Note on Modulared Function Spaces

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Introduction

In 1950, H. Nakano [3] constructed, on a finite measure space[1], the modulared function space[2] as a concrete example of abstract modulared semi-ordered linear spaces which was defined also by the same author, and showed further conversely that an arbitrary abstract modulared semi-ordered linear space could be represented by a product space, in a sense, of such defined modulared function spaces.

However, if we define them on a locally finite measure space, we have more general modulared function space. And then, any abstract modulared space is, now, considered to be equivalent to one of latter modulared function spaces. In this note, we give detailed proofs to the process of definition of this modulared function space (§ 1), and do to the construction of its complementary space (2. 1 in § 2) as well as to the possibility of integral representation of the modulared conjugate space (2. 2 in § 2), founded on the theory constructed by H. Nakano. And we add a sufficient condition for completely continuity of an integral operator as an application of this integral representation (§ 3).

On the space of all the measurable functions defined on a measure space, if we define an order-relation by such a manner that it will appear in 1. 1 of § 1, the space get the structure of the semi-ordered linear space which is of super-universal continuity or of universal continuity[3] according as the measure space being of finiteness or of local finiteness respectively.

§ 1. Modulared function spaces.

1. 1. The definition of modulared function spaces.

Let Ω be an abstract space and μ be a complete[4] σ-additive measure defined on a Borel field $\mathfrak{B}$ of subsets of Ω satisfying local finiteness :

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1) For measure and integration, see, for instance, H. Nakano [4].

2) Terminologies and notations used in this note are due to H. Nakano [3].

3) Semi-ordered linear space R is said to be continuous if for any sequence $0 \leqq x_\nu \in R\,(\nu=1, 2, \cdots)$ there exists $\bigcap_{\nu=1}^{\infty} x_\nu$, R is said to be universally continuous if for any system $0 \leqq x_\lambda \in R\,(\lambda \in \Lambda)$ there exists $\bigcap_{\lambda \in \Lambda} x_\lambda$ and R is said to be super-universally continuous if R is universally continuous and further for any $0 \leqq x_\lambda \in R\,(\lambda \in \Lambda)$ there exists a sequence $\lambda_\nu \in \Lambda\,(\nu=1, 2, \cdots)$ such that $\bigcap_{\lambda \in \Lambda} x_\lambda = \bigcap_{\nu=1}^{\infty} x_{\lambda_\nu}$.

4) $A \in \mathfrak{B}$ is said to be locally negligible if $\mu(E \cap A)=0$ for all $E \in \mathfrak{B}$ with $\mu(E) < \infty$. μ is said to be complete if, for any $E \subset \Omega$, $E \subset A$ for some locally negligible $A \in \mathfrak{B}$ implies $E \in \mathfrak{B}$.

μ 1) $\cup\{E;\ \mu(E)<\infty,\ E\in\mathfrak{B}\}=\Omega$;

μ 2) for any $F\subset\Omega$, $F\cap E\in\mathfrak{B}$ for all $E\in\mathfrak{B}$ with $\mu(E)<\infty$ implies $F\in\mathfrak{B}$.

Let $\Phi(\xi,\omega)$ $(\xi\geqq 0,\ \omega\in\Omega)$ be a function satisfying the following conditions:

Φ 1) $0\leqq\Phi(\xi,\omega)\leqq\infty$ for all $\xi\geqq 0,\ \omega\in\Omega$;

Φ 2) $\Phi(\xi,\omega)$ is a measurable[5] function on Ω for all $\xi\geqq 0$;

Φ 3) $\Phi(\xi,\omega)$ is a non-decreasing convex functions of $\xi\geqq 0$ for all $\omega\in\Omega$;

Φ 4) $\Phi(0,\omega)=0$ for all $\omega\in\Omega$;

Φ 5) $\Phi(\xi-0,\omega)=\Phi(\xi,\omega)$ for all $\omega\in\Omega$;

Φ 6) $\Phi(\xi,\omega)\to\infty$ as $\xi\to\infty$ for all $\omega\in\Omega$;

Φ 7) for any $\omega\in\Omega$, there exists $\xi_\omega>0$ such that $\Phi(\xi_\omega,\omega)<\infty$.

(Instead of writing $\Phi(|\xi|,\omega)$, we denote merely $\Phi(\xi,\omega)$ for $\xi<0$, if there is no confusion in the following).

For any measurable function $x(\omega)$ $(\omega\in\Omega)$, $\Phi(x(\omega),\omega)$ is measurable too. We denote by L_Φ the system of all measurable functions $x(\omega)$ $(\omega\in\Omega)$ satisfying

$$\int_\Omega \Phi(\alpha\, x(\omega),\omega)\ d\mu(\omega)<\infty \qquad \text{for some } \alpha=\alpha_x>0\,.$$

Here, $\int_\Omega\cdot d\mu=\sup\left\{\int_E\cdot d\mu;\ \mu(E)<\infty,\ E\in\mathfrak{B}\right\}$ where $\int_E\cdot d\mu$ is ordinary Lebesgue integral on E. We write '$x(\omega)\leqq y(\omega)$ a.e. (almost everywhere) on E' if $x(\omega)\leqq y(\omega)$ on a measurable set E except for some locally negligible set A[6] $(\subset E)$. We define $x\leqq y$ for any $x,\ y\in L_\Phi$ if $x(\omega)\leqq y(\omega)$ a.e. on Ω.

First we show that L_Φ *constitutes a universally continuous semi-ordered linear space* by above introduced order.

In fact, L_Φ is linear, because, for any $x,\ y\in L_\Phi$

$$\Phi(x(\omega)+y(\omega),\omega)\leqq\frac{1}{2}\Phi(x(\omega),\omega)+\frac{1}{2}\Phi(y(\omega),\omega)$$

by Φ 3) which implies

$$\int_\Omega \Phi(\alpha(x(\omega)+y(\omega)),\omega)\ d\mu<\infty \qquad \text{for some } \alpha>0\,.$$

L_Φ is a lattice because $x\in L_\Phi$ implies $|x|\in L_\Phi$ and $L_\Phi\ni x,\ x\geqq y$ implies $y\in L_\Phi$.

Thus, L_Φ is a semi-ordered linear space.

L_Φ is continuous, because for a system $0\leqq x_\nu,\ x\in L_\Phi$ $(\nu=1,2,\cdots)$, if $x_\nu\leqq x$ $(\nu=1,2,\cdots)$ then $\sup_{\nu\geqq 1} x_\nu(\omega)\leqq x_\nu(\omega)$ a.e. on Ω and so $\bigcup_{\nu=1}^\infty x_\nu\in L_\Phi$.

