On Representation of Semi-ordered Linear Spaces

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Introduction

Representation theories of semi-ordered linear spaces were discussed by H. Nakano and I. Amemiya in [1], [2] and [3] in the case that the spaces are continuous or lattice ordered.

In this paper we shall consider their theories to the case that the space is not lattice ordered.

To every element a of semi-ordered linear space R there exist b and $c \in R^+$ (1) such that $a=b-c$ and we define two sets P_a and N_a in R such that

$$P_a = \{x : 0 \leq x \leq b \text{ for every } b, c \in R^+ \text{ and } a=b-c\},$$

$$N_a = P_{-a} = \{x : 0 \leq x \leq c \text{ for every } b, c \in R^+ \text{ and } a=b-c\}.$$

Now we suppose in R the following postulates:

1) $P_a = \{0\}$ implies $a \leq 0$,

2) $P_{a+b} \subset P_a + P_b$ (2) for every element $a, b \in R$.

We shall call such a semi-ordered linear space *semi-lattice ordered* (we write s.l.o.l space).

We shall investigate a methode of imbedding an Archimedean s.l.o.l. space in a C-space on a compact Hausdorff space.

Principles and methods in this paper due to I. Amemiya in [3].

1. Properties of semi-lattice ordered linear spaces

Let R be a s.l.o.l. space. By the definition of P_a in R, we have the following:

(1) $a \leq b$ implies $P_a \subset P_b$,

(2) $x \in P_a$ implies $P_x \subset P_a$,

(3) $a \in P_a$, if and only if $a \geq 0$,

(4) $P_{\alpha a} = \alpha P_a$ for every $\alpha \geq 0$.

We have by the postulate 1):

(5) $P_a \cap P_b = \{0\}$, $a, b \geq 0$ implies $a \frown b = 0$.

(1) $R^+ = \{x : 0 \leq x \in R\}$

(2) $P_a + P_b = \{x+y : x \in P_a \text{ and } y \in P_b\}$

The following relations (6) and (7) are evident:

(6) $a \frown b = 0$ implies $\alpha a \frown \beta b = 0$ for every $\alpha, \beta \geq 0$,

(7) $a \frown b = 0$ implies $a + b = a \smile b$.

Theorem 1. *If* $a \leq 0$, $x \in P_b$ and $a - x \leq b$, then we have $a \leq b$.

Proof. By the assumption, we have

$$0, b \leq b - a + x \text{ and } x \leq b - a + x .$$

Hence we have $a \leq b$.

Theorem 2. *If R is Archimedean, then we have*

$$P_a \cap N_a = \{0\} \text{ and } P_a \perp N_a{}^{(1)} \text{ for every element } a \in R.$$

Proof. If $P_a \cap N_a \ni x$ and $a = b - c$, $b, c \geq 0$, then

$$a = (b - x) - (c - x) = (b - 2x) - (c - 2x) = \cdots = (b - \nu x) - (c - \nu x) = \cdots$$

and

$$b - x \geq 0,\ b - 2x \geq 0, \cdots, b - \nu x \geq 0, \cdots.$$

Since $x \geq 0$ we have $x = 0$ and $P_a \cap N_a = \{0\}$.

Consequently we have $P_a \perp N_a$ by (5).

2. Ideals

Let R be an Archimedean s.l.o.l. space. A linear manifold S of R is said to be an *ideal* if it satisfies the following conditions:

3) $S \ni s$ implies $P_s \subset S$,

4) to every $a \in R$, if $S \ni s$, $P_a \ni x$ and $s - x \leq a$,

then there exists $t \in S$ such that $t \leq a$.

As such examples, we see easily that R and $\{0\}$ are ideals by the *Theorem* 1.. To a set $A \subset R$ and an element $a \in R$ we define that

$$A[a] = \{x : b + \alpha a \leq x \leq c + \beta a \text{ for some } b, c \in A \text{ and } \alpha, \beta\} .$$

By this definition, we see $A \subset A[a]$. Let S be an ideal in R.

Theorem 3. *If $a \geq 0$, then $S[a]$ is an ideal.*

Proof. The linearity in $S[a]$ is evident. If $x \in S[a]$, then $x \leq s + \alpha a$ for some $s \in S$ and $\alpha \geq 0$.

