\gamma\beta||\alpha}$ vanish in general. Indeed we may deduce the equations

$$g_{ji||\beta} = 2C_{jih} H^h_{\beta\alpha} \dot{u}^\alpha \tag{9}$$

and

$$g_{\gamma\beta||\alpha} = g_{ji||\alpha} B^j_\gamma B^i_\beta = 2C_{jih} B^j_\gamma B^i_\beta H^h_{\alpha\varepsilon} \dot{u}^\varepsilon \tag{10}$$

with $C_{jih} = \dfrac{1}{2}\dfrac{\partial g_{ji}}{\partial \dot{x}^h}$. These are the indirect illustrations of the fact that our covariant derivatives differ from induced or intrinsic ones. Differentiating the equations $g_{ji} B^j_\beta C^i_S = 0$ covariantly, we obtain a relation between the Euler-Schouten's tensors:

$$L^\alpha_{\beta S} = -g^{\gamma\alpha}\left(g_{TS}\delta^\delta_\gamma + \frac{\partial g_{TS}}{\partial \dot{u}^\gamma}\dot{u}^\delta\right) H^T_{\delta\beta}. \tag{11}$$

Now we intend to see some relations between the various curvature tensors. Calculations of the commutation formulae of our covariant derivatives to B^h_α and C^h_S give

$$2H^h_{\gamma[\beta||\alpha]} = B^k_\gamma\left(K^h_{k\cdot ji} B^j_\beta B^i_\alpha + 2\frac{\partial \Gamma^{*h}_{kj}}{\partial \dot{x}^i} B^j_{[\beta} H^i_{\alpha]\delta} \dot{u}^\delta\right) - B^h_\delta K^\delta_{\gamma\cdot\beta\alpha} \tag{12}$$

and

$$L^h_{\beta S||\alpha} - L^h_{\alpha S||\beta} = C^k_S\left(K^h_{k\cdot ji} B^j_\beta B^i_\alpha + 2\frac{\partial \Gamma^{*h}_{kj}}{\partial \dot{x}^i} B^j_{[\beta} H^i_{\alpha]\gamma} \dot{u}^\gamma\right)$$
$$-C^h_R K^R_{S\cdot\beta\alpha} - \frac{\partial C^h_S}{\partial \dot{u}^\varepsilon} K^\varepsilon_{\gamma\cdot\beta\alpha} \dot{u}^\gamma. \tag{13}$$

On the other hand, the followings are also valid by means of (8):

$$H^h_{\gamma\beta||\alpha} = B^h_\delta H^R_{\gamma\beta} L^\delta_{\alpha R} + C^h_R H^R_{\gamma\beta||\alpha}, \tag{14}$$
$$L^h_{\beta S||\alpha} = B^h_\gamma L^\gamma_{\beta S||\alpha} + C^h_R L^\gamma_{\beta S} H^R_{\alpha\gamma}. \tag{15}$$

Using the relation (14) and decomposing the equations (12) with respect to the index h, we can deduce the Gauss-Codazzi's equations

$$K^\alpha_{\delta\cdot\gamma\beta} + 2H^R_{\delta[\gamma} L^\alpha_{\beta]R} = B^k_\delta B^j_\gamma B^i_\beta K^h_{k\cdot ji} B^\alpha_h + 2B^k_\delta B^j_{[\gamma} H^R_{\beta]\varepsilon} \dot{u}^\varepsilon C^i_R \frac{\partial \Gamma^{*h}_{kj}}{\partial \dot{x}^i} B^\alpha_h \tag{16}$$

and

$$2H^R_{\gamma[\beta||\alpha]} = B^k_\gamma B^j_\beta B^i_\alpha K^h_{k\cdot ji} C^R_h + 2B^k_\gamma B^j_{[\beta} H^S_{\alpha]\delta} \dot{u}^\delta C^i_S \frac{\partial \Gamma^{*h}_{kj}}{\partial \dot{x}^i} C^R_h. \tag{17}$$

