Some Note on Classic Integral

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Introduction

Let a function $f(x)$ $(a \leqq x \leqq b)$ be bounded over the interval $[a, b]$ and $\varDelta$ be any division of the interval $[a, b]$ into a finite number of subintervals, designating the points of subdivision by

$$\varDelta: \ a = x_0 < x_1 \cdots < x_n = b.$$

Putting

$$I_i = \left\{ \sum_{i=1}^{n} f(x)(x_i - x_{i-1}) : x_{i-1} \leqq x \leqq x_i \right\} \qquad (i = 1, 2, \cdots, n)$$

$$S_\varDelta = \left\{ \sum_{i=1}^{n} \xi_i : \xi_i \in I_i \right\}$$

for every $S_\varDelta$, if their intersection is composed of only single point ; namely, $\bigcap_\varDelta S_\varDelta = \{\xi\}$, then we shall call this point ξ *the mean value of the function* f *over the interval* $[a, b]$.

In this paper we shall consider the relation between Riemann integrability of the function f and existibility of mean value of it over the interval $[a, b]$.

1. Intervals

Let I be an interval, i. e., $I = [a, b]$. For any positive number ξ, putting

$$\xi I = \{\xi x : x \in I\}$$

we see easily that

(A) ξI *is an interval, too, i. e.,* $\qquad I = [\xi a, \xi b]$

and

(B) $I_1 \subset I_2$ *implies* $\xi I_1 \subset \xi I_2$ *for two intervals* I_1 *and* I_2.

Putting

$I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ and

$I_1 + I_2 = \{x_1 + x_2 : x_1 \in I_1$ *and* $x_2 \in I_2\}$

we see easily that

$I_1 + I_2 \subset [a_1 + a_2, \ b_1 + b_2]$.

Conversely for any real number z satisfying $a_1 + a_2 \leqq z \leqq b_1 + b_2$, we obtain

$$a_2 \leqq z - a_1 \text{ and } z - b_1 \leqq b_2 .$$

If $z-a_1 \leqq b_2$, then $z=a_1+(z-a_1)\in(I_1+I_2)$.

And if $z-a_1>b_2$, then putting

$$z-w=b_2<z-a_1$$

we obtain immediately

$$a_1<w,\ z-b_1\leqq z-w \text{ and } a_1<w\leqq b_1.$$

Hence $z=w+(z-w)\in(I_1+I_2)$.

Consequently we conclude that

(C) I_1+I_2 is an interval, too, i.e., $I_1+I_2=[a_1+a_2,\ b_1+b_2]$.

For an interval $I=[a,b]$ we define $|I|$ to mean its length $b-a$. When the interval I is divided into two intervals $I_1=[a,c]$ and $I_2=[c,b]$, for any interval J we obtain obviously the following relations (D) by (A) and (C).

(D) $|I_1+I_2|J=|I_1|J+|I_2|J=|I|J.$

2. Continuous Function and Interval Ranged Function

Let a function $f(x)$ $(a\leqq x\leqq b)$ be continuous over the interval $[a,b]$.

For any interval I, which is included in the interval $[a,b]$, $f(I)$; namely,

$$f(I)=\{f(x): \textit{for every } x\in I\},$$

is an interval, too. Such a function f shall be called *an interval ranged function over the interval* $[a,b]$.

Let Δ and Δ' be two divisions of the interval $[a,b]$, given by

$$\Delta:\ a=x_0<x_1<\cdots<x_l=b$$

$$\Delta':\ a=x_0'<x_1'<\cdots<x_m'=b.$$

We define $\Delta\geqq\Delta'$ to mean that

$$\{x_0,x_1,\cdots,x_l\}\subset\{x_0',x_1',\cdots,x_m'\}.$$

And we obtain by (D)

(E) $\Delta\geqq\Delta'$ implies $S_\Delta\supset S_{\Delta'}$.

Furthermore for a finite number of divisions Δ_i $(i=1,2,\cdots,n)$ of the interval $[a,b]$, given by

$$\Delta_i:\mathrm{a}=x_0^{(i)}<x_1^{(i)}<\cdots<x_{m_i}^{(i)}=b,$$

if we put the division Δ of the interval $[a,b]$ such that

$$\Delta: a=x_0<x_1<\cdots<x_p=b$$

$$\bigcup_{i=1}^{n}\{x_0^{(i)},x_1^{(i)},\cdots,x_{m_i}^{(i)}\}=\{x_0,x_1,\cdots,x_p\},$$

then $\Delta_i\geqq\Delta$ $(i=1,2,\cdots,n)$.