Since $P_{\alpha a} \subset S[a]$ we have by the postulate 2)

$$P_x \subset P_{s + \alpha a} \subset P_s + P_{\alpha a} \subset S[a] .$$

Finally if $x \in S[a]$, $y \in P_b$ and $x - y \leq b$, then there exist $s \in S$ and $\beta \leq 0$ such that $s + \beta a - y \leq b$ and we have by (1)

$$s - y \leq b - \beta a ,\ y \in P_{b - \beta a} .$$

(1) $P_a \perp N_a$ means that $x \frown y = 0$ for every $x \in P_a$ and $y \in N_a$.

By the condition 4) we have $t \in S$ such that $t \leq b - \beta a$ and hence $t + \beta a \leq b$.

An ideal S in R is said to be *maximal* if $S \neq R$ and there is no other ideal T such that $S \subsetneqq T \subsetneqq R$. Now we assume that R has the unit element $e \in R$.(1) By virtue of maximal theorem there exists a maximal ideal in R. By the previous theorem, if S is maximal, then we have the following:

(8) $S \ni a \geq 0$ implies $S[a] = R$.

Theorem 4. *If S is maximal and $S \ni a \geq 0$, $a \frown b = 0$, then we have $b \in S$.*

Proof. By (8) we have that $S[a] = R$, $0 \leq b \leq s + \alpha a$ for some $s \in S$ and $\alpha \geq 0$. Putting $s = x - y$, x, $y \in R^+$ we have $0 \leq b \leq x + \alpha a$.

Since $b \frown \alpha a = 0$ by (6) we have by (7)

$$b + \alpha a \leq x + \alpha a, \ 0 \leq b \leq x, \ b \in P_s \subset S \text{ and } b \in S.$$

Theorem 5. *If S is maximal and P_a, $N_a \subset S$, then $a \in S$.*

Proof. First we shall prove that if $a \bar{\in} S$, then $S[a] = R$.

We need only that $S[a]$ is an ideal. The linearity and the condition 3) in $S[a]$ are clear. We shall prove that $S[a]$ satisfies the condition 4). If $x \in S[a]$ and $x - y \leq b$, $y \in P_b$, then there exist $s \in S$ and α such that $s + \alpha a - y \leq b$. If $y \in S$, then $S[a]$ satisfies the condition 4). And if $y \bar{\in} S$, then $S[y] = R$ by (8) and we have

$$t + \beta y \leq b \text{ for some } t \in S \text{ and } \beta \leq 0.$$

If $-1 \leq \beta \leq 0$, then $t - y \leq t + \beta y \leq b$. Since the condition 4) is satisfied in S we have

$$u \leq b, \ u \in S \subset S[a].$$

And if $\beta < -1$, then we can take some natural number ν such that $\nu \leq -\beta < \nu + 1$ and we have

$$t - y < t + (\beta + \nu)\, y \leq b + \nu y, \ y \in P_{b + \nu y}.$$

And we have by the condition 4) in S

$$u_1 \leq b + \nu y, \ u_1 \in S.$$

Consequently we have the following:

$$u_2 \leq b + (\nu - 1)\, y, \ u_2 \in S \text{ for } u_1 - y \leq b + (\nu - 1)\, y, \ y \in P_{b + (\nu - 1) y},$$

$$u_3 \leq b + (\nu - 2)\, y, \ u_3 \in S \text{ for } u_2 \ \ y \leq b + (\nu - 2)\, y, \ y \in P_{b + (\nu - 2) y}$$

and so forth and we have that $v \leq b$, $v \in S \subset S[a]$.

Next since $S[a] = R$, there exist $s \in S$ and $\alpha (\neq 0)$ such that $0 \leq e \leq s + \alpha a$ and we see obviously

$$s + \alpha a \in P_{s + \alpha a} \subset P_s + P_{\alpha a} \subset S \text{ and } a \in S.$$

This fact contradicts to $a \bar{\in} S$.

(1) To every $a \in R$ we have $\alpha \geq 0$ such that $a \leq \alpha e$.