In a continous semi-ordered linear space R, we define the projector $[x]$ $(x\in R)$ by

$$[x]\,y=\bigcup_{\nu=1}^\infty(\nu\,|x|\cap y) \qquad \text{for any } 0\leqq y\in R\,.$$

We denote by χ_E or χ_x the characteristic function of the point set E or $\{\omega;\ x(\omega)\neq 0\}$

5) A function $f(\omega)$ $(\omega\in\Omega)$ is said to be measurable on Ω if $\{\omega;\ f(\omega)\leqq\alpha\}\cap E\in\mathfrak{B}$ for all real number α and for all $E\in\mathfrak{B}$ with $\mu(E)<\infty$.

6) See 4).

in Ω for a measurable function $x(\omega)$ $(\omega\in\Omega)$ respectively. We see, for the projector $[x]$ of $x\in L_\Phi$, by definition,

$$[x]\,y=\chi_x y \qquad \text{for all } y\in L_\Phi$$

under the convention $(\pm\infty)\cdot 0=0$.

In order to show universal continuity of L_Φ, we use the following theorem (see H. Nakano [3 ; Th. 13. 1]).

(T. 1) *In a continuous semi-ordered linear space R, if, for any system of projectors* $[x_\lambda]$ $(\lambda\in\Lambda)$, *there exists* $\bigcup_{\lambda\in\Lambda}[x_\lambda]$ *then R is universally continuous.*

L_Φ is univerally cotinuous: We need only show that for the system of measurable sets E_λ, $E\in\mathfrak{B}$ $(\lambda\in\Lambda)$ satisfying $E_\lambda\subset E$ $(\lambda\in\Lambda)$, $\bigcup_{\lambda\in\Lambda}E_\lambda$ belongs to $\mathfrak{B}$, by the theorem (T. 1). Here, we may regard E_λ $(\lambda\in\Lambda)$ as a directed system $E_\lambda\uparrow_{\lambda\in\Lambda}$ with $E_\lambda\subset E$ $(\lambda\in\Lambda)$ considering the system of all the finite unions of the original system.
First let $\mu(E)<\infty$ and $\gamma=\sup_{\lambda\in\Lambda}\mu(E_\lambda)\leqq\mu(E)<\infty$. We choose a countable system E_{λ_ν} $(\nu=1, 2, \cdots)$ from $\{E_\lambda\}$ such that

$$E_{\lambda_\nu}\uparrow_{\nu=1}^{\infty} \qquad \text{and} \qquad \lim_{\nu\to\infty}\mu(E_{\lambda_\nu})=\gamma\,.$$

Putting also $E_0=\bigcup_{\nu=1}^{\infty}E_{\lambda_\nu}$ we have $E_0\in\mathfrak{B}$ and $\mu(E_0)=\gamma$ because $E_{\lambda_\nu}\subset E$ and $E_{\lambda_\nu}\uparrow_{\nu=1}^{\infty}E_0$. If there exists $\delta>0$ such that $\mu(E_{\lambda_0}-E_0)=\delta$ for some $\lambda_0\in\Lambda$ then

$$\gamma\geqq\mu(E_{\lambda_0}\cup E_{\lambda_\nu})\geqq\mu(E_{\lambda_\nu})+\delta \qquad \text{for all } \lambda_\nu>\lambda_0\,.$$

Therefore E_0 includes E_λ except for some measurable set with null measure for all $\lambda\in\Lambda$. And since any measurable set including E_λ for all $\lambda\in\Lambda$ includes E_0 except on some locally negligible set, $\bigcup_{\lambda\in\Lambda}E_\lambda$ must belong to $\mathfrak{B}$. Next let $\mu(E)=\infty$, the general case. Let $E^{(\sigma)}$ $(\sigma\in\Sigma)$ be the largest subsystem of $\mathfrak{B}$ such that $E^{(\sigma)}\subset E$ with $\mu(E^{(\sigma)})<\infty$ for all $\sigma\in\Sigma$, then $E=\bigcup_{\sigma\in\Sigma}E^{(\sigma)}$ by μ 1). Putting $F_\sigma\equiv\bigcup_{\lambda\in\Lambda}(E_\lambda\cap E^{(\sigma)})$ for all $\sigma\in\Sigma$ and $F=\bigcup_{\sigma\in\Sigma}F_\sigma$, we have $F_\sigma\in\mathfrak{B}$ with $\mu(F^{(\sigma)})<\infty$ for all $\sigma\in\Sigma$ by the same way stated above because $E_\lambda\cap E^{(\sigma)}\uparrow_{\lambda\in\Lambda}$ and $E_\lambda\cap E^{(\sigma)}\subset E^{(\sigma)}$ for all $\lambda\in\Lambda$. Therefore we see, for any $\sigma\in\Sigma$, $F\cap E^{(\sigma)}=F_\sigma\,(\in\mathfrak{B})$ which implies $F\,(=\bigcup_{\lambda\in\Lambda}E_\lambda)\in\mathfrak{B}$ by μ 2). Thus L_Φ is universally continuous.

Next we show that L_Φ contains, in a sense, in plenty of non-trivial elements. That is ;

(A) *for any* $E\in\mathfrak{B}$ *with* $\mu(E)<\infty$ *and for any* $\varepsilon>0$ *there exists* $E_0=E(\varepsilon)\in\mathfrak{B}$ *such that* $\chi_{E_0}\in L_\Phi$ *with*

$$E_0\subset E \qquad \textit{and} \qquad \mu(E_0)\geqq\mu(E)-\varepsilon\,.$$

In fact, putting

$$(*) \qquad \sup\{\xi\,;\,\Phi(\xi,\omega)<\infty\}=e_1(\omega) \qquad (\omega\in\Omega)\,,$$

$e_1(\omega)$ is measurable because

$$\{\omega;\ e_1(\omega)\leqq\eta\}=\bigcup_{\eta'<\eta}\{\omega;\ \Phi(\eta',\omega)<\infty\}\qquad\text{for every } \eta\geqq 0$$

and $0<e_1(\omega)\leqq\infty$ by Φ 7).

Then for any $E\in\mathfrak{B}$ with $\mu(E)<\infty$ we have

$$E\cap\left\{\omega;\ \frac{1}{n}<e_1(\omega)\right\}\equiv E_n\uparrow_{n=1}^{\infty} E$$

and so, for any $\varepsilon>0$, we can select n_0 such that

$$\mu(E_{n_0})\geqq\mu(E)-\frac{\varepsilon}{2}$$

for which $\Phi\left(\frac{1}{n_0}\chi_{E_0}(\omega),\omega\right)<\infty\ (\omega\in\Omega)$ by (*).