Analogously the equations (13) and (15) yeild the Codazzi-Ricci's equations

$$L^\alpha_{\gamma S||\beta} - L^\alpha_{\beta S||\gamma} = C^k_S B^j_\gamma B^i_\beta K^h_{k\cdot ji} B^\alpha_h + 2C^k_S B^j_{[\gamma} H^R_{\alpha]\delta} \dot{u}^\delta C^i_R \frac{\partial \Gamma^{*h}_{kj}}{\partial \dot{x}^i} B^\alpha_h$$
$$+2\dot{u}^\delta K^\varepsilon_{\delta\cdot\gamma\beta} B^j_\varepsilon C^i_S C^h_{ji} B^\alpha_h \tag{18}$$

and

$$K^R_{S\cdot\beta\alpha}-2H^R_{\gamma[\beta}L^\gamma_{\alpha]S}=C^k_S B^j_\beta B^i_\alpha K^h_{k\cdot ji}C^R_h+2C^k_S B^j_{[\beta}H^T_{\alpha]\gamma}\dot{u}^\gamma C^i_T\frac{\partial\Gamma^{*h}_{kj}}{\partial\dot{x}^i}C^R_h. \tag{19}$$

To conclude the equations (18), we utilzed the identities

$$\frac{\partial C^h_S}{\partial\dot{u}^\beta}B^\alpha_h=-2B^j_\beta C^i_S C^h_{ji}B^\alpha_h \tag{20}$$

which follow from $C^h_S B^\alpha_h=0$. Especially the equations (16) are denoted with (11) as follows:

$$K_{\delta\gamma\beta\alpha}=B^k_\delta B^j_\gamma B^i_\beta B^h_\alpha K_{kjih}+2g_{ji}H^j_{\delta[\beta}H^i_{\alpha]\gamma}+4H^k_{\delta[\beta}H^j_{\alpha]\varepsilon}\dot{u}^\varepsilon B^i_\gamma C_{kji}$$
$$+2B^k_\delta B^i_{[\beta}H^i_{\alpha]\varepsilon}\dot{u}^\varepsilon\frac{\partial\Gamma^{*h}_{kj}}{\partial\dot{x}^i}g_{lh}B^l_\gamma. \tag{21}$$

§ 2. A vector with components

$$N^h(u,\dot{u})=\frac{H^h_{\beta\alpha}(u,\dot{u})\dot{u}^\beta\dot{u}^\alpha}{F^2(x,\dot{x})} \tag{22}$$

is normal to F_m, We call it the normal curvature vector at (u^α) for a direction $(\dot{u}^\alpha)$ and its "length"

$$N(u,\dot{u})=\Big(g_{ji}(x,\dot{x})N^j(u,\dot{u})N^i(u,\dot{u})\Big)^{\frac{1}{2}} \tag{23}$$

the normal curvature at (u^α) for a direction $(\dot{u}^\alpha)$. The normal curvature $N(u,\dot{u})$ is the first curvature in F_n of the geodesic of F_m through the point (u^α) in the direction $(\dot{u}^\alpha)$ which is defined by [5] p. 152 (1.8). If the normal curvature $N(u,\dot{u})$ is constant for all directions $(\dot{u}^\alpha)$ at a point (u^α), we call the point (u^α) the umbilical point of first kind. By the equations

$$\frac{\partial H^R_{\beta\alpha}}{\partial\dot{u}^\gamma}=C^R_h\frac{\partial\Gamma^{*h}_{ji}}{\partial\dot{x}^k}B^k_\gamma B^j_\beta B^i_\alpha \tag{24}$$

and the well known identities $\dfrac{\partial\Gamma^{*h}_{ji}}{\partial\dot{x}^k}\dot{x}^j\dot{x}^i=0$ ([1] p. 35, [5] p. 81), we can prove the following property.