Consequently we conclude the following (F) by (E).

(F) For every division Δ of the interval $[a,b]$, the set system $\{S_\Delta\}_\Delta$

satisfies the finite intersection property : for any finite number of $\varDelta_i$ $(i=1, 2, \cdots, n)$, there exists some $\varDelta$ such that

$$\bigcap_{i=1}^{n} S_{\varDelta_i} \supset S_{\varDelta} \neq \emptyset .$$

Of course every $S_\varDelta$ is a bounded closed set, because it is an interval by (A) and (C), and we denote $S_\varDelta = [a_\varDelta, b_\varDelta]$. Hence we conclude that

(G) $$\bigcap_{\varDelta} S_\varDelta = [\sup_{\varDelta} a_\varDelta, \inf_{\varDelta} b_\varDelta] \neq \emptyset.$$ [1]

The function f is uniformly continuous over the interval $[a, b]$ since it is continuous over the interval $[a, b]$. Consequently if two distinct points α and β belong to the set $\bigcap_{\varDelta} S_\varDelta$, then for positive number ε satisfying $0<\varepsilon<\dfrac{|\alpha-\beta|}{b-a}$, there exists some positive number δ with respect to ε such that

$$|x-x'|<\delta \text{ implies } \quad |f(x)-f(x')|<\varepsilon.$$

Let $\varDelta$ be division of the interval $[a, b]$, given by

$$\varDelta: \ a=x_0<x_1<\cdots<x_n=b.$$

$$\operatorname*{Max}_{i=1,2,\cdots,n} \{x_i-x_{i-1}\}<\delta.$$

For such $\varDelta$ we have by assumption

$$\alpha=\sum_{i=1}^{n} f(\xi_i)(x_i-x_{i-1})$$

$$\beta=\sum_{i=1}^{n} f(\eta_i)(x_i-x_{i-1}) \qquad \text{for some } x^{i-1} \leqq \xi_i, \ \eta_i \leqq x_i \ .$$

Hence

$$|\alpha-\beta|=\left|\sum_{i=1}^{n} \{f(\xi_i)-f(\eta_i)\}(x_i-x_{i-1})\right|<|\alpha-\beta|$$

Consequently we have the following :

(H) *If a function $f(x)$ $(a \leqq x \leqq b)$ is continuous over the interval $[a, b]$, then there exists the mean volue of f over the interval $[a, b]$, denoting it by ξ, it is evident that*

$$\xi=\int_b^a f(x)dx.$$

The following theorem is well known : [2]

Let a function $f(x)$ $(a \leqq x \leqq b)$ be bounded over the interval $[a, b]$. The function f is Riemann integrable over the interval $[a, b]$, if and only if for any positive number ε there exists some dsvision $\varDelta$ of the interval $[a, b]$ with respect to ε, given by

1) See [2], pp. 55–56
2) See [1], pp. 8–9

$$\Delta: \quad a=x_0<x_1<\cdots<x_n=b$$

such that

$$\sum_{i=1}^{n} \omega_i(x_i-x_{i-1})<\varepsilon,$$

in here each ω_i *is oscillation of* f *over the interval* $[x_{i-1}, x_i]$.

Let a function $f(x)$ $(a \leqq x \leqq b)$ be an interval ranged function over the interval $[a, b]$ (not always continuous), of course it is bounded over the interval $[a, b]$.

We see similarly that (G) is established in this case.

By the use of the previous theorem, it is evident with similar treatment as we introduce (H) that the function f has the mean value over interval $[a, b]$ if it is Riemann integrable over the interval $[a, b]$.

Conversely if f has the mean value over the interval $[a, b]$, then for every division Δ of the interval $[a, b]$, putting

$$S_\Delta=[a_\Delta, b_\Delta]$$

we have by (F) and (G)

$$\sup_\Delta a_\Delta=\inf_\Delta b_\Delta \quad \text{and} \quad \inf_\Delta (b_\Delta-a_\Delta)=0.$$

Hence, for any positive number ε there exists some division Δ of the interval $[a, b]$, given by

$$\Delta: \quad a=x_0<x_1<\cdots<x_n=b$$

such that

$$\sum_{i=1}^{n} \omega_i(x_i-x_{i-1})=b_\Delta-a_\Delta<\varepsilon.$$

Consequently we conclude the following.