For this E_{n_0}, we have

$$E_{n_0}\cap\left\{\omega;\ \Phi\left(\frac{1}{n_0},\omega\right)\leqq p\right\}\equiv E_{n_0}^{(p)}\uparrow_{p=1}^{\infty} E_0$$

and also, for any $\varepsilon>0$, there exists p_0 such that

$$\mu(E_{n_0}^{(p_0)})\geqq\mu(E_{n_0})-\frac{\varepsilon}{2}.$$

Therefore $\mu(E_{n_0}^{(p_0)})\geqq\mu(E)-\varepsilon$, furthermore

$$\int_\Omega\Phi\left(\frac{1}{n_0}\chi_{E_{n_0}}^{(p_0)}(\omega),\ \omega\right)d\mu=\int_{E_{n_0}^{(p_0)}}\Phi\left(\frac{1}{n_0},\ \omega\right)d\mu\leqq$$

$$\leqq p_0\mu(E_{n_0}^{(p_0)})<\infty\,.$$

Thus $E_{n_0}^{(p_0)}$ is E_0 in this problem.

Putting

$$m_\Phi(x)=\int_\Omega\Phi(x(\omega),\omega)\,d\mu\qquad\textit{for all}\quad x\in L_\Phi$$

m_Φ satisfies the following properties, so-called modular conditions. (m_Φ will be sometimes denoted merely by m if there is no confusion in the following).

m 1) $0\leqq m(x)\leqq\infty$ for all $x\in L_\Phi$;

m 2) if $m(\xi x)=0$ for all $\xi\geqq 0$ then $x=0$;

m 3) for any $x\in L_p$ there exists $\alpha_x>0$ such that $m(\alpha_x x)<\infty$;

m 4) for every $x\in L_p$, $m(\xi x)$ is a convex function of $\xi\geqq 0$;

m 5) $|x|\leqq|y|\ (x,\ y\in L_\Phi)$ implies $m(x)\leqq m(y)$;

m 6) $|x|\cap|y|=0\ (x,\ y\in L_p)$ implies $m(x+y)=m(x)+m(y)$;

m 7) $0\leqq x_\lambda\uparrow_{\lambda\in\Lambda}x\ (x_\lambda,\ x\in L_\Phi)$ implies $m(x)=\sup_{\lambda\in\Lambda}m(x_\lambda)$.

In fact, m 1) or m 3) is clear by the definition of m or L_p respectively.

m 2): If $m(\xi x)=\int_\Omega\Phi(\xi x(\omega),\omega)\,d\mu=0$ for all $\xi\geqq 0$, then $\Phi(\xi x(\omega),\omega)=0$ except on some locally negligible set E_ξ for all $\xi\geqq 0$, and therefore it also holds except on the locally negligible set $\bigcup_{\xi\geqq 0}E_\xi$. Thus $x(\omega)=0$ a.e. on Ω by Φ 6).

$m\,3)$ and $m\,5)$ are immediate consequences of convexity and monotony of $\Phi(\xi, \omega)$ as a function of $\xi \geqq 0$, respectively.

$m\,6)$: Putting $E_x \equiv \{\omega\,;\, x(\omega) \neq 0\}$ and $E_y \equiv \{\omega\,;\, y(\omega) \neq 0\}$, $|x| \cap |y| = 0$ implies $\mu(E_x \cap E_y) = 0$, and so

$$\int_\Omega \Phi(x(\omega) + y(\omega), \omega)\, d\mu = \int_{E_x} \Phi(x(\omega), \omega)\, d\mu + \int_{E_y} \Phi(y(\omega), \omega)\, d\mu\,.$$

$m\,7)$: $0 \leqq x_\lambda(\omega) \uparrow_{\lambda \in \Lambda} x(\omega)$ implies

$$\Phi(x_\lambda(\omega), \omega) \uparrow_{\lambda \in \Lambda} \Phi(x(\omega), \omega) \qquad \text{a.e. on } \Omega\,.$$

Under such an assumption,

$$\int_\Omega \Phi(x(\omega), \omega)\, d\mu = \sup_{\lambda \in \Lambda} \int_\Omega \Phi(x_\lambda(\omega), \omega)\, d\mu$$

is valid by the theory of Lebesgue integral.

A functional on a universally continuous semi-ordered linear space satisfying from $m\,1)$ to $m\,7)$ is called a *modular*. The space L_Φ with a modular $m_\Phi(x)$ $(x \in L_\Phi)$ is called a *modulared function space* by Φ and is sometimes denoted by (L_Φ, m_Φ).

1.2. Monotone completeness of m_Φ.

(B) *m_Φ is monotone complete; i.e. for $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} (x_\lambda \in L_\Phi)$ with $\sup_{\lambda \in \Lambda} m(x_\lambda) < \infty$, there exists $\bigcup_{\lambda \in \Lambda} x_\lambda$ in L_Φ.*

In fact, putting

$$(\sharp) \qquad f(\omega) = \sup_{\lambda \in \Lambda} \Phi(x_\lambda(\omega), \omega)\,.$$

we have $\int_\Omega f(\omega)\, d\mu < \infty$ by theory of Lebesgue integral and so $f(\omega) < \infty$ a.e. on Ω. We choose a sequence $\lambda_\nu \in \Lambda$ $(\nu = 1, 2, \cdots)$ such that

$$x_{\lambda_\nu} \uparrow_{\nu=1}^\infty \quad \text{and} \quad \Phi(x_{\lambda_\nu}(\omega), \omega) \uparrow_{\nu=1}^\infty f(\omega) \qquad \text{a.e. on } \Omega$$

so $x_0(\omega) \equiv \sup_{\nu \geqq 1} x_{\lambda_\nu}(\omega) \in L_\Phi$.

Here

$$(\sharp\sharp) \qquad \Phi(x_0(\omega), \omega) = f(\omega) \qquad \text{a.e. on } \Omega\,.$$

Putting

$$(\sharp\sharp\sharp) \qquad e_0(\omega) = \sup\{\xi\,;\, \Phi(\xi, \omega) = 0\} \qquad (\omega \in \Omega)\,,$$

then $e_0(\omega)$ is measurable because

$$\{\omega\,;\, e_0(\omega) \geqq \eta\} = \bigcap_{\eta' < \eta} \{\omega\,;\, \Phi(\eta', \omega) = 0\} \qquad \text{for every } \eta \geqq 0,$$

and further $e_0 \in L_\Phi$ because $\Phi(e_0(\omega), \omega) = 0$ a.e. on Ω.

