Remark to Relations between Spectral Systems in Semi-ordered Linear Space

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Introduction

Spectral theory of semi-ordered linear space was fully discussed by H. Nakano in [1] and [2]. Relations between spectral systems are easy and simple if we apply the second spectral theory in [1]. In this paper we shall consider it by the first spectral theory in [1]. All notations and therminology of this paper will be followed to [1]. Let R be a continuous semi-ordered linear space.

Corresponding to every pair of elements a and $b \in R$ there exists uniquely a resolution[1] $[p_\lambda]$ $(-\infty<\lambda<+\infty)$ of the projecter[2] $[a]$ such that

$$[a]\,b=\int_{-\infty}^{+\infty}\lambda d[p_\lambda]a \quad {}^{3)}$$

Such a resolution of the projecter $[a]$ is called the spectral system of an element $b \in R$ by an element $a \in R$ and it is given by

$$[p_\lambda]=[(\lambda a-b)^+]\,[a^+]+[(\lambda a-b)^-]\,[a^-] \quad (-\infty<\lambda<+\infty)$$

Relations between spectral systems.

Theorem 1. If two systems of projecters $[p_\lambda]$ and $[q_\lambda]$ $(-\infty<\lambda<+\infty)$ are respectively the spectral systems of elements $b \in R$ and $-b \in R$ by an element $a \in R$ then we have for every real number λ

$$[q_\lambda]=[a]-\bigcap_{\rho>-\lambda}[p_\rho]$$

Proof. For any real number λ we need only to prove that

$$[q_\lambda]\bigcap_{\rho>-\lambda}[p_\rho]=0 \text{ and } [a]=[q_\lambda]+\bigcap_{\rho>-\lambda}[p_\rho]$$

For any real number λ we have by assumption

$$[p_\lambda]=[(\lambda a-b)^+]\,[a^+]+[(\lambda a-b)^-]\,[a^-]$$
$$[q_\lambda]=[(\lambda a+b)^+]\,[a^+]+[(\lambda a+b)^-]\,[a^-]$$

Therefore we obtain for every real number λ

$$(*) \qquad [q_{-\lambda}]\,[p_\lambda]=0$$
$$(**) \qquad [q_\lambda]+[p_{-\lambda}]=[\lambda a+b]\,[a]$$

Putting $[p]=[q_\lambda]\bigcap_{\rho>-\lambda}[p_\rho]$ we have by (*)

$$[p]\,[q_{-\sigma}]=0 \qquad (\sigma>-\lambda)$$

This relation yields by assumption

$$0=[p]\bigcup_{\sigma>-\lambda}[q_{-\sigma}]=[p][q_\lambda]=[p]$$

Putting $[q]=[a]-([q_\lambda]+\bigcap_{\rho>-\lambda}[p_\rho])$ we have by assumption and (**)

$$[q]([p_{-\lambda}]+[q_\lambda])=[q][\lambda a+b]=0 \text{ and } [q](\lambda a+b)=0$$

For any positive real number ε this relation yields

$$[q][(-\lambda a-b+\varepsilon a)^+][a^+]=[q][a^+]$$
$$[q][(-\lambda a-b+\varepsilon a)^-][a^-]=[q][a^-]$$

Concequently we conclude for any positive real number ε

$$[q]\leqq[p_{-\lambda+\varepsilon}] \qquad \text{and hence } [q]\bigcap_{\rho>-\lambda}[p_\rho]=[q]=0$$

Theorem 2. *For any real number $\alpha\neq 0$ let two systems of projecters $[p_\lambda]$ and $[q_\lambda]$ $(-\infty<\lambda<+\infty)$ be respectively the spectral systems of elements b and $\alpha b\in R$ by an element $a\in R$. If $\alpha>0$ then we have $[q_\lambda]=[p_{\lambda/\alpha}]$ and if $\alpha<0$ then we have $[q_\lambda]=[a]-\bigcap_{\rho>\frac{\lambda}{\alpha}}[p_\rho]$ for every real number λ.*

Proof. For $\alpha>0$ we have by assumption

$$[q_\lambda]=[(\lambda a-\alpha b)^+][a^+]+[(\lambda a-\alpha b)^-][a^-]$$
$$=\left[\left(\frac{\lambda}{\alpha}a-b\right)^+\right]\left[a^+\right]+\left[\left(\frac{\lambda}{\alpha}a-b\right)^-\right]\left[a^-\right]=\left[p_{\frac{\lambda}{\alpha}}\right]$$

for any real number λ.

For $\alpha<0$ we have by assumption and Theorem 1.

$$[q_\lambda]=[(\lambda a-(-\alpha)(-b))^+][a^+]+[(\lambda a-(-\alpha)(-b))^-][a^-]$$
$$=\left[\left(-\frac{\lambda}{\alpha}a+b\right)^+\right]\left[a^+\right]+\left[\left(-\frac{\lambda}{\alpha}a+b\right)^-\right]\left[a^-\right]=\left[a\right]-\bigcap_{\rho>\frac{\lambda}{\alpha}}\left[p_\rho\right]$$

Lemma 1. *If three systems of projecters $[p_\lambda]$, $[q_\lambda]$ and $[r_\lambda]$ $(-\infty<\lambda<+\infty)$ are respectively the spectral systems of b, c and $b+c\in R$ by an element $a\in R$ we have*

$$(\sharp) \qquad [p_\sigma][q_\rho]\leqq[r_\lambda] \qquad \text{for } \sigma+\rho<\lambda$$
$$(\sharp\sharp) \qquad [r_\lambda](1-[p_\sigma])\leqq[q_\rho] \qquad \text{for } \sigma+\rho>\lambda$$

Proof. In the case $a\geqq 0$ we prove ($\sharp$).