THEOREM 1. In order that a point (u^α) on F_m be an umbilical point of first kind, it is necessary and sufficient that the equations

$$N^2(u,\dot{u})\,g_{\beta\alpha}(u,\dot{u})\,\dot{u}^\alpha=g_{ji}(x,\dot{x})\,N^j(u,\dot{u})\,H^i_{\beta\alpha}(u,\dot{u})\,\dot{u}^\alpha$$
$$+\frac{1}{2}F^2(x,\dot{x})\,C_{kji}(x,\dot{x})\,N^k(u,\dot{u})\,N^j(u,\dot{u})B^i_\beta \tag{25}$$

are valid for all directions $(\dot{u}^\alpha)$ at (u^α).

The equations (25) are so complicated that we can seldom expect further investigations about an umbilical point of first kind. To conquer this difficulty, we will adopt another definition of umbilicality. Let us define the mean curvature normal and the mean curvature at a line element $(u^\alpha,\dot{u}^\alpha)$ on F_m by a vector with components

$$M^h(u,\dot{u})=\frac{1}{m}H^h_{\beta\alpha}(u,\dot{u})g^{\beta\alpha}(u,\dot{u}) \tag{26}$$

and a scalar

$$M(u, \dot{u}) = \left(g_{ji}(x, \dot{x})\, M^j(u, \dot{u})\, M^i(u, \dot{u})\right)^{\frac{1}{2}}. \tag{27}$$

Our mean curvatures depend not only the points but the directional arguments. If the equations

$$H^h_{\beta\alpha}(u, \dot{u}) = g_{\beta\alpha}(u, \dot{u})\, M^h(u, \dot{u}) \tag{28}$$

consist for all directions $(\dot{u}^\alpha)$ at a point (u^α), then we call the point (u^α) the umbilical point of second kind. This definition is formally a natural extension of that in Riemannian geometry. The following properties tell us some relations between the two umbilicalities defined above.

THEOREM 2. Let (u^α) be an umbilical point of second kind. Then, for every direction at that point, the mean curvature normal coincides with the normal curvature vector and consequently the mean curvature is the same as the normal curvature. The point (u^α) is also the umbilical point of first kind if and only if the equations

$$C_{kji}(x, \dot{x})\, N^k(u, \dot{u})\, N^j(u, \dot{u})\, B^i_\alpha = 0 \tag{29}$$

consist for all directions $(\dot{u}^\alpha)$ at (u^α).

The proof of **THEOREM 2** follows from (25) and (28). It is also obtainable from (24), (28) and the identities $\dfrac{\partial \Gamma^{*h}_{ji}}{\partial \dot{x}^k}\dot{x}^i = C^h_{jk|i}\dot{x}^i$ ([1] p. 35, [5] p. 81) that the equations

$$C^R_i \frac{\partial M^i(u, \dot{u})}{\partial \dot{u}^\alpha} = 0,$$

$$C^k_S(u, \dot{u})\, B^j_\beta B^i_\alpha C_{kji|h}(x, \dot{x})\, \dot{x}^h = 0 \tag{30}$$

are valid for all directions $(\dot{u}^\alpha)$ at an umbilical point (u^α) of second kind where the single bar denotes E. Cartan's covariant derivative ([1] p. 18, [5] p. 74). A totally umbilical subspace F_m of second kind is defined by the property that all points on F_m are umbilical points of second kind. Noticing the fact that a totally geodesic subspace F_m be characterized by $H^h_{\beta\alpha}(u, \dot{u})\, \dot{u}^\beta \dot{u}^\alpha = 0$, we can prove the following theorem.

THEOREM 3. Let F_m be a totally umbilical subspace of second kind of F_n. Then F_m is the totally geodesic subspace of F_n if and only if the mean curvature (i. e. normal curvature) of F_m vanishes identically.