Since $\Phi(x_\lambda(\omega), \omega) \leqq \Phi(x_0(\omega), \omega)$ a.e. on Ω $(\lambda \in \Lambda)$ by $(\sharp)$ and $(\sharp\sharp)$, we have

$$x_\lambda(\omega) \leqq x_0(\omega) \qquad \text{a.e. on } \{\omega\,;\, \Phi(x_0(\omega), \omega) \neq 0\}$$

by $\Phi\,3)$ and

$$x_\lambda(\omega) \leqq e_0(\omega) \qquad \text{a.e. on } \{\omega\ ,\ \Phi(x_0(\omega),\ \omega)=0\}$$

by (###). That is

$$x_\lambda(\omega) \leqq x_0(\omega)+e_0(\omega) \qquad \text{a.e. on } \Omega\ (\lambda \in \Lambda)\ .$$

Thus for an upper bounded system $x_\lambda\ (\lambda \in \Lambda)$, there exists $\bigcup_{\lambda \in \Lambda} x_\lambda$ in L_Φ.

§ 2. Modulared conjugate space of $(L_\Phi,\ m_\Phi)$.

2.1. Complementary modulared function space of $(L_\Phi,\ m_\Phi)$.

Putting, for all $\omega \in \Omega$

$$\pi(\xi,\ \omega)=\begin{cases} \lim\limits_{\nu\to\infty} \nu\left\{\Phi(\xi,\ \omega)-\Phi\left(\xi-\dfrac{1}{\nu},\ \omega\right)\right\} & (\xi>0) \\ \lim\limits_{\xi\to+0} \pi(\xi,\ \omega) & (\xi=0)\ , \end{cases}$$

$\pi(\xi,\ \omega)$ is measurable on Ω with a parameter $\xi \geqq 0$ as well as is non-decreasing and left-continuous as a function of $\xi \geqq 0$. Here, the measurability is deduced by that $\pi(\xi,\ \omega)$ is a limit function of measurable functions, non-decreasingness by $\Phi\ 3)$ on Ω and left-continuity by that $\pi(\xi,\ \omega)$ is the left-side derivative of the convex function $\Phi(\xi,\ \omega)$. (For detailed properties of convex functions, see M. A. Krasnoselskii and Y. B. Rutickii [2]).

By inverse operation, we have

$$(\times) \qquad \Phi(\xi,\ \omega)=\int_0^\xi \pi(\xi',\ \omega)\ d\xi' \qquad (\omega \in \Omega)\ .$$

Let $\bar{\pi}(\eta,\ \omega)$ be the left-continuous inverse function of $\pi(\xi,\ \omega)$ in the sense of W. Young; i.e.

$$(\times\times) \qquad \bar{\pi}(\eta,\ \omega)=\sup\{\xi\ ;\ \pi(\xi,\ \omega)<\eta\}$$

then $\bar{\pi}(\eta,\ \omega)$ is also measurable on Ω with a parameter $\xi \geqq 0$ and is non-decreasing for $\xi \geqq 0$ too, where the measurability is due to the relation

$$\{\omega\ ;\ \bar{\pi}(\eta,\ \omega)<\xi\}=\bigcap_{\xi>\xi'}\{\omega\ ;\ \pi(\xi',\ \omega)\leqq\eta\} \qquad \text{for all } \xi \geqq 0$$

and the non-decreasingness is known by that $\eta_1<\eta_2$ implies

$$\bar{\pi}(\eta_1,\ \omega)=\sup\{\xi\ ;\ \pi(\xi,\ \omega)<\eta_1\} \leqq$$
$$\leqq \sup\{\xi\ ;\ \pi(\xi,\ \omega)<\eta_2\}=\pi(\eta_2,\ \omega)\ .$$

It is known (M. A. Krasnoselskii and Y. B. Rutickii [2]) that for all $\omega \in R$

$$\bar{\pi}(\eta-0,\ \omega)\leqq\xi\leqq\bar{\pi}(\eta+0,\ \omega) \qquad \text{for } \eta=\pi(\xi,\ \omega)$$
$$\pi(\xi-0,\ \omega)\leqq\eta\leqq\pi(\xi+0,\ \omega) \qquad \text{for } \xi=\bar{\pi}(\eta,\ \omega)\ .$$

The complementary function $\bar{\Phi}(\eta,\ \omega)\ (\eta \geqq 0,\ \omega \in \Omega)$ of Φ is defined by

$$\bar{\Phi}(\eta,\ \omega)=\int_0^\eta \bar{\pi}(\eta',\ \omega)\ d\eta' \qquad (\eta \geqq 0,\ \omega \in \Omega)\ .$$

With respect to $\bar{\Phi}$, all the corresponding properies from $\Phi\ 1)$ to $\Phi\ 7)$; i.e. from

$\bar{\Phi}$ 1) to $\bar{\Phi}$ 7), are valid too. For $\bar{\Phi}$ 1) and from $\bar{\Phi}$ 3) to $\bar{\Phi}$ 6), these are immediate consequences of the definition of $\bar{\Phi}$. For $\bar{\Phi}$ 2); for every fixed $\eta \geqq 0$, the measurability of $\bar{\Phi}(\eta, \omega)$ with respect to $\omega \in \Omega$ is due to the fact that it is a limit function of finite combinations of measurable functions by the definition of $\bar{\Phi}$. For $\bar{\Phi}$ 7); because for any $\omega \in \Omega$, there exists $\xi_\omega > 0$ such that $\Phi(\xi_\omega, \omega) < \infty$ by Φ 7), we can find $\alpha_\omega > 0$ such that $\xi_\omega \geqq \alpha_\omega > 0$ with $\pi(\alpha_\omega, \omega) < \infty$ by $(\times)$, and so $\bar{\pi}(\pi(\alpha_\omega, \omega), \omega) \leqq \alpha_\omega$ by $(\times\times)$. Thus $\bar{\Phi}(\pi(\alpha_\omega, \omega), \omega) = \int_0^{\pi(\alpha_\omega, \omega)} \bar{\pi}(\eta, \omega)\, d\eta \leqq \alpha_\omega \pi(\alpha_\omega, \omega) < \infty$.

