Some Note on Classic Integral (II)

by Kiro ISOBE

(Received October 9, 1965)

Introduction

We considered about a mean value of a function over an interval $[a, b]$ in [1]. Namely, let a function $f(x)$ be bounded over an interval $[a, b]$ and Δ be any division of the interval $[a, b]$ into a finite number of subintervals, designating the points of subdivision by

$$\Delta : \quad a = x_0 < x_1 < \cdots < x_n = b .$$

We put

$$I_i = \{ f(x)\,(x_i - x_{i-1}) \ : \ x_{i-1} \leqq x \leqq x_i \} ,$$

$$S_\Delta = \left\{ \sum_{i=1}^{n} \xi_i : \ \xi_i \in I_i \right\} .$$

For every set S_Δ, if their intersection is composed of only single point, i.e., $\bigcap_\Delta S_\Delta = \{\xi\}$, then we shall call this point ξ the mean value of the function f over the interval $[a, b]$.

The *norm* of the division Δ is defined such that

$$|\Delta| = \operatorname*{Max}_{i=1,2,\cdots,n} \{x_i - x_{i-1}\}$$

and we put

$$S = \bigcap_{\varepsilon>0} \overline{\bigcup_{|\Delta|<\varepsilon} S_\Delta} .$$

In this paper, we shall consider the following case that the set S is composed of only single point.

Main Result

Let a bounded function $f(x)$ be Riemann integrable over an interval $[a, b]$ and $\xi_0 = \int_a^b f(x)\,dx$. For any division Δ of the interval $[a, b]$, given by

$$\Delta : \quad a = x_0 < x_1 < \cdots < x_n = b ,$$

putting

$$G_i = \sup_{x_{i-1} \leqq x \leqq x_i} f(x) , \qquad g_i = \inf_{x_{i-1} \leqq x \leqq x_i} f(x) ,$$

$$G_\Delta = \sum_{i=1}^{n} G_i\,(x_i - x_{i-1}) \text{ and } g_\Delta = \sum_{i=1}^{n} g_i\,(x_i - x_{i-1}) ,$$

it is well known that

(*) *for any positive number ε, there exists some positive number δ such that*

$$|\Delta|<\delta \textit{ implies } |G_\Delta-\xi_0|<\varepsilon \text{ and } |g_\Delta-\xi_0|<\varepsilon$$

and

$$\inf_\Delta G_\Delta=\sup_\Delta g_\Delta=\xi_0\,.$$

It is trivial that

$$S=\bigcap_{\varepsilon>0}\overline{\bigcup_{|\Delta|<\varepsilon} S_\Delta}\neq\emptyset\,.$$

If $S\ni\xi,\xi'$ and $\xi\neq\xi'$, then we adopt some positive number ε such that

$$\frac{|\xi-\xi'|}{6}>\varepsilon>0\,.$$

For such a ε there exists some positive number δ by (*) such that

$$|\Delta|<\delta \text{ implies } |G_\Delta-\xi_0|<\varepsilon \text{ and } |g_\Delta-\xi_0|<\varepsilon\,.$$

Since $S\ni\xi,\xi'$, we have some two numbers ξ_1 and ξ_1' satisfying

$$|\xi-\xi_1|<\varepsilon\,,\qquad \xi_1\in\bigcup_{|\Delta|<\delta}S_\Delta \text{ and } |\xi'-\xi_1'|<\varepsilon\,,\qquad \xi_1'\in\bigcup_{|\Delta|<\delta}S_\Delta$$

respectively.

Hence adopting some two divisions $\Delta(|\Delta|<\delta)$ and $\Delta'(|\Delta'|<\delta)$ of the interval $[a, b]$ such that

$$\xi_1\in S_\Delta \text{ and } \xi_1'\in S_{\Delta'}\,,$$

we have

$$G_\Delta\geqq\xi_1\geqq g_\Delta \text{ and } G_{\Delta'}\geqq\xi_1'\geqq g_{\Delta'}$$

respectively. And we obtain by (*)

$$|\xi_1-\xi_0|<2\varepsilon \text{ and } |\xi_1'-\xi_0|<2\varepsilon$$

and hence

$$|\xi_1-\xi_1'|<4\varepsilon\,.$$

Consequently we have following contradiction

$$|\xi-\xi'|\leqq|\xi-\xi_1|+|\xi_1-\xi_1'|+|\xi_1'-\xi'|<6\varepsilon<|\xi-\xi'|\,.$$

Hence the set S is composed of only single point.

Furthermore for any positive number ε, we have some positive number δ by (*) such that

$$|\Delta|<\delta \text{ implies } |G_\Delta-\xi_0|<\frac{\varepsilon}{4} \text{ and } |g_\Delta-\xi_0|<\frac{\varepsilon}{4}\,.$$

Since $S\ni\xi$, we have some point ξ' and division Δ' of the interval $[a, b]$ such that

$$|\xi-\xi'|<\varepsilon\,,\qquad \xi'\in S_{\Delta'} \text{ and } |\Delta'|<\delta$$

and it is that

$$|\xi'-\xi_0|<\frac{\varepsilon}{2}\,,$$

because

$$G_{\Delta'} \geqq \xi', \qquad \xi_0 \geqq g_{\Delta'}.$$

Consequently we conclude that

$$|\xi - \xi_0| \leqq |\xi - \xi'| + |\xi' - \xi_0| \text{ and } \xi = \xi_0.$$

Conversely we assume that the set $S = \bigcap_{\varepsilon > 0} \overline{\bigcup_{|\Delta| < \varepsilon} S_\Delta}$ is composed of only single point ξ for a bounded function $f(x)$ over an interval $[a, b]$.

Let Δ be any division of the interval $[a, b]$, given by

$$\Delta : \quad a = x_0 < x_1 < \cdots < x_n = b.$$

For any positive number ε, there exists some numbers ξ_i $(i = 1, 2, \cdots, n)$ such that

$$G_i - \frac{\varepsilon}{b-a} < f(\xi_i) \qquad (x_{i-1} \leqq \xi_i \leqq x_i)$$

and we obtain

$$G_\Delta \geqq \sum_{i=1}^{n} f(\xi_i)\ (x_i - x_{i-1}) > G_\Delta - \varepsilon$$

hence

$$G_\Delta \in \overline{S}_\Delta.$$

Consequently if $|\Delta| < \delta$, then $G_\Delta \in \overline{\bigcup_{|\Delta| < \delta} S_\Delta}$.

It is trivial by (*) that

$$\inf_\Delta G_\Delta = \inf_{|\Delta| < \delta} G_\Delta$$

and hence

$$\inf_\Delta G_\Delta \in \overline{\bigcup_{|\Delta| < \delta} S_\Delta}$$

for any positive number δ. Consequently we obtain that

$$\inf_\Delta G_\Delta \in S \text{ and } \inf_\Delta G_\Delta = \xi.$$

We have in the same way to the above

$$\sup_\Delta g_\Delta \in S \text{ and } \sup_\Delta g_\Delta = \xi.$$

Consequently we conclude that the function $f(x)$ is Riemann integrable over the interval $[a, b]$.

Remark. *A necessary and sufficient condition that the bounded function $f(x)$ be Riemann integrable over the interval $[a, b]$ is that the set $S = \bigcap_{\varepsilon > 0} \overline{\bigcup_{|\Delta| < \varepsilon} S_\Delta}$ is composed of only single point. And putting $S = \{\xi\}$, we have $\xi = \int_a^b f(x)\, dx$.*

(September 30, 1965)

Reference

[1] K. Isobe: Some Note on Classic Integral. 1964, Mem. Kitami Col. Tech. No. 3.