The scalar

$$R(x, \dot{x}, X) = \frac{K_{kjih}(x, \dot{x})\, \dot{x}^k X^j \dot{x}^i X^h}{(g_{ki}(x, \dot{x})\, g_{jh}(x, \dot{x}) - g_{kj}(x, \dot{x})\, g_{ih}(x, \dot{x}))\, \dot{x}^k X^j \dot{x}^i X^h} \tag{31}$$

is called the Riemannian curvature with respect to a pair of two directions $(\dot{x}^h; X^h)$ at a point (x^h) on F_n ([5] p. 130.) If the Riemannian curvature $R(x, \dot{x}, X)$ is always independent of the choice of (X^h) for every line element $(x^h, \dot{x}^h)$, F_n is said to be of isotropic curvature $R(x, \dot{x}) = R(x, \dot{x}, X)$. The necessary and sufficient condition for an F_n to be of isotropic curvature $R(x, \dot{x})$ is that F_n admits a system of equations

$$K_{kjih}(x, \dot{x})\dot{x}^k\dot{x}^i = R(x, \dot{x})\{F^2(x, \dot{x})\, g_{jh}(x, \dot{x}) - (g_{kj}(x, \dot{x})\dot{x}^k)(g_{ih}(x, \dot{x})\dot{x}^i)\} \tag{32}$$

for every line element $(x^h, \dot{x}^h)$ ([5] p. 131). Analogous definitions may be applied for a subspace F_m with connexion coefficients $\Gamma^{*\alpha}_{\gamma\beta}$. We shall define the Riemannian curvature with respect to a pair of two directions $(\dot{u}^\alpha\,;\,X^\alpha)$ at a point on a subspace F_m by

$$\bar{R}(u, \dot{u}, X) = \frac{K_{\delta\gamma\beta\alpha}(u, \dot{u})\,\dot{u}^\delta X^\gamma \dot{u}^\beta X^\alpha}{(g_{\delta\beta}(u, \dot{u})\, g_{\gamma\alpha}(u, \dot{u}) - g_{\delta\gamma}(u, \dot{u})\, g_{\beta\alpha}(u, \dot{u}))\dot{u}^\delta X^\gamma \dot{u}^\beta X^\alpha}. \tag{33}$$

But our Riemannian curvature $\bar{R}(u, \dot{u}, X)$ or F_m is not the intrinsic one. If the Riemannian curvature $\bar{R}(u, \dot{u}, X)$ of F_m is determined only by the line element on F_m, we shall call it the isotropic curvature $\bar{R}(u, \dot{u})$ of F_m. Now the equations (21) yeild

$$K_{\delta\gamma\beta\alpha}\dot{u}^\delta\dot{u}^\beta = K_{kjih}\dot{x}^k B^j_\gamma \dot{x}^i B^h_\alpha + g_{ji}\{F^2 N^j H^i_{\gamma\alpha} - (H^j_{\delta\gamma}\dot{u}^\delta)(H^i_{\beta\alpha}\dot{u}^\beta)\} - F^2 N^k B^i_\gamma B^i_\alpha C_{kji|h}\dot{x}^k. \tag{34}$$

Noticing the symmetry of $K_{kjih}\dot{x}^k\dot{x}^i$ with respect to the indices j and h ([5] p. 109), we can see that $K_{\delta\gamma\beta\alpha}\dot{u}^\delta\dot{u}^\beta$ is also symmetric with respect to the indices γ and α. With this property, we know that F_m is of isotropic curvature $\bar{R}(u, \dot{u})$ if and only if F_m admits the equations

$$K_{\delta\gamma\beta\alpha}(u, \dot{u})\,\dot{u}^\delta\dot{u}^\beta = \bar{R}(u, \dot{u})\{F^2(x, \dot{x})\, g_{\gamma\alpha}(u, \dot{u}) - (g_{\delta\gamma}(u, \dot{u})\dot{u}^\delta)(g_{\beta\alpha}(u, \dot{u})\dot{u}^\beta)\}. \tag{35}$$

Its proof is formally the same as H. Rund's proof of (32) ([5] p. 131).