We have so-called W. Young's inequality (M. A. Krasnoselskii and Y. B. Rutickii [2]) for $\xi, \eta \geqq 0$

$$(Y_1) \qquad \xi\eta \leqq \Phi(\xi, \omega) + \bar{\Phi}(\eta, \omega) \qquad (\omega \in \Omega),$$

in particular

$$(Y_2) \qquad \alpha\beta = \Phi(\alpha, \omega) + \bar{\Phi}(\beta, \omega) \qquad (\omega \in \Omega)$$

for

$$\pi(\alpha - 0, \omega) \leqq \beta \leqq \pi(\alpha + 0, \omega) \quad \text{and} \quad \bar{\pi}(\beta - 0, \omega) \leqq \alpha \leqq \bar{\pi}(\beta + 0, \omega).$$

$\bar{\bar{\Phi}}(\xi, \omega)$ coincides with $\Phi(\xi, \omega)$, because the process $\bar{\Phi} \to \bar{\pi} \to \bar{\bar{\pi}} \to \bar{\bar{\Phi}}$ is equivalent to that of $\bar{\Phi} \to \bar{\pi} \to \pi \to \Phi$.

For $\bar{\Phi}(\eta, \omega)$ $(\eta \geqq 0, \omega \in \Omega)$, we obtain also a universally continuous semi-ordered linear space $L_{\bar{\Phi}}$ with the monotone complete modular $m_{\bar{\Phi}}$:

$$m_{\bar{\Phi}}(a) = \int_\Omega \bar{\Phi}(a(\omega), \omega)\, d\mu \qquad (a \in L_{\bar{\Phi}})$$

by the same method we defined (L_Φ, m_Φ). $(L_{\bar{\Phi}}, m_{\bar{\Phi}})$ is called the *complementary modulared function space* of (L_Φ, m_Φ).

2.2. Integral representation of modulared conjugate space of (L_Φ, m_Φ).

We can consider the modulared conjugate space (H. Nakano [3 ; § 38]) $(\bar{L}_\Phi^m, \bar{m})$[7] of (L_Φ, m); i.e. $\bar{L}_\Phi^m$ is such a normal subspace of the conjugate space $\bar{L}$, which is the system of all universally continuous linear functional $u(x)$ $(x \in L_\Phi)$ that satisfying

$$\sup\{|u(x)| ;\ m(x) \leqq 1,\ x \in L_\Phi\} < \infty$$

and in which a modular $\bar{m}$ is defined as

$$◎) \qquad \bar{m}(\bar{a}) = \sup_{x \in L_\Phi} \{\bar{a}(x) - m(x)\} \qquad (\bar{a} \in \bar{L}_\Phi^m).$$

Now we prove that

(C) *the modular conjugate space* $(\bar{L}_\Phi^m, \bar{m})$ *of* (L_Φ, m) *is isometric to the complementary modulared function space* $(L_{\bar{\Phi}}, m_{\bar{\Phi}})$ *of* (L_Φ, m). And consequently, this fact could be considered that $(L_{\bar{\Phi}}, m_{\bar{\Phi}})$ *gives an integral representation of* $(\bar{L}_\Phi^m, \bar{m})$.

We proceed to prove by deviding it into the three steps:

7) In this paragraph, m and $\bar{m}$ mean also m_Φ and $\bar{m}_\Phi$ respectively.

2.2.1.

We cite the following theorem (H. Nakano [3; Th. 22.7]) which plays an essential rôle on the conjugate space $\bar{L}_\Phi$.

(T. 2) *If for two universally continuous semi-ordered linear spaces R and $\hat{R}$, a bilinear functional*

$$(x, \hat{a}) \qquad \text{for } x\in R,\ \hat{a}\in\hat{R}$$

is defined such that

c 1) *bilinearity*;

$$(\alpha x+\beta y,\ \hat{a})=\alpha(x,\ \hat{a})+\beta(y,\ \hat{a})$$
$$(x,\ \alpha\hat{a}+\beta\hat{b})=\alpha(x,\ \hat{a})+\beta(x,\ \hat{b}),$$

c 2) *positivity*;

$$(x,\ \hat{a})\geqq 0 \qquad \text{for } x\geqq 0,\ \hat{a}\geqq 0,$$

c 3) *universal continuity*;

$$x_\lambda\downarrow_{\lambda\in\Lambda}0 \text{ implies } \inf_{\lambda\in\Lambda}(x_\lambda,\ \hat{a})=0 \qquad \text{for } \hat{a}\geqq 0$$
$$\hat{a}_\lambda\downarrow_{\lambda\in\Lambda}0 \text{ implies } \inf_{\lambda\in\Lambda}(x,\ \hat{a}_\lambda)=0 \qquad \text{for } x\geqq 0,$$

c 4) *for any positive $\hat{a}>0$, there exists $x\geqq 0$ for which*

$$(x,\ \hat{a})\geqq 0 \quad \text{and} \quad (x,\ \hat{b})=0 \text{ for } \hat{b}\cap\hat{a}=0,$$

c 5) *for any positive $x\geqq 0$ and $\hat{a}\geqq 0$, there exists $\hat{b}\geqq 0$ such that*

$$([x]\,y,\ \hat{a})=(y,\ \hat{b}) \qquad \text{for all } y\in R,$$

c 6) *for any positive $x>0$, there exists $\hat{a}\geqq 0$ such that $(x,\ \hat{a})>0$*

then $\hat{R}$ is isomorphic to a complete semi-normal manifold [8] *of the conjugate space $\bar{R}$ of R by the correspondance*

$$\hat{R}\ni\hat{a}\to\hat{a}^R\in\bar{R},\quad \hat{a}^R(x)=(x,\ \hat{a}) \qquad \text{for all } x\in R.$$

For the modulared function space L_Φ and for its complementary modulared function space $L_{\bar{\Phi}}$, we put

$$(x,\ a)=\int_\Omega x(\omega)\ a(\omega)\ d\mu \qquad (x\in L_\Phi,\ a\in L_{\bar{\Phi}})$$

where the existance of the integral in the right term is lead by (Y_1) and m 3). Then we have a bilinear functional $(x,\ a)$ on $(L_\Phi,\ L_{\bar{\Phi}})$ satisfying from c 1) to c 6). In fact, c 1), c 2) and c 3) are evident.

c 4): For any $0<a\in L_{\bar{\Phi}}$ with $0<\mu(E)<\infty$ where $E=\{\omega;\ a(\omega)\neq 0\}$, there exists $E_1\in\mathfrak{B}$ such that

$$E\supset E_1,\ 0<\mu(E_1)<\infty \quad \text{and} \quad \chi_{E_1}\in L_\Phi$$

---

8) A linear manifold M of a semi-ordered linear space R is said to be complete semi-normal if for any $y\in R$ with $|x|\cap|y|=0$ for all $x\in M$ implies $y=0$ and if $|x|\geqq|y|$, $x\in M$ implies $y\in M$.

by (A). Therefore

$$\int_{\Omega}\chi_{E_1}(\omega)\ a(\omega)\ d\mu>0 \quad\text{and}\quad \int_{\Omega}\chi_{E_1}(\omega)\ b(\omega)\ d\mu=0$$

for $b\in L_{\bar{\Phi}}$ with $a\cap b=0$.