(I) *In order that an interval ranged function* f *has the mean value* ξ *over the interval* $[a, b]$, *it is necessary and sufficient that it is Riemann integrable over the interval* $[a, b]$ *and of course*

$$\xi=\int_b^a f(x)\,dx.$$

Furthermore let a function $f(x)$ $(a \leqq x \leqq b)$ be bounded over the interval $[a, b]$. The function f has the mean value over the interval $[a, b]$ if it is Riemann integrable over the interval $[a, b]$ and the set $\bigcap_\Delta S_\Delta \neq \emptyset$ for every division Δ of the interval $[a, b]$ into a finite number of subintervals.

3. Some Examples.

i) If we put the function $f(x)$ $(0 \leqq x \leqq 1)$ such that

$$f(x)=\begin{cases} x & (x\neq \frac{1}{2}) \\ 0 & (x=\frac{1}{2}) \end{cases}$$

then it has not the mean value over the interval $[0, 1]$.

Because, putting two functions $g(x)$ and $h(x)$ $(0 \leqq x \leqq 1)$ respectively

$$g(x)=x \quad \text{and} \quad h(x)=\begin{cases} x\,(x\neq \frac{1}{2}) \\ 0 \text{ and } \frac{1}{2} \ (h(x) \text{ is two valued at } x=\frac{1}{2}) \end{cases}$$

and denoting S_Δ with respect to f, g and h by $S_\Delta(f)$, $S_\Delta(g)$ and $S_\Delta(h)$ respectively, it is evident that the set $\bigcap_\Delta S_\Delta(h)$ includes the set $\bigcap_\Delta S_\Delta(g)$ and $\bigcup_\Delta S_\Delta(f)$ in it. And we see easily that the set $\bigcap_\Delta S_\Delta(h)$ is composed of only single point. For $\bigcap_\Delta S_\Delta(g) = \{\frac{1}{2}\}$, we have $\bigcap_\Delta S_\Delta(h)=\{\frac{1}{2}\}$. Consequently if $\bigcap_\Delta S_\Delta(f)\neq\emptyset$, then it is composed of only point $\frac{1}{2}$, but $f([0, 1]) \ni \frac{1}{2}$.

ii) Putting the function $f(x)$ $(0 \leqq x \leqq 1)$

$$f(x)=\begin{cases} 1 & (x\neq \frac{1}{2}) \\ 0 & (x=\frac{1}{2}) \end{cases}$$

it is evident that the function f has the mean value 1 over interval $[0, 1]$.

iii) Let A_0 be the interval $[0, 1]$. We remove the open interval $(\frac{1}{3}, \frac{2}{3})$ from A_0 and denote the remaining closed set by A_1. Then we remove the open intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ from A_1 and denote the remaining closed set (consisting of four intervals) by A_2. From each of these four intervals we remove the middle open interval of length $(\frac{1}{3})^3$, and so forth. If we continue this process, we obtain the decreasing sequence of closed sets A_n. We set $F^{(0)}=\bigcap_{n=0}^{\infty} A_n$.

The set $F^{(0)}$ is named "Cantor's set" and it has the cardinal number of the continum as well known.(3) The set $[0, 1]-F_{(0)}$ is the union of a countable number of disjoint open intervals U_n $(n=1, 2, \cdots)$.

We can consistute the set $F_n^{(1)}$, removing a countable number of disjoint open intervals from each U_n similarly to the above and put $F^{(1)}=\bigcup_{n=1}^{\infty} F_n^{(1)}$. If we continue this process, then we obtain the sequence of the disjoint sets $F^{(n)}$, and each set $F^{(n)}$ has the cardinal number of the continum. And for any interval I, included in the interval $[0, 1]$, there exists some $F^{(n_0)}$ *such that* $I \supsetneqq F^{(n_0)}$.

For each set $F^{(n)}$ has the cardinal number of the continum, we have one-to-one mapping f_n of the each set $F^{(n)}$ onto the interval $[0, 1]$. Putting

$$f(x)=\begin{cases} f_n(x) & (x\in \bigcup_{n=1}^{\infty} F^{(n)}) \\ 0 & (x \,\overline{\in}\, \bigcup_{n=1}^{\infty} F^{(n)}) \end{cases}$$

we see easily that the function f is interval ranged function over the interval $[0, 1]$, but it is has not the mean value over the interval $[0, 1]$.

3) See [2], pp. 32–33

Reference

[1] K. Kunugi: Zitukansuron oyobi Sekibunron 1959, (Kyoritu Syuppan).
[2] K. Kolmogorov and Fomin: Functional Analysis Vol. 1 1963 (Graylock).