Let F_m be a totally umbilical subspace of second kind. Then the equations (34) are reduced to

$$K_{\delta\gamma\beta\alpha}\dot{u}^\delta\dot{u}^\beta = K_{kjih}\dot{x}^k B^j_\gamma \dot{x}^i B^h_\alpha + M^2\{F^2 g_{\gamma\alpha} - (g_{\delta\gamma}\dot{u}^\delta)(g_{\beta\alpha}\dot{u}^\beta)\} \tag{36}$$

because of the last relation of (30) and **THEOREM 2.** Furthermore, assuming that F_n be of isotropic curvature $R(x, \dot{x})$, substitution of (32) into (36) gives

$$K_{\delta\gamma\beta\alpha}\dot{u}^\delta\dot{u}^\beta = (R(x, \dot{x}) + M^2(u, \dot{u}))\{F^2 g_{\gamma\alpha} - (g_{\delta\gamma}\dot{u}^\delta)(g_{\beta\alpha}\dot{u}^\beta)\}. \tag{37}$$

Comparing this with (35), we obtain a conclusion.

THEOREM 4. Let F_n be of isotropic curvature and F_m its totally umbilical subspace of second kind. Then F_m has an isotropic curvature which is equal to the sum of square of mean curvature (i. e. normal curvature) on F_m and isotropic curvature of F_n.

§ 3. If, around each point on F_n, there exists a system of coordinates (x^h) in which the fundamental metric function $F(x, \dot{x})$ is independent of the variables (x^h), our space F_n is called the Minkowskian space M_n. For an F_n to be Minkowskian, it is necessary and sufficient that there consist two systems of equations ([1] p. 39, [5] p. 136)

$$K^{h}_{k \cdot ji} = 0 \text{ and } \frac{\partial \Gamma^{*h}_{ji}}{\partial \dot{x}^k} = 0. \tag{38}$$

Let F_m be a subspace of M_n. Then the equations (24) become $\frac{\partial H^R_{\beta\alpha}}{\partial \dot{u}^r} = 0$ by virtue of (38). If F_m admits an umbilical point (u^α) of second kind, the equations

$$M^2(u, \dot{u}) \frac{\partial g_{\beta\alpha}(u, \dot{u})}{\partial \dot{u}^r} = 0 \tag{39}$$

are valid for every direction $(\dot{u}^\alpha)$ at (u^α). Taking account of the homogeneity of (39) with respect to $(\dot{u}^\alpha)$, we have the last theorem.

THEOREM 5. Let (u^α) be an umbilibal point of second kind on a subspace F_m of a Minkowskian space M_n. If the complement of the carrier of mean curvature (i. e. normal curvature) as a function on the indicatrix at (u^α) is a void set, the Minkowskian metric on the tangent space of F_m at (u^α) must be Euclidean. Especially, if an F_m is the totally umbilical subspace of second kind of a Minkowskian space M_n and every point on F_m satisfies the foregoing condition, the Finslerian structure of F_m must be Riemannian.

Contrasting to this theorem, a totally umbilical subspace of second kind of a Minkowskian space which has vanishing mean curvature (i. e. normal curvature) is Minkowskian. It can be verified from **THEOREM 3.**

References

[1] E. Cartan: Les espaces de Finsler. Hermann, Paris (1934).
[2] L. P. Eisenhart: Riemannian geometry. Princeton Univ. Press (1926).
[3] H. A. Eliopoulos: Subspaces of a generalized metric space. Canad. J. Math. 11 (1959), 235–255.
[4] H. Rund: Hypersurfaces of a Finsler space. Canad. J. Math. 8 (1956), 487–503.
[5] H. Rund: The differential geometry of Finsler spaces. Springer Verlag (1959).
[6] H. Rund: Curvature properties of hypersurfaces of Finsler and Minkowskian spaces. Tensor (N, S) 14 (1963), 501–523.