$c\,5)$: For any $0\leqq x\in L_{\Phi}$ and $0\leqq a\in L_{\Phi}$ putting

$$E=\{\omega\ ;\ x(\omega)=0\} \quad\text{and}\quad b(\omega)=\chi_E\, a(\omega)$$

then $b\in L_{\bar{\Phi}}$ by $|a|\geqq|b|$. Therefore

$$\int_{\Omega}(\chi_E(\omega)\ y(\omega))\ a(\omega)\ d\mu=\int_{\Omega}y(\omega)\ b(\omega)\ d\mu$$

for all $y\in L_{\bar{\Phi}}$.

$c\,6)$: For any $0\neq x\in L_{\Phi}$, putting $E=\{\omega\ ;\ x(\omega)\neq 0\}$ we can find $E_1\in\mathfrak{B}$ similarly to $c\,4)$ such that

$$E\supset E_1 \qquad\text{with}\qquad 0<\mu(E_1)\leqq\mu(E)$$

and $a(\omega)\equiv\chi_{E_1}\in L_{\Phi}$.
Therefore

$$\int_{\Omega}x(\omega)\ \chi_{E_1}(\omega)\ d\mu=\int_{E}a(\omega)\ d\mu>0\,.$$

Thus, by the theorem (T. 2), $L_{\bar{\Phi}}$ could be considered as a complete semi-normal manifold of $\bar{L}_{\Phi}$ up to isomorphism.

2. 2. 2.

Here we prove that on $L_{\bar{\Phi}}\,(\subset\bar{L}_{\Phi})$, $m_{\bar{\Phi}}$ coincides with $\bar{m}$; i.e.

$$(+)\qquad m_{\bar{\Phi}}(a)=\bar{m}(a)\qquad (a\in L_{\bar{\Phi}})\,.$$

For this purpose, it suffices to show only

$$(++)\qquad m(x)=\sup_{a\in L_{\bar{\Phi}}}\{(x,\ a)-m_{\bar{\Phi}}(a)\}\qquad (x\in L_{\Phi})$$

because $(++)$ deduces $(+)$ by the relation Φ and $\bar{\Phi}$ in 2. 1. and by ◎).

For the proof, we classfy the elements of $L_{\bar{\Phi}}$ into following four cases.

1_1. On the case $0\leqq x\in L_{\Phi}$ is domestic ; i.e. $m(\alpha x)<\infty$ for some $\alpha>1$.

Putting $a(\omega)=\pi(x(\omega),\ \omega)$ then $a\in L_{\Phi}$, because we have, for domestic $x\geqq 0$,

$$\sup_{0\leqq\xi<1}\{(m(x)-m(\xi\, x))/(1-\xi)\}<\infty$$

by $m\,4)$ and consequently

$$\int_{\Omega}x(\omega)\ \pi(x(\omega),\ \omega)\ d\mu=$$

$$=\int_{\Omega}\{\sup_{0\leqq\xi<1}\{(\Phi(x(\omega),\ \omega)-\Phi(\xi\, x(\omega),\ \omega))/(1-\xi)\ x(\omega)\}\ x(\omega)\ d\mu<\infty\,.$$

By (Y_2), we have

$$x(\omega)\ a(\omega)=\Phi(x(\omega),\ \omega)+\bar{\Phi}(a(\omega),\ \omega)$$

and so

$$\infty > \int_\Omega x(\omega)\, a(\omega)\, d\mu = \int_\Omega \Phi(x(\omega), \omega)\, d\mu + \int_\Omega \bar{\Phi}(a(\omega), \omega)\, d\mu .$$

Therefore there exists $a \in L_{\bar{\Phi}}$ such that

$$m(x) = (x, a) - m_{\bar{\Phi}}(a) \quad (x \in L_\Phi) .$$

1_2. On the case $0 \leqq x \in L_\Phi$ with $m(x) < \infty$.

In this case, every ξx with $0 \leqq \xi < 1$ is domestic and $\xi x \uparrow_{0 \leqq \xi < 1} x$. Therefore, by 1_1

$$m(x) = \sup_{0 \leqq \xi < 1} m(\xi x) = \sup_{0 \leqq \xi < 1} \sup_{0 \leqq a \in L_{\bar{\Phi}}} \{(\xi x, a) - m_{\bar{\Phi}}(a)\} =$$

$$= \sup_{0 \leqq a \in L_{\bar{\Phi}}} \sup_{0 \leqq \xi < 1} \{(\xi x, a) - m_{\bar{\Phi}}(a)\} = \sup_{0 \leqq a \in L_{\bar{\Phi}}} \{(x, a) - m_{\bar{\Phi}}(a)\} .$$

Thus

$$m(x) = \sup_{a \in L_{\bar{\Phi}}} \{(x, a) - m_{\bar{\Phi}}(a)\} .$$

2_1. On the case $0 \leqq x \in L_{\bar{\Phi}}$, $m(x) = \infty$ with $\sup_{y \in H} m(y) = \infty$ where

$$H \equiv \{y \,;\, 0 \leqq y < x,\ m_{\bar{\Phi}}(y) < \infty\} .$$

By 1_2, $m(y) = \sup_{0 \leqq a \in L_{\bar{\Phi}}} \{(y, a) - m_{\bar{\Phi}}(a)\}$ for $y \in H$. Therefore

$$\infty = \sup_{y \in H} m(y) = \sup_{y \in H} \{ \sup_{0 \leqq a \in L_{\bar{\Phi}}} (y, a) - m_{\bar{\Phi}}(a)\} =$$

$$= \sup_{0 \leqq a \in L_{\bar{\Phi}}} \sup_{y \in H} \{(y, a) - m_{\bar{\Phi}}(a)\} \leqq \sup_{0 \leqq a \in L_{\bar{\Phi}}} \{(x, a) - m_{\bar{\Phi}}(a)\} .$$

Thus

$$\sup_{0 \leqq a \in L_{\bar{\Phi}}} \{(x, a) - m_{\bar{\Phi}}(a)\} = \infty .$$

2_2. On the case $0 \leqq x \in L_\Phi$, $m(x) = \infty$ with $\sup_{y \in H} m(y) \equiv \gamma < \infty$.