Putting $[p]=[p_\sigma][q_\rho]$ we have by assumption

$$[p](\lambda a-b-c)\geqq[p](\sigma a-b+\rho a-c)=[p](\sigma a-b)^++[p](\rho a-c)^+\geqq 0$$

Therefore we obtain

$$[p](\sigma a-b)^+\leqq(\lambda a-b-c)^+$$

This relation yields obviously $[p]\leqq[r_\lambda]$

Now we can prove the general case of ($\sharp$). We obtain just above

$$[(\sigma a-b)^+][(\rho a-c)^+][a^+]\leqq[(\lambda a-b-c)^+][a^+]$$
$$[(\sigma a-b)^-][(\rho a-c)^-][a^-]\leqq[(\lambda a-b-c)^-][a^-]$$

because two systems $[(\lambda a-b)^-][a^-]$ and $[(\lambda a-c)^-][a^-]$ are respectively the spectral systems of $-b$ and $-c$ by $a.^-$

Therefore we have

$$\begin{aligned}[p_\sigma][q_\rho]&=([(\sigma a-b)^+][a^+]+[(\sigma a-b)^-][a^-])([(\rho a-c)^+][a^+]+[(\rho a-c)^-][a^-])\\&=[(\sigma a-b)^+][(\rho a-c)^+][a^+]+[(\sigma a-b)^-][(\rho a-c)^-][a^-]\\&\leqq[(\lambda a-b-c)^+][a^+]+[(\lambda a-b-c)^-][a^-]=[r_\lambda]\end{aligned}$$

If $\sigma+\rho>\lambda$ then $\rho>\lambda-\sigma$ and there exists σ' such that $\rho>\lambda-\sigma'$ and $\sigma'<\sigma$. Therefore we conclude by Theorem 1. and (#)

$$[q_\rho]\geqq[r_\lambda](1-\bigcap_{\varepsilon>\sigma'}[p_\varepsilon])\geqq[r_\lambda](1-[p_\sigma])$$

Theorem 3. *If two systems* $[p_\lambda]$ *and* $[q_\lambda]$ $(-\infty<\lambda<+\infty)$ *are respectively the spectral systems* b *and* $c\in R$ *by an element* $a\in R$, *then putting*

$$[r_\lambda]=\bigcup_{\sigma+\rho<\lambda}[p_\sigma][q_\rho]$$

we obtain the speceral system $[r_\lambda]$ $(-\infty<\lambda<+\infty)$ *of* $b+c$ *by* a.

Proof. Let $[r'_\lambda]$ $(-\infty<\lambda<+\infty)$ be the spectral system of $b+c$ by a. We have by assumption and Lemma 1. (#)

$[p_\sigma][q_\rho]\leqq[r_\lambda]$ for $\sigma+\rho<\lambda$ and $[r'_\lambda]\geqq[r_\lambda]$

For any real number $\varepsilon>0$, we have by assumption

$$[a]=\bigcup_{\nu=-\infty}^{+\infty}(1-[p_{(\nu-1)\varepsilon}])[p_{\nu\varepsilon}]$$

for a real number δ such that $\lambda-\varepsilon<\delta<\lambda$ since $\lambda-2\varepsilon<(\nu-1)\varepsilon+\delta-\nu\varepsilon$ we have hence by Lemma 1. (##)

$$\begin{aligned}[r'_{\lambda-2\varepsilon}]&=\bigcup_{\nu=-\infty}^{+\infty}[r'_{\lambda-2\varepsilon}](1-[p_{(\nu-1)\varepsilon}])[p_{\nu\varepsilon}]\\&\leqq\bigcup_{\nu=-\infty}^{+\infty}[q_{\delta-\nu\varepsilon}][p_{\nu\varepsilon}]\leqq\bigcup_{\sigma+\rho<\lambda}[p_\sigma][q_\rho]=[r_\lambda]\end{aligned}$$

Therefore we conclude $[r'_\lambda]\leqq[r_\lambda]$ for every real number λ since $[r']$ $(-\infty<\lambda<+\infty)$ is left side continuous.

Note

(1) A monotone increasing system $[p_\lambda]$ $(-\infty<\lambda<+\infty)$ is called a resolution of projecter $[p]$ if

(*a*) $[p_\lambda]=\bigcup_{\rho<\lambda}[p_\rho]$ for every real number λ

(*b*) $[p]=\bigcup_{-\infty<\lambda<+\infty}[p_\lambda]$

(2) In continuous semi-ordered linear space, for an element $p\in R$ the projecter $[p]$ is define such that

$$[p]a=\bigcup_{\nu=1}^{\infty}\nu|p|\cap a^+-\bigcup_{\nu=1}^{\infty}\nu|p|\cap a^-$$

For two projecter $[p]$ and $[q]$ we define $[p]\geqq[q]$ if and only if $[p]a\geqq[q]a$ for every positive element $a \in R$. Totality of all projecters constitutes a σ-conditionary distributive lattice with 0 and $[p] \cap [q] = [p][q]$, $[p] \cup [q] = [|p| + |q|]$

(3) See [1]. § 9.

Reference

[1] Nakano, H.: Modern Spectral Theory, Tokyo Math. Book Ser. *Vol. 2* (1950)

[2] Nakano, H.: Modulared Semi-ordered Linear Space Tokyo Math. Book Ser. *Vol. 1* (1950)