If $a\,(\geq 0)$ is not an unit, then $\{0\}[a](\ni e)$ is an ideal and we have a maximal ideal S including $\{0\}[a]$. Such a S contains a.

Next let $a\,(>0)$ be not an unit. Putting $\inf_{a \leq \lambda e} \lambda = \alpha\,(>0)$, we see easily that $\alpha e - a > 0$ and $\alpha e - a$ is not an unit. Consequently by previous result, there exists a maximal ideal S which contains $\alpha e - a$. Such a S does not contain a and we have next theorem.

Theorem 6. *If $a\,(>0)$ is not an unit, then there exists a maximal ideal which does not contain a.*

3. Quotient spaces

A semi-ordered linear space R is called *one-dimensional* if for every elements $a\,(\neq 0)$ and $b \in R$ there exists α such that $b = \alpha a$.

A semi-ordered linear space R is one-dimensional, if and only if R is linearly ordered space and every element $a > 0$ of R is an unit.

Let R be an Archimedean s.l.o.l. space with the unit element $e \in R$ and S be a maximal ideal in R. The quotient space R/S is defined as a semi-ordered linear space. In fact we define $A \leq B$, $A,\ B \in R/S$ if there exist two elements $a \in A$ and $b \in B$ such that $a \leq b$. (1)

Theorem 7. *The quotient space R/S is one-dimensional.*

Proof. If $R/S \ni A \neq 0$ and $a \in A$, then $a \,\bar{\in}\, S$ and by the *Theorem* 5. there exists some element $b \,\bar{\in}\, S$ such that $b \in P_a \cup N_a$.

If such an element b belongs to P_a, then $S[b] = R$ and we have that $s + \alpha b \leq a$ for some $s \in S$ and $\alpha < 0$. By the methode which is used for proof of the *Theorem* 5., we have some element $t \in S$ satisfying $t \leq a$. Consequently we see that $0 \leq A$.

Similarly if $b \in N_a$, then we have that $A \leq 0$. Consequently R/S is a linearly ordered space. Next if $R/S \ni A > 0$, then we have some elements $s \in S$ and $a > 0$ such that $s + a \in A$. Since $S[a] = R$, we have that $e \leq t + \beta a$ for some $t \in S$ and $\beta > 0$.

Consequently A is an unit in R/S, because the element $(e \in)\, E \in R/S$ is an unit in R/S.

4. Homomorphisms

Let R be an Archimedean s.l.o.l. space with the unit element $e \in R$ and S be a maximal ideal in R. A non-constant real valued function φ on R is called a *homomorphism* if it satisfies the following conditions:

5) $\varphi(\alpha a + \beta b) = \alpha\varphi(a) + \beta\varphi(b)$ for every $a, b \in R$ and α, β,

6) $0 \leq a$ implies $0 \leq \varphi(a)$,

7) $\varphi(a) = 0$ implies $\varphi(x) = 0 \quad (x \in P_a)$.

(1) cf. [4] Satz 1.

If R is one-dimensional (of course an one-dimensional space is an Archimedean s.l.o.l. space), then there exists a homomorphism on R. Consequently there exists a homomorphism φ on R/S by previous theorem. Now we can make the real valued function ψ on R such that $\psi(a)=\varphi(A)$ if $a\in A(\in R/S)$. It is evident that such ψ on R is homomorphism. And we have next theorem.

Theorem 8. *If S is a maximal ideal in R, then there exists a homomorphism φ on R such that $\varphi^{-1}(0)=S$.*

Theorem 9. *If φ is a homomorphism on R, then $\varphi^{-1}(0)$ is a maximal ideal in R.*

Proof. It is clear that linearity and the condition 3) are satisfied in $\varphi^{-1}(0)$. Putting

$$s-x\leq a\,,\ \varphi(s)=0 \text{ and } x\in P_a\,,$$

if $\varphi(x)=0$, then the condition 4) is satisfied in $\varphi^{-1}(0)$ and if $\varphi(x)\neq 0$, then $\varphi(a)\neq 0$ by the condition 7).

Consequently we have that $\alpha\varphi(a)=\varphi(x)$ for some $\alpha>0$.

Since $\alpha a-x=t\in\varphi^{-1}(0)$, we obtain $a\geq\frac{1}{\alpha}t\in\varphi^{-1}(0)$.