In order to avoid complexity, we prove next two lemmas (L. 1) and (L. 2) for this case.

(L. 1) *We have*

$$\chi_{(x - y_0)}\, y_0 = \chi_{(x - y_0)}\, e_1 \tag{1}$$

where $\bigcup_{y \in H} y = y_0$[9] *and* e_1 *is the same of* (*) *in* § 1.

Proof. Putting $\chi_0 = \chi_{(x - y_0)}$ and $y_1 = \chi_0 y_0$, (1) is

$$y_1 = \chi_0 e_1 . \tag{1'}$$

Since

$$\int_\Omega \Phi(y_1(\omega), \omega)\, d\mu = \int_\Omega \Phi(\chi_0 y_0(\omega), \omega)\, d\mu < \infty$$

9) For the existance of y_0, we verify such that because H could be considered a directed system $y \uparrow_{y \in H}$ with $\sup_{y \in H} m(y) < \infty$, consequently $y_0 = \bigcup_{y \in H} y \in L_\Phi$.

by $m(y_1)<\infty$, we have

$$\Phi(y_1(\omega),\ \omega)<\infty \qquad \text{a.e. on } \Omega .$$

Therefore

$$\chi_0(\omega)\ e_1(\omega)\geqq y_1(\omega) \qquad \text{a.e. on } \Omega$$

by (*).

If $\chi_0 e_1 \neq y$, there exist $\varepsilon>0$ and $E(\varepsilon)=E\in\mathfrak{B}$ such that $\mu(E)<\infty$, $\chi_E\leqq\chi_0$ and

$$\chi_0(\omega)\ e_1(\omega)>y_1(\omega)+\varepsilon\ \chi_E(\omega) \qquad \text{a.e. on } \Omega$$

with

$$\Phi(y_1(\omega)+\varepsilon\ \chi_E(\omega),\ \omega)<\infty \qquad \text{a.e. on } \Omega .$$

Therefore, there exists $E_1\subset E$ with $\mu(E_1)>0$ and

$$\int_\Omega \Phi(\chi_{E_1}(\omega)\ y(\omega)+\varepsilon\ \chi_{E_1}(\omega),\ \omega)\ d\mu<\infty .$$

On the other hand, this relation implies $\chi_0(\chi_{E_1}y_1+\varepsilon\ \chi_{E_1})\leqq y_1$ because for any $0\leqq z$ with $m(z)<\infty$, $z\cap x\leqq y_0$ we have $\chi_0 z\leqq y$.

That contradicts to that

$$\chi_{E_1}y_1+\varepsilon\ \chi_{E_1}\leqq\chi_{E_1}y_1 \quad \text{implies} \quad \varepsilon\ \chi_{E_1}=0 .$$

Thus (1′), accordingly (1) too, are valid.

(L. 2) *There exist* $y_1\in L_\Phi$ *and* $a_1\in L_{\bar\Phi}$ *such that*

$$(2) \qquad ((x-y_1),\ a_1)>0$$

$$(3) \qquad m_{\bar\Phi}(\xi\ a_1)\leqq\xi\ (y_1,\ a_1) \qquad (\xi\geqq 0) .$$

Proof. Let y_1 and χ_0 be the same in (1′) of (L. 1). Conveniently there exists $0\leqq a_1\in L_{\bar\Phi}$ such that $\chi_0 a_1=a_1$ by (A).

These y_1 and a_1 satisfy (L. 2) as follows:

For (2);

$$((x-y_1),\ a_1)=\int_\Omega (x-y_1)(\omega)\ a_1(\omega)d\mu>0 ,$$

and for (3);

$$m_{\bar\Phi}(\xi\ a_1)=\int_\Omega \bar\Phi(\xi\ \chi_0(\omega)\ a_1(\omega),\ \omega)\ d\mu\leqq$$

$$\leqq\int_\Omega \xi\ a_1(\omega)\ \chi_0(\omega)\ e_1(\omega)\ d\mu=\int_\Omega \xi\ a_1(\omega)\ y_1(\omega)\ d\mu=(\xi\ a_1,\ y_1)$$

for $\xi\geqq 0$, because we have

$$\bar\Phi(\eta,\ \omega)\leqq\eta\ e_1(\omega) \qquad \text{for } \eta>\eta_0 \text{ and a.e. } \omega \text{ on } \Omega$$

where η_0 is some positive number, by (*) and the definition of $\bar\Phi$.

Now we can calculate the value of the modular for the case 2_2 by (2) and (3):

$$\bar m_{\bar\Phi}(x)=\sup_{0\leqq a\in L_{\bar\Phi}}\{(x,\ a)-m_{\bar\Phi}(a)\}=\sup_{\xi>0}\{(x,\ \xi\ a_1)-m_{\bar\Phi}(\xi\ a_1)\}\geqq$$

$$\geqq\sup_{\xi>0}\{(\xi\ x,\ a_1)-\xi\ (y_1,\ a_1)\}=\sup_{\xi>0}\{\xi\ ((x-y_1),\ a_1)\}$$

Thus, that completes the proof of $(++)$.

2.2.3.

Finally we show $L_{\bar{\Phi}} \cong \bar{L}^m_{\Phi}$; i.e. $L_{\bar{\Phi}}$ is isomorphic to $\bar{L}^m_{\Phi}$.

In fact, for any $0 \leqq a \in \bar{L}^m_{\Phi}$, there exists $\alpha > 0$ such that $\bar{m}(\alpha a) < \infty$ by m 3). For this αa, there exists a system $0 \leqq \hat{a}_\lambda \in L_{\bar{\Phi}}$ $(\lambda \in \Lambda)$ with $a_\lambda \uparrow_{\lambda \in \Lambda} \alpha a$ and so

$$\sup_{\lambda \in \Lambda} m_{\bar{\Phi}}(\hat{a}_\lambda) = \sup_{\lambda \in \Lambda} \bar{m}(\hat{a}_\lambda) \leqq \bar{m}(\alpha a) < \infty$$

by 2.2.2.

Since $m_{\bar{\Phi}}$ is monotone complete by (B), there exists $\hat{a} \in L_{\bar{\Phi}}$ such that $\hat{a}_\lambda \uparrow_{\lambda \in \Lambda} \hat{a}$. Thus $\alpha a = \hat{a}$ and therefore $a = \dfrac{1}{\alpha} \hat{a} \in L_{\bar{\Phi}}$.

§ 3. Application to an integral operator.

We assume $\Omega = [0, 1]$ in this section. As an application of the integral representation shown in § 2, we give a sufficient condition for the complete continuity of an integral operator defined on L_p.

As for the case on Orlicz spaces, A. C. Zaanen [6] discussed on this kind of problems. (See further M. A. Krasnoselskii and Y. B. Rutickii [2]). The follow is considered a special generalization of one of those results to the case L_p.

That is; *for* $\Phi(\xi, t)$ $(\xi \geqq 0,\ t \in [0, 1])$ *and for its complementary function* $\bar{\Phi}(\eta, s)$ $(\eta \geqq 0,\ s \in [0, 1])$ *if*

(1)[10] Φ *and* $\bar{\Phi}$ *satisfy* Δ_2'*-condition; i.e. there exists* $\gamma > 0$ *and* $h(t) \in L_1([0, 1])$ *such that*

$$(\Delta_2') \qquad \Phi(2\xi, t) \leqq \gamma \Phi(\xi, t) + h(t)$$

for all $\xi \geqq 0$ *and a.e.* t *on* $[0, 1]$ *and likewise the analogous condition is satisfied for* $\bar{\Phi}$ *too;*

for $K(s, t)$ $(0 \leqq s,\ t \leqq 1)$ *if*

(2) $K(s, t)$ *is measurable on* $[0, 1] \times [0, 1]$;

(3) $K(s, t) \in L_{\bar{\Phi}}$ *as a function of* t *for a.e.* s *on* $[0, 1]$;

(4) $k(s) = m_{\bar{\Phi}}(K(s, t)) \in L_{\bar{\Phi}}$

then the operation $x = Ky$ $(y \in L_\Phi)$; *i.e.*

$$x(s) = \int_0^1 K(s, t)\, y(t)\, dt \qquad \text{for every } y \in L_\Phi$$

is a completely continuous linear operator on L_Φ.

In fact, linearity of K is clear.

10) (1) is equivalent to that m_Φ (or $m_{\bar{\Phi}}$) is *finite*; i.e. $m_\Phi(x) < \infty$ for all $x \in L_\Phi$ by [1; § 4. Th. 1′]. Refer also [5; § 5. Th. 5. 2].

For $z \in L_{\bar{\Phi}}$ and $y \in L_{\bar{\Phi}}$, we have

$$\int_0^1\int_0^1 |K(s,\ t)\ z(s)\ y(t)|\ ds\ dt = \int_0^1\left(\int_0^1 K(s,\ t)\ y(t)\ dt\right)|z(s)|\ ds \leqq$$

$$\leqq \int_0^1 (m_{\bar{\Phi}}(k(s,\ t)) + m(y))|z(s)|\ ds =$$

$$= \int_0^1 k(s)|z(s)|\ ds + m(y)\int_0^1 |z(s)|\ ds$$

by $z(s) \in L_{\bar{\Phi}} \subset L_1$, (Y_1) of § 2 and assumptions.
Thus, by (C) of § 2

$$x(s) = \int_0^1 K(s,\ t)\ y(t)\ dt\ \in \bar{L}_{\bar{\Phi}}^{m_{\bar{\Phi}}} = L_{\Phi}\ ,$$

that is, the range of K is also in L_{Φ}.

Conditionally modular convergence[11] of m_{Φ}, in this case, is equivalent to norm[12] (by modular) convergence and L_p is reflexive as a Banach space by (1). (Refer [3]). Let $y_n \in L_{\Phi}\ (n=1,\ 2,\ \cdots)$ be a (norm) bounded sequence; i.e. $m(y_n) \leqq M$ for some $0 < M < \infty$, then we can choose a weakly convergent sub-sequence $y_p(t)$ $(p = n_1,\ n_2,\ \cdots)$ such that

$$\lim_{p\to\infty}\int_0^1 z(t)\ y_p(t)\ dt = \int_0^1 z(t)\ y_0(t)\ dt$$

for all $z \in L_p$ and for some $y_0 \in L_{\Phi}$.
Putting $z(t) \equiv K(s,\ t)\ (\in L_{\bar{\Phi}})$, we have

$$\lim_{p\to\infty} x_p(s) \equiv \lim_{p\to\infty}\int_0^1 K(s,\ t)\ y_p(t)\ dt =$$

$$= \int_0^1 K(s,\ t)\ y_0(t)\ dt \equiv x_0(s)\ ,$$

that is, $\lim_{p\to\infty} |x_p(s) - x_0(s)| = 0$ a.e. on $[0,\ 1]$.

Therefore $\lim_{p\to\infty} \Phi(x_p(s) - x_0(s),\ s) = 0$ a.e. on $[0,\ 1]$.

Furthermore we have, by (Y_2)

$$|x_p(s) - x_0(s)| \leqq m_{\bar{\Phi}}(K(s,\ t)) + m(y_p - y_0) \leqq k(s) + M'$$

because we can calculate and define M' such that

$$m(y_p - y_0) \leqq \frac{1}{2}\{m(2y_p) + m(2y_0)\} \leqq$$

$$\leqq \frac{1}{2}\{\gamma M + \gamma_1 + m(2y_0)\} \equiv M'$$

11) In modulared sapce R, a sequence $x_\nu \in R\ (\nu=1,\ 2,\ \cdots)$ is said to be conditionally modular convergent to x, if $\lim_{\nu\to\infty} m(\alpha(x_\nu - x)) = 0$ for some $\alpha > 0$.

12) In a modulared space R, we can introduce a norm by

$$\|x\| = \inf\left\{\frac{1}{|\xi|}\ ;\ m(\xi x) \leqq 1\right\} \qquad (x \in R).$$

for some $0 \leqq \gamma_1 < \infty$ by (1).
Therefore we have

$$\Phi((x_p(s)-x_0(s)),\ s) \leqq \Phi(k(s)+M',\ s).$$

Where the right term is integrable by (4). Thus

$$\lim_{p\to\infty}\int_0^1 \Phi(x_p(s)-x_0(s),\ s)\ ds = 0$$

by theory of Lebesgue integral.
Consequently the sequence

$$x_p = Ky_p \qquad (p=n_1,\ n_2,\ \ldots)$$

converges in norm-topology. Therefore K is completely continuous.

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