Maximality of $\varphi^{-1}(0)$ is evident.

5. Proper space

Let R be an Archimedean s.l.o.l. space with the unit element $e\in R$. Considering every homomorphism φ (which satisfies that $\varphi(e)=1$) on R as a point, we obtain a space X.

Corresponding to every element $a\in R^+$, we define that

$$U_a=\{\varphi:\varphi\in X \text{ and } \varphi(a)>0\}\,.$$

We see easily the following relations:

(9) $U_e=X$ and $U_0=\emptyset$,

(10) $U_a\cup U_b=U_{a+b}$ for every $a,\ b\in R^+$.

By the *Theorem* 4. and 9., we have

(11) $a\frown b=0$ implies $U_a\cap U_b=\emptyset$.

If $P_a\cap P_b\ni c>0$, then we have that $U_a\cap U_b\supset U_c\neq\emptyset$ by the *Theorem* 6. and 8.. Consequently we can introduce a topology into X such that the totality of all U_a $(a\in R^+)$ constitutes a neighbourhood system. This topological space X is called the *proper space* of R.

Theorem 10. *The proper space X of R is compact Hausdorff space.*

Proof. Putting an open covering of X

$$X\subset\bigcap_{\lambda\in\Lambda}U_{a_\lambda}\,,$$

we can make the least linear manifold A of R including a_λ $(\lambda\in\Lambda)$ and $A[0]$ is an ideal. If $e\bar{\in} A[0]$, then there exists a maximal ideal S in R which includes $A[0]$ and there exists a homomorphism $\varphi\in X$ such that $\varphi^{-1}(0)=S$ by the *Theorem* 8.. Such a φ does not belong to $\bigcup_{\lambda\in\Lambda} U_{a_\lambda}$. Consequently we have that $e\in A[0]$ and $e\leq\alpha_1 a_{\lambda_1}+\cdots+\alpha_\kappa a_{\lambda_\kappa}$ for some $\lambda_\nu\in\Lambda$ and $\alpha_\nu>0$ $(\nu=1,\cdots,\kappa)$.

Hence we have that $X\subset\bigcup_{\nu=1}^{\kappa} U_{a_{\lambda_\kappa}}$, i.e., X is compact.

Next if $\varphi\neq\psi$, $\varphi,\psi\in X$, then we have some element a such that $\varphi(a)>1$ and $\psi(a)=0$ and we have that $\varphi(a-e)>0$ and $\psi(e-a)=1$. Consequently by the *Theorem* 5. and 9., there exist two elements $x\in P_{a-e}$ and $y\in N_{a-e}$ such that $\varphi(x)>0$ and $\psi(y)>0$.

Since $x\frown y=0$ by the *Theorem* 2., we have that $\varphi\in U_x$, $\psi\in U_y$ and $U_x\cap U_y=\emptyset$ by (11), namely, X is a Hausdorff space.

6. Representation

Let R be an Archimedean s.l.o.l. space with the unit element $e\in R$. Corresponding to every element $a\in R$, we define a real valued function $f_a(\varphi)$ $(\varphi\in X)$ such that

$$f_a(\varphi)=\varphi(a)\ (\varphi\in X).$$

We see easily the following relations:

(12) $f_{\alpha a+\beta b}=\alpha f_a+\beta f_b$ for every $a,b\in R$ and α,β,

(13) $a\leq b$ implies $f_a\leq f_b$.

Theorem 11. *To every element* $a\in R$, $f_a(\varphi)$ $(\varphi\in X)$ *is continuous.*

Proof. We put for every α

$$Y=\{\varphi:f_a(\varphi)<\alpha \text{ and } \varphi\in X\}=\{\varphi:\varphi(\alpha e-a)>0 \text{ and } \varphi\in X\}.$$

If $\varphi\in Y$, then we have that $\varphi(x)>0$ for some $x\in P_{\alpha e-a}$ and $\varphi\in U_x\subset Y$, namely, Y is an open set. Similarly

$$Z=\{\varphi:f_a(\varphi)>\alpha \text{ and } \varphi\in X\} \text{ is an open set.